Supplemental Material to "Multi-Frequency Phase Synchronization"

Tingran Gao*

Zhizhen Zhao†

1 Technical Proofs

1.1 Proof of Lemma 1

Proof of Lemma 1. The conclusion of this lemma is identical to [Zhong and Boumal, 2018, Theorem 8]; the only difference is that the event probability is slightly larger — in [Zhong and Boumal, 2018, Theorem 8] the event probability is $1-\mathcal{O}\left(n^{-2}\right)$. This can be done by straightforwardly modifying the arguments in the proof of [Zhong and Boumal, 2018, Theorem 8], and at the expense of increasing the absolute constant picked in that proof. Actually, this is already stated by the authors of [Zhong and Boumal, 2018] on page 998 of the published version, in the paragraph right below their Theorem 5. We document here how this modification can be done.

The randomness in the proof of [Zhong and Boumal, 2018, Theorem 8] arises only from the dependence of Lemma 9 and Lemma 10 of [Zhong and Boumal, 2018], so it is sufficient to track the failure probability of the events there. These modifications only need to be stated for real sub-Gaussian random variables, as the trivial passage from real to complex cases is the same as detailed in the proof of [Zhong and Boumal, 2018, Lemma 9].

[Zhong and Boumal, 2018, Lemma 9] is based on the well-known concentration results on the maximum singular value of sub-Gaussian random matrices, in particular, [Rudelson and Vershynin, 2010, Proposition 2.4], which states for any sub-Gaussian random matrix A of dimension n-by-n with independent, zero mean sub-Gaussian entries (whose subgaussian moments are bounded by 1) that, for any $t>0$,

$$\mathbb{P}\left(\sigma_{\max}\left(A\right)>C\sqrt{n}+t\right)\leq 2e^{-ct^2}$$

where $c,C>0$ are positive absolute constants. We take here $t=C\sqrt{n}$, so $\|A\|_2\lesssim\sqrt{n}$ with probability at least $1-2e^{-cC^2n}$. Obviously, there exists sufficiently large absolute constant $C_2>0$ such that

$$e^{-cC^2n}\leq\frac{C_2}{n^{2+\epsilon}}\quad\forall n\in\mathbb{N},$$

where $\epsilon\in(0,2]$ is the arbitrarily chosen but fixed constant in the statement of our Lemma 1.

[Zhong and Boumal, 2018, Lemma 10] attains the event probability $1-\mathcal{O}\left(n^{-2}\right)$ by taking a union bound, over n instances of $1\leq m\leq n$ and $|\mathcal{U}_m|$ instances of $u\in\mathcal{U}_m$, for individual event probabilities of $1-4en^{-5}-4e^{-c_2n/4}$, where c_2 is an absolute positive constant. However, note that in the case of eigenvectors, we have $|\mathcal{U}_m|=1$ (consisting of a singleton, cf. the second paragraph on pp.1000 of [Zhong and Boumal, 2018], right above section title "Introducing auxiliary eigenvector problems"), which is two orders

*Committee on Computational and Applied Mathematics, Department of Statistics, The University of Chicago, Chicago IL, USA (tingrangao@galton.uchicago.edu).

†Department of Electrical and Computer Engineering, Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana IL, USA (zhizhenz@illinois.edu).

of magnitude smaller than the bound $|\mathcal{U}_m| \leq 3n^2$ stated in [Zhong and Boumal, 2018, Lemma 10]. The union bound thus yields the success probability of at least $1 - 4en^{-4} - 4ne^{-c_2 n/4}$, which is $1 - \mathcal{O}\left(n^{-4}\right)$.

Combining both ends lead to the success probability of $1 - \mathcal{O}\left(n^{-(2+\epsilon)}\right)$ for any $\epsilon \in (0, 2]$.

For the last inequality, note that $z = (e^{\iota\theta_1}, \cdots, e^{\iota\theta_n})^\top$, $e^{\iota k(\theta_i - \theta_j)} = z_i^k \overline{z_j^k}$, and $W_{ij}^{(k)} = u_i^{(k)} \overline{u_j^{(k)}}$, and note that $\left|z_i^k\right| = 1$ for all $1 \leq i \leq n$ and $1 \leq k \leq k_{\max}$. We have

$$\begin{aligned}
\left|W_{ij}^{(k)} - e^{\iota k(\theta_i - \theta_j)}\right| &= \left|u_i^{(k)}\overline{u_j^{(k)}} - z_i^k \overline{z_j^k}\right| \leq \left|u_i^{(k)}\right| \cdot \left|u_j^{(k)} - z_j^k\right| + \left|z_j^k\right| \cdot \left|u_i^{(k)} - z_i^k\right| \\
&\overset{\text{(Lemma 1)}}{\leq} \left(1 + C_0 \sigma \sqrt{\frac{\log n}{n}} + 1\right) \cdot C_0 \sigma \sqrt{\frac{\log n}{n}} < (2 + C_0 c_0)\, C_0 \sigma \sqrt{\frac{\log n}{n}}
\end{aligned}$$

where in the last inequality we used the assumption $\sigma < c_0 \sqrt{n / \log n}$ ∎

1.2 Proof of Lemma 2

Proof of Lemma 2. The proof starts with some elementary observations for the Dirichlet kernel $\mathrm{Dir}_m : [0, 2\pi] \to \mathbb{R}$, defined as

$$\mathrm{Dir}_m(x) = \sum_{k=-m}^{m} e^{\iota k x} = \frac{\sin\left[(m + 1/2)\, x\right]}{\sin(x/2)}. \tag{1}$$

Note the following (cf. Figure 1):

(1) $|\mathrm{Dir}_m(x)|$ is upper bounded by $1/\sin(x/2)$;

(2) $|\mathrm{Dir}_m(x)|$ vanishes at $2\pi\ell/(2m+1)$, for $\ell \in [2m]$;

(3) A unique local maximum exists between each pair of consecutive zeros on $\mathbb{R}/2\pi$.

Let θ_* be the local maximizer attaining the highest "side lobe" of $|\mathrm{Dir}_m(x)|$ between $2\pi/(2m+1)$ and $4\pi/(2m+1)$ in Figure 1. When $\phi \in [\theta_*, 2\pi - \theta_*]$, by Lemma 1, the periodogram $\left|\mathrm{Re}\left\{\sum_{k=1}^{k_{\max}} W_{ij}^{(k)} e^{-\iota k \phi}\right\}\right|$ will not exceed

$$\begin{aligned}
&\frac{1}{2}\left[\sin(\theta_*/2)\right]^{-1} - \frac{1}{2} + 2C_2 k_{\max} \sigma \sqrt{\log n / n} \\
&\leq \frac{1}{2}\left[\sin\left(\frac{\pi}{2m+1}\right)\right]^{-1} - \frac{1}{2} + 2C_2 k_{\max} \sigma \sqrt{\log n / n}.
\end{aligned} \tag{2}$$

On the other hand, again by Lemma 1, the periodogram $\left|\mathrm{Re}\left\{\sum_{k=1}^{k_{\max}} W_{ij}^{(k)} e^{-\iota k \phi}\right\}\right|$ stays above

$$\begin{aligned}
&\frac{1}{2}\left|\mathrm{Dir}_m(0) - 1\right| - 2C_2 k_{\max} \sigma \sqrt{\log n / n} \\
&= m - 2C_2 k_{\max} \sigma \sqrt{\log n / n}.
\end{aligned} \tag{3}$$

Therefore, as long as the upper bound (2) is no greater than the lower bound (3), which one can check is satisfied if condition (19) in the state of the lemma holds, i.e., if

$$\left[2k_{\max} \sin\left(\frac{\pi}{2k_{\max}+1}\right)\right]^{-1} + 4C_2 \sigma \sqrt{\log n / n} < 1$$

then the peak location of the periodogram $\left|\mathrm{Re}\left\{\sum_{k=1}^{k_{\max}} W_{ij}^{(k)} e^{-\iota k \phi}\right\}\right|$ can occur nowhere other than within $[0, \theta_*] \cup [2\pi - \theta_*, 2\pi]$, which gives the conclusion

$$\left|\hat{\theta}_{ij} - (\theta_i - \theta_j)\right| \leq \theta_* < \frac{4\pi}{2m+1}$$

with $m = k_{\max}$. This completes the proof.

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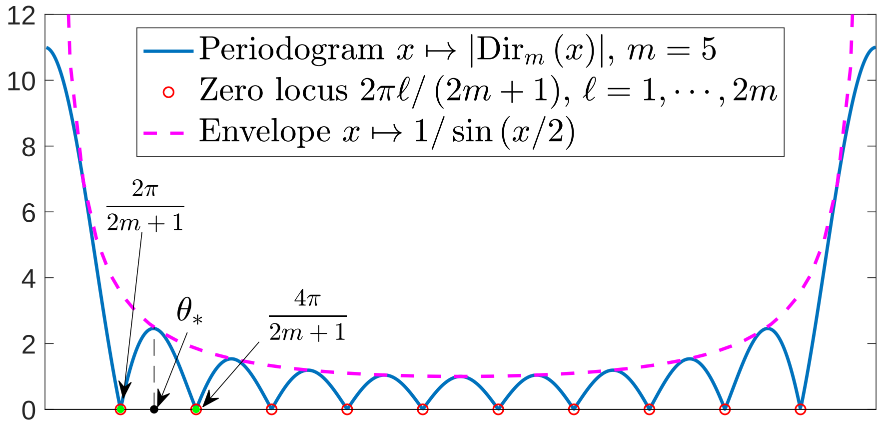


Figure 1: Illustration of the periodogram of a Dirichlet kernel function with $m = 5$ on $[0, 2\pi]$, for the proof of Lemma 2.

1.3 Proof of Theorem 2

Proof of Theorem 2. First, we note that the second part of the theorem about $\hat{x}$ follows directly from [Liu et al., 2017, Proposition 1], as in the proof of [Zhong and Boumal, 2018, Lemma 8].

Assuming for the moment that the key assumption in Lemma 2 is satisfied, namely, n and $k_{\max}$ have been chosen such that

$$\frac{1}{2k_{\max}\sin\left(\dfrac{\pi}{2k_{\max}+1}\right)} + 4C_2\sigma\sqrt{\frac{\log n}{n}} < 1. \tag{4}$$

With a union bound over each of the $\mathcal{O}\left(n^2\right)$ estimated relative phases $\hat{\theta}_{ij}$ obtained at the end of the Step 2 of Algorithm 1, with probability at least $1 - \mathcal{O}\left(n^2 \cdot n^{-(2+\epsilon)}\right) = 1 - \mathcal{O}\left(n^{-\epsilon}\right)$ we have for all $(i,j) \in E$

$$\left|\hat{\theta}_{ij} - (\theta_i - \theta_j)\right| \leq \frac{4\pi}{2k_{\max}+1}$$

and thus

$$\left|\widehat{H}_{ij} - z_i\overline{z}_j\right| = \left|e^{\iota\hat{\theta}_{ij}} - e^{\iota(\theta_i-\theta_j)}\right| \leq \left|\hat{\theta}_{ij} - (\theta_i - \theta_j)\right| \leq \frac{4\pi}{2k_{\max}+1}. \tag{5}$$

Therefore,

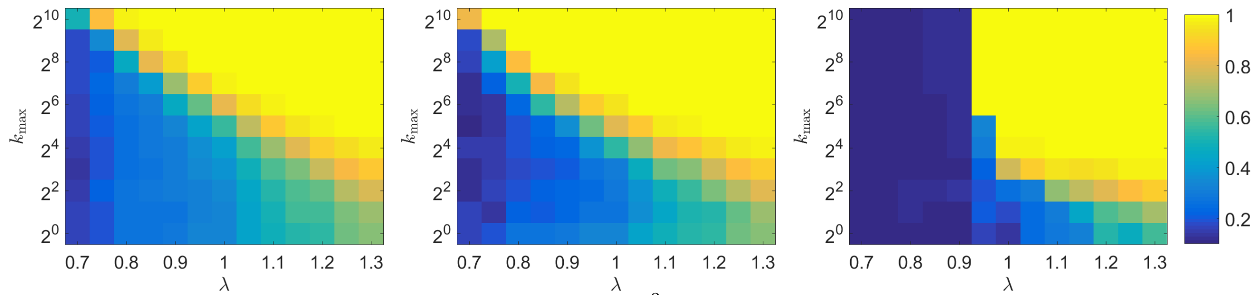
$$\left\|\widehat{H} - zz^*\right\|_2 \leq \left\|\widehat{H} - zz^*\right\|_{\mathrm{Frob}} \leq \frac{4\pi n}{2k_{\max}+1} \tag{6}$$

where the last equality follows from bounding each entry of $H - zz^*$ individually using the rightmost term in (5). (Note that by doing so we do not need any information on the randomness of $H - zz^*$.) By the Davis–Kahan $\sin\Theta$ Theorem in [Zhong and Boumal, 2018, Lemma 11], as long as $n > \left\|\widehat{H}_{ij} - zz^*\right\|_2$, which we know from (6) that can be guaranteed if $k_{\max} > 2\pi - 1/5 \approx 5.7832$, the angle $\theta\left(\hat{u}, z\right)$ between $\hat{u}$ and z satisfies

$$\sin\theta\left(\hat{u}, z\right) \leq \frac{\left\|\widehat{H} - zz^*\right\|_2}{n - \left\|\widehat{H} - zz^*\right\|_2} \leq \frac{\dfrac{4\pi n}{2k_{\max}+1}}{n - \dfrac{4\pi n}{2k_{\max}+1}} = \frac{4\pi}{2k_{\max}+1-4\pi} < \frac{400\sqrt{2}\,\pi}{k_{\max}}$$

where in the last inequality we used the fact that $(2-0.01)\,k_{\max} \geq 4\pi - 1$ for all $k_{\max} \geq 6$. Therefore, setting $C_3 := \left(400\sqrt{2}\,\pi\right)^2$, we have

$$\frac{|\hat{u}^* z|}{\|\hat{u}\|_2 \|z\|_2} = \left|\cos\theta\left(\hat{u}, z\right)\right| \geq \cos^2\theta\left(\hat{u}, z\right) = 1 - \sin^2\theta\left(\hat{u}, z\right) \geq 1 - \frac{C_3}{k_{\max}^2}.$$



(a) PPE-SPC (b) PPE-SPC^3 (c) AMP

Figure 2: U(1) synchronization under Gaussian noise model with $\sigma = \sqrt{n}/\lambda$ for $n = 100$ vertices. Every data point is the median over 20 trials.

Now we seek lower bound for n and $k_{\max}$ that satisfies (4) under the condition $\sigma < c_0\sqrt{n/\log n}$ imposed in Lemma 1. Obviously, (4) is satisfied if

$$2k_{\max}\sin\left(\frac{\pi}{2k_{\max}+1}\right) > \frac{1}{1-4C_2\sigma\sqrt{\log n/n}}. \tag{7}$$

Using the elementary inequality [Kroopnick, 1997]

$$\sin x > \frac{x}{\sqrt{1+x^2}}, \qquad \forall x > 0.$$

we know that a sufficient condition for (7) to hold is

$$\frac{2k_{\max}\cdot\dfrac{\pi}{2k_{\max}+1}}{\sqrt{1+\dfrac{\pi^2}{(2k_{\max}+1)^2}}} > \frac{1}{1-4C_2\sigma\sqrt{\log n/n}} \quad\Leftrightarrow\quad \frac{2k_{\max}\pi}{\sqrt{(2k_{\max}+1)^2+\pi^2}} > \frac{1}{1-4C_2\sigma\sqrt{\log n/n}}. \tag{8}$$

Note that for all $k_{\max} \geq 2$ we have $2k_{\max}+1 > \pi$, and thus $(2k_{\max}+1)^2 + \pi^2 < 2\left(2k_{\max}+1\right)^2$. Therefore, a sufficient condition for the rightmost inequality of (8) to hold is

$$\frac{\sqrt{2}\,\pi k_{\max}}{2k_{\max}+1} > \frac{1}{1-4C_2\sigma\sqrt{\log n/n}} \quad\Leftrightarrow\quad k_{\max} > \frac{1}{\sqrt{2}\,\pi\left(1-4C_2\sigma\sqrt{\log n/n}\right)-2}$$

■

2 Extra Numerical Results

We consider the incomplete graph structure with $n = 100$ vertices under Erdős-Renyi graph model and the edge connection probability $p = 0.23$ for the following experiments. Figure 2 shows that Algorithm 1 (PPE-SPC) is also robust for incomplete graphs.

Figures 3 and 4 show the performance of our PPE-SPC and its variant PPE-SPC^3 on complete graph with $n = 500$ vertices.

References

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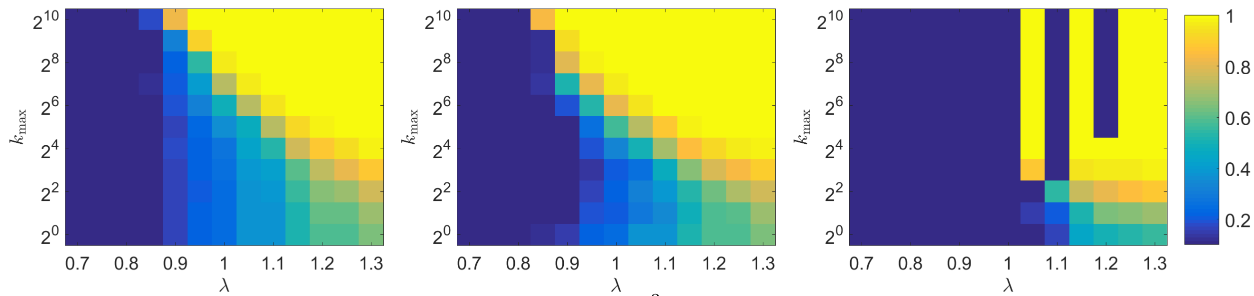
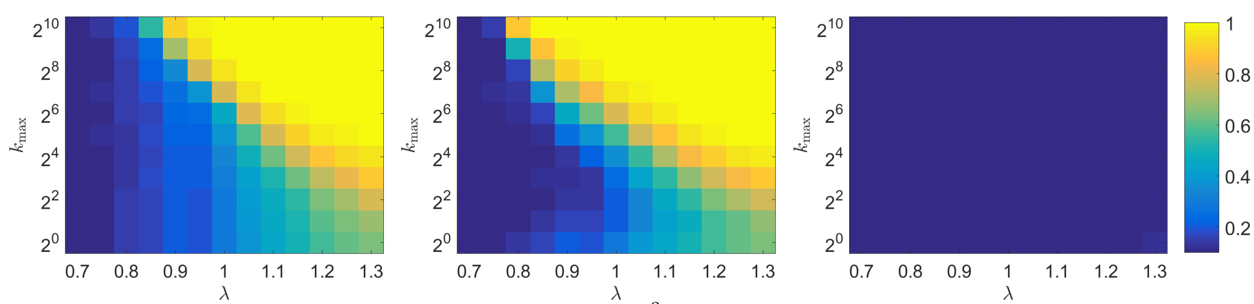


(a) PPE-SPC (b) PPE-SPC^3 (c) AMP

Figure 3: Correlation value for U(1) synchronization under Gaussian noise model for $n = 500$ vertices.



(a) PPE-SPC (b) PPE-SPC^3 (c) AMP

Figure 4: Correlation value for U(1) synchronization under random corruption model with $r = \frac{\lambda}{\sqrt{n}}$ for $n = 500$ vertices and fully connected graph.

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