A. Appendix

A.1. Proof of Lemma 2

Proof. Recall that \( V_n = (X_n^\top X_n + \lambda I_d) \). Note that

\[
\hat{\phi}_n = (X_n^\top X_n + \lambda I_d)^{-1} X_n^\top f^{-1}(\bar{\varepsilon} \circ \bar{\varepsilon})
\]

(39)

\[
= V_n^{-1} X_n^\top f^{-1}(\bar{\varepsilon} \circ \bar{\varepsilon})
\]

(40)

\[
= V_n^{-1} X_n^\top (f^{-1}(\bar{\varepsilon} \circ \bar{\varepsilon}) - X_n \phi_* + X_n \phi_* + \lambda V_n^{-1} \phi_* + \lambda V_n^{-1} \phi_* + \phi_*).
\]

(41)

\[
= V_n^{-1} X_n^\top (f^{-1}(\bar{\varepsilon} \circ \bar{\varepsilon}) - X_n \phi_* - \lambda V_n^{-1} \phi_*).
\]

(42)

Therefore, for any \( x \in \mathbb{R}^d \), we know

\[
|x^\top \hat{\phi}_n - x^\top \hat{\phi}_a| = |x^\top V_n^{-1} X_n^\top (f^{-1}(\bar{\varepsilon} \circ \bar{\varepsilon}) - X_n \phi_* - \lambda x^\top V_n^{-1} \phi_*| \leq \|x\|_{V_n^{-1}} \left( \lambda \|\phi_*\|_{V_n^{-1}} \right)
\]

(43)

Moreover, by rewriting \( \bar{\varepsilon} = \bar{\varepsilon} - \varepsilon + \varepsilon \), we have

\[
f^{-1}(\bar{\varepsilon} \circ \bar{\varepsilon}) = f^{-1}((\bar{\varepsilon} - \varepsilon + \varepsilon) \circ (\bar{\varepsilon} - \varepsilon + \varepsilon)) + f^{-1}(\varepsilon \circ \varepsilon) + M_f^{-1} \left( 2(\varepsilon \circ X_n(\theta_* - \hat{\theta}_n)) + (X_n(\theta_* - \hat{\theta}_n) \circ X_n(\theta_* - \hat{\theta}_n)) \right).
\]

(44)

where (50)-(51) follow from the fact that both \( f(\cdot) \) and \( f^{-1}(\cdot) \) are linear with a slope \( M_f \) and \( M_f^{-1} \), respectively, as described in Section 3. Therefore, by (44)-(51) and the Cauchy-Schwarz inequality, we have

\[
|x^\top \hat{\phi}_n - x^\top \hat{\phi}_a| \leq \|x\|_{V_n^{-1}} \left( \lambda \|\phi_*\|_{V_n^{-1}} + \|X_n^\top (f^{-1}(\varepsilon \circ \varepsilon) - X_n \phi_*)\|_{V_n^{-1}} + 2M_f^{-1} \|X_n(\varepsilon \circ X_n(\theta_* - \hat{\theta}_n))\|_{V_n^{-1}} + M_f^{-1} \|X_n(\theta_* - \hat{\theta}_n) \circ X_n(\theta_* - \hat{\theta}_n)\|_{V_n^{-1}} \right).
\]

(52)

A.2. Proof of Lemma 3

We first introduce the following useful lemmas.

Lemma A.1 (Lemma 8.2 in (Erdős et al., 2012)) Let \( \{a_i\}_{i=1}^N \) be \( N \) independent random complex variables with zero mean and variance \( \sigma^2 \) and having uniform sub-exponential decay, i.e., there exists \( \kappa_1, \kappa_2 > 0 \) such that

\[
\mathbb{P}\{|a_i| \geq x^{\kappa_1}\} \leq \kappa_2 e^{-x}.
\]

(53)

where \( \kappa_1 \) and \( \kappa_2 \) are positive constants that depend only on \( \kappa_1, \kappa_2 \). Moreover, for the standard \( a_1^2 \)-distribution, \( \kappa_1 = 1 \) and \( \kappa_2 = 2 \).

For any \( p \times q \) matrix \( A \), we define the induced matrix norm as \( \|A\|_2 := \max_{v \in \mathbb{R}^q, \|v\|_2 = 1} \|Av\|_2 \).

Lemma A.2

\[
\|V_n^{-1/2} X^\top\|_2 \leq 1, \forall n \in \mathbb{N}.
\]

(54)

Proof. By the definition of induced matrix norm,

\[
\|V_n^{-1/2} X^\top\|_2 = \max_{\|v\|_2 = 1} \sqrt{v^\top V_n^{-1} X^\top v} \leq \lambda_{\max}(X^\top X + \lambda I_d)^{-1} \lambda_{\max}(X^\top X) + \lambda \leq 1,
\]

(55)

where (63) follows from the singular value decomposition and \( \lambda_{\max}(X^\top X) \geq 0 \).
To simplify notation, we use \( X \) and \( V \) as a shorthand for \( X_n \) and \( V_n \), respectively. For convenience, we rewrite \( V^{-1/2}X^\top = [v_1 \cdots v_n] \) as the matrix of \( n \) column vectors \( \{v_i\}_{i=1}^n \) (each \( v_i \in \mathbb{R}^d \)) and show the following property.

**Lemma A.3** Let \( v_i \in \mathbb{R}^d \) be the \( i \)-th column of the matrix \( V^{-1/2}X^\top \), for all \( 1 \leq i \leq n \). Then, we have

\[
\sum_{i=1}^n \|v_i\|_2^2 \leq d. \tag{64}
\]

**Proof of Lemma A.3.** Recall that \( \lambda_{\text{max}}(\cdot) \) denotes the largest eigenvalue of a square matrix. We know

\[
\sum_{i=1}^n \|v_i\|_2^2 = \text{tr} \left( (XV^{-1/2})(V^{-1/2}X^\top) \right) \tag{65}
\]

\[
= \text{tr} \left( (V^{-1/2}X)(X^\top V^{-1/2}) \right) \tag{66}
\]

\[
\leq d \cdot \lambda_{\text{max}} \left( (V^{-1/2}X)(X^\top V^{-1/2}) \right) \tag{67}
\]

where (66) follows from the trace of a product being commutative, and (67) follows since the trace is the sum of all eigenvalues. Moreover, we have

\[
\lambda_{\text{max}} \left( (XV^{1/2})(X^\top V^{-1/2}) \right) \tag{68}
\]

\[
= \left\| (XV^{1/2})(X^\top V^{-1/2}) \right\|_2 \tag{69}
\]

\[
\leq \left\| (XV^{1/2}) \right\|_2 \left\| (X^\top V^{-1/2}) \right\|_2 \leq 1, \tag{70}
\]

where (70) follows from the fact that the \( \ell_2 \)-norm is sub-multiplicative. Therefore, by (65)-(70), we conclude that

\[
\sum_{i=1}^n \|v_i\|_2^2 \leq d. \tag{62}
\]

We are now ready to prove Lemma 3.

**Proof of Lemma 3.** To simplify notation, we use \( X \) and \( V \) as a shorthand for \( X_n \) and \( V_n \), respectively. To begin with, we know \( f^{-1}(\varepsilon \circ \varepsilon) - X\phi_\ast = \frac{1}{M_f}((\varepsilon \circ \varepsilon) - f(X\phi_\ast)) \). Therefore, we have

\[
\|X(f^{-1}(\varepsilon \circ \varepsilon) - X\phi_\ast)\|_{V^{-1}} \tag{71}
\]

\[
= \frac{1}{M_f} \sqrt{\varepsilon \circ \varepsilon - f(X\phi_\ast)\top XV^{-1}X^\top (\varepsilon \circ \varepsilon - f(X\phi_\ast))} \tag{72}
\]

where each element in the vector \( (\varepsilon \circ \varepsilon - f(X\phi_\ast)) \) is a centered \( \chi^2 \)-distribution with a scaling of \( f(\phi_\ast, x_i) \). Defining \( W = \text{diag}(f(x_1^\top \phi_\ast), \ldots, f(x_n^\top \phi_\ast)) \), we have

\[
\|X(f^{-1}(\varepsilon \circ \varepsilon) - X\phi_\ast)\|_{V^{-1}} \tag{73}
\]

\[
= \frac{1}{M_f} \sqrt{\varepsilon \circ \varepsilon - f(X\phi_\ast)\top XV^{-1}X^\top (\varepsilon \circ \varepsilon - f(X\phi_\ast))} \tag{74}
\]

We use \( \eta = W^{-1}(\varepsilon \circ \varepsilon - f(X\phi_\ast)) \) as a shorthand and define \( U = (U_{ij}) = WXX^\top X^\top W \). By Lemma A.1 and the fact that \( \varepsilon(x_1), \ldots, \varepsilon(x_n) \) are mutually independent given the contexts \( \{x_i\}_{i=1}^n \), we have

\[
\mathbb{P}\left\{ \eta^\top U \eta - 2 \cdot \text{tr}(U) \geq 2s \left( \sum_{i=1}^n |U_{ii}|^2 \right)^{1/2} \right\} \leq C_1 \exp(-C_2 \sqrt{s}). \tag{77}
\]

Recall that \( V^{-1/2}X^\top = [v_1 \cdots v_n] \). The trace of \( U \) can be upper bounded as

\[
\text{tr}(U) = \text{tr}(WXX^\top X^\top W) \tag{78}
\]

\[
= \text{tr}(V^{-1/2}X^\top WWXX^\top V^{-1/2}) \tag{79}
\]

\[
\leq \sum_{i=1}^n f(x_i^\top \phi_\ast)^2 \cdot \|v_i\|_2^2 \tag{80}
\]

\[
\leq (\sigma_{\text{max}}^2)^2 \sum_{i=1}^n \|v_i\|_2^2 \leq (\sigma_{\text{max}}^2)^2 d, \tag{81}
\]

where the last inequality in (81) follows directly from Lemma A.3. Also by the commutative property of the trace operation, we have

\[
\sum_{i=1}^n |U_{ii}|^2 \leq (\sum_{i=1}^n U_{ii})^2 \leq ((\sigma_{\text{max}}^2)^2 d)^2, \tag{82}
\]

where (a) follows from \( U \) being positive semi-definite (all diagonal elements are nonnegative), and (b) follows from (81). Therefore, by (76)-(82), we have

\[
\mathbb{P}\left\{ \eta^\top U \eta \geq 2s \cdot (\sigma_{\text{max}}^2)^2 d + 2(\sigma_{\text{max}}^2)^2 d \right\} \leq C_1 \cdot \exp(-C_2 \sqrt{s}). \tag{83}
\]

By choosing \( s = \left( \frac{1}{C_2} \ln \frac{C_1}{\delta} \right)^2 \), we have

\[
\mathbb{P}\left\{ \eta^\top U \eta \geq 2(\sigma_{\text{max}}^2)^2 d \left( \left( \frac{1}{C_2} \ln \frac{C_1}{\delta} \right)^2 + 1 \right) \right\} \leq \delta. \tag{85}
\]
We first introduce a useful lemma.

Therefore, we conclude that with probability at least \(1 - \delta\), the following inequality holds

\[
\|X(f^{-1}(\varepsilon \circ X) - X\phi_*)\|_V^{-1} \leq \frac{1}{M_f} \sqrt{2(\sigma_{\text{max}}^2)^2 \cdot d \left( \left( \frac{1}{C_2} \ln \frac{C_1}{\delta} \right)^2 + 1 \right)}.
\]

(87)

A.3. Proof of Lemma 4

We first introduce a useful lemma.

**Lemma A.4 (Theorem 4.1 in (Tropp, 2012))** Consider a finite sequence \(\{A_k\}\) of fixed self-adjoint matrices of dimension \(d \times d\), and let \(\{\gamma_k\}\) be a finite sequence of independent standard normal variables. Let \(\sigma^2 = \|\sum_k A_k^2\|_2\). Then, for all \(s \geq 0\),

\[
P\left\{ \lambda_{\text{max}}\left( \sum_k \gamma_k A_k \right) \geq s \right\} \leq d \cdot \exp\left(-\frac{s^2}{2\sigma^2}\right),
\]

(88)

where \(\lambda_{\text{max}}(\cdot)\) denotes the largest eigenvalue of a square matrix.

Now we are ready to prove Lemma 4.

**Proof of Lemma 4.** To simplify notation, we use \(X\) and \(V\) as a shorthand for \(X_n\) and \(V_n\), respectively. Recall that \(V^{-1/2}X^T = [v_1, v_2, ..., v_n]\) and define \(A_i = v_i v_i^T\) for all \(i = 1, ..., n\). Note that \(A_i\) is symmetric, for all \(i\). Define an \(n \times n\) diagonal matrix \(D = \text{diag}(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)\). Then we have:

\[
\|X(f^{-1}(\varepsilon \circ (X \theta_* - \hat{\theta}))\|_V^{-1} \leq \|V^{-1/2}X^T f^{-1}(\varepsilon \circ (X \theta_* - \hat{\theta}))\|_2 \leq \|V^{-1/2}X^T DX(\theta_* - \hat{\theta})\|_2 \leq \|V^{-1/2}X^T DXV^{-1/2}V^{1/2}(\theta_* - \hat{\theta})\|_2 \leq \|V^{-1/2}X^T DXV^{-1/2}\|_2 \cdot \|V^{1/2}(\theta_* - \hat{\theta})\|_V.
\]

(94)

Next, the first term in (94) can be expanded into

\[
\|V^{-1/2}X^T DXV^{-1/2}\|_2 \leq \sum_{i=1}^n \varepsilon_i v_i v_i^T \|_2 = \sum_{i=1}^n \varepsilon_i \sqrt{\frac{f(\hat{x}_i^\top \phi_*)}{f(x_i^\top \phi_*)}} \cdot \left( f(x_i^\top \phi_*) A_i \right) \|_2
\]

(96)

Note that \(\frac{\xi_i}{\sqrt{f(\hat{x}_i^\top \phi_*)}}\) is a standard normal random variable, for all \(i\). We also define a \(d \times d\) matrix \(\Sigma = \sum_{i=1}^n f(x_i^\top \phi_*) A_i^2\). Then, we have

\[
\Sigma = \sum_{i=1}^n f(x_i^\top \phi_*) \left( v_i v_i^T \right) \left( v_i v_i^T \right) = \sum_{i=1}^n f(x_i^\top \phi_*) \|v_i\|^2 v_i v_i^T.
\]

(97)

(98)

We also know

\[
\|\sum_{i=1}^n A_i\|_2 = \left\| \sum_{i=1}^n v_i v_i^T \right\|_2 
\leq \left\| V^{-1/2} X^T \left( X V^{-1/2} \right) \right\|_2 \leq \left\| \left( V^{-1/2} X^T \right) \left( X V^{-1/2} \right) \right\|_2 \leq 1,
\]

(100)

where (101) follows from Lemma A.2. Moreover, we know

\[
\|\Sigma\|_2 = \left\| \sum_{i=1}^n f(x_i^\top \phi_*) \|v_i\|^2 v_i v_i^T \right\|_2 
\leq d \cdot \sigma_{\text{max}}^2 \|\sum_{i=1}^n A_i\|_2 
\leq d \cdot \sigma_{\text{max}}^2 \|\sum_{i=1}^n A_i\|_2 \leq d \cdot \sigma_{\text{max}}^2,
\]

(102)

where (103) follows from Lemma A.2-A.3, \(f(x_i^\top \phi_*) \leq \sigma_{\text{max}}^2\), and that \(\|v_i v_i^T\|\) is positive semi-definite, and the last inequality follows directly from (101). By Lemma A.4 and the fact that \(\varepsilon(x_1), \cdots, \varepsilon(x_n)\) are mutually independent given the contexts \(\{x_i\}_{i=1}^n\), we know that

\[
P\left\{ \lambda_{\text{max}}\left( \sum_{i=1}^n \varepsilon_i A_i \right) \geq 2 \sqrt{\frac{\|\Sigma\|_2}{d^2}} \right\} \leq d \cdot e^{-s}. \quad (105)
\]

Therefore, by choosing \(s = \ln(d/\delta)\) and the fact that \(\lambda_{\text{max}}\left( \sum_{i=1}^n \varepsilon_i A_i \right) = \|\sum_{i=1}^n \varepsilon_i A_i\|_2\), we obtain

\[
P\left\{ \left\| \sum_{i=1}^n \varepsilon_i A_i \right\|_2 \geq \sqrt{\frac{\sigma_{\text{max}}^2 d \ln \frac{d}{\delta}}{\delta}} \right\} \leq \delta. \quad (106)
\]

Finally, by applying Lemma 1 and (106) to (94), we conclude that for any \(n \in \mathbb{N}\), for any \(\delta > 0\), with probability at least \(1 - \delta\), we have

\[
\|X_n(f^{-1}(\varepsilon \circ X_n(\theta_* - \hat{\theta}_n)))\|_{V_n}^{-1} \leq a_n^{(1)}(\delta) \cdot a_n^{(3)}(\delta). \quad (107)
\]
A.4. Proof of Lemma 5

We first introduce a useful lemma on the norm of the Hadamard product of two matrices.

**Lemma A.5** Given any two matrices $A$ and $B$ of the same dimension, the following holds:

$$\|A \circ B\|_F \leq \text{tr}(AB^\top) \leq \|A\|_2 \cdot \|B\|_2,$$

where $\|\cdot\|$ denotes the Frobenius norm. When $A$ and $B$ are vectors, the above degenerates to

$$\|A \circ B\|_2 \leq \|A\|_2 \cdot \|B\|_2.$$  

**Proof of Lemma 5.** To simplify notation, we use $X$ and $V$ as a shorthand for $X_n$ and $V_n$, respectively. Let $M$ be a positive definite matrix. We have

$$\|AV\|_M = \|M^{1/2}AV\|_2 \leq \|M^{1/2}A\|_2 \cdot \|V\|_2,$$

where the last inequality holds since $\ell_2$-norm is sub-multiplicative. Meanwhile, we also observe that

$$(\theta_\ast - \hat{\theta})^\top X^\top \left(\theta_\ast - \hat{\theta}\right)$$

$$= (\theta_\ast - \hat{\theta})^\top V^{1/2}V^{-1/2}X^\top XV^{-1/2}V^{1/2}(\theta_\ast - \hat{\theta})$$

$$\leq \left\| (\theta_\ast - \hat{\theta})^\top V^{1/2}V^{-1/2}X^\top \right\|_2^2$$

$$\leq \left\| (\theta_\ast - \hat{\theta})^\top \right\|_2^2 \|V^{-1/2}X^\top \|_2^2$$

Therefore, we know

$$\left\| X^\top \left(X(\theta_\ast - \hat{\theta}) \circ X(\theta_\ast - \hat{\theta})\right) \right\|_{V^{-1}}$$

$$\leq \left\| V^{-1/2}X^\top \right\|_2 \left\| (X(\theta_\ast - \hat{\theta}) \circ X(\theta_\ast - \hat{\theta})\right\|_2$$

$$\leq 1 \cdot \left\| X(\theta_\ast - \hat{\theta}) \right\|_2^2$$

$$\leq 1 \cdot \left( (\theta_\ast - \hat{\theta})^\top X^\top X(\theta_\ast - \hat{\theta}) \right)$$

$$\leq \left\| (\theta_\ast - \hat{\theta}) \right\|_V^2 \leq (\alpha_n^{(1)}(\delta))^2,$$

where (118) follows from Lemma A.2 and A.5, and (120) follows from Lemma 1. The proof is complete. □

A.5. Proof of Theorem 2

Recall that $h_\beta(u, v) = \left(\Phi(\beta - \frac{v}{\sqrt{f(v)}})\right)^{-1}$. We first need the following lemma about Lipschitz smoothness of the function $h_\beta(u, v)$.

**Lemma A.6** The function $h_\beta(u, v)$ defined in (31) is (uniformly) Lipschitz smooth on its domain, i.e., there exists a finite $M_h > 0$ ($M_h$ is independent of $u, v, \beta$) such that for any $\beta$ with $|\beta| \leq B$, for any $u_1, u_2 \in [-1, 1]$ and $v_1, v_2 \in [\sigma_{\min}^2, \sigma_{\max}^2]$, we have

$$\|\nabla h_\beta(u_1, v_1) - \nabla h_\beta(u_2, v_2)\| \leq M_h \left\| (u_1) - (u_2) \right\|_2,$$

Moreover, we have

$$h_\beta(u_2, v_2) - h_\beta(u_1, v_1) \leq \left( u_2 - u_1 \right)^\top \nabla h_\beta(u_1, v_1) + \frac{M_h}{2} \left\| u_2 - u_1 \right\|_2^2$$

**Proof of Theorem 2.** Define

$$q_u := \sup_{u_0 \in (-1, 1)} \left| \frac{\partial h_\beta}{\partial u} \right|_{u = u_0},$$

$$q_v := \sup_{v_0 \in [\sigma_{\min}^2, \sigma_{\max}^2]} \left| \frac{\partial h_\beta}{\partial v} \right|_{v = v_0}.$$

By the discussion in the proof of Lemma A.6, we know that $q_u$ and $q_v$ are both positive real numbers. By substituting $u_1 = \theta_1^\top x$, $u_2 = \theta_2^\top x$, $v_1 = f(\phi_1^\top x)$, and $v_2 = f(\phi_2^\top x)$ into (123), we have

$$h_\beta(\theta_1^\top x, \phi_2^\top x) - h_\beta(\theta_1^\top x, \phi_1^\top x)$$

$$\leq \left( (\theta_2 - \theta_1)^\top x \right)^\top \nabla h_\beta(\theta_1^\top x, f(\phi_1^\top x))$$

$$+ \frac{M_h}{2} \left\| (\theta_2 - \theta_1)^\top x \right\|_2^2$$

into (123), we have

$$h_\beta(\theta_1^\top x, \phi_2^\top x) - h_\beta(\theta_1^\top x, \phi_1^\top x)$$

$$\leq \left( (\theta_2 - \theta_1)^\top x \right)^\top \nabla h_\beta(\theta_1^\top x, f(\phi_1^\top x))$$

$$+ \frac{M_h}{2} \left\| (\theta_2 - \theta_1)^\top x \right\|_2^2$$

into (123), we have

$$h_\beta(\theta_1^\top x, \phi_2^\top x) - h_\beta(\theta_1^\top x, \phi_1^\top x)$$

$$\leq \left( (\theta_2 - \theta_1)^\top x \right)^\top \nabla h_\beta(\theta_1^\top x, f(\phi_1^\top x))$$

$$+ \frac{M_h}{2} \left\| (\theta_2 - \theta_1)^\top x \right\|_2^2$$
where (130)-(131) follow from the Cauchy-Schwarz inequality and the fact that \( f(\cdot) \) is Lipschitz continuous, and (132)-(133) follow from the facts that \( \|x\|_2 \leq 1, \|\theta_2 - \theta_1\|_2 \leq 2 \), and \( \|\phi_2 - \phi_1\|_2 \leq 2L \). By letting \( C_3 = q_0 + M_h \) and \( C_4 = M_f(q_0 + M_hM_fL) \), we conclude (32)-(33) indeed holds with \( C_3 \) and \( C_4 \) being independent of \( \theta_1, \theta_2, \phi_1, \phi_2, \) and \( \beta \).

\[ \text{A.6. Proof of Lemma 6} \]

**Proof.** By Theorem 2 and (35), we know

\[
Q_{t+1}^{HR}(x) - h_{\beta_{t+1}}(\theta^*_x, \phi^*_x) \leq h_{\beta_{t+1}}(\theta^*_t x, \phi^*_t) + \xi_t(\delta) \|x\|_{V^{-1}} - h_{\beta_{t+1}}(\theta^*_x, \phi^*_x) \leq 2\xi_t(\delta) \|x\|_{V^{-1}}.
\]

Similarly, by switching the roles of \( \theta^*_x, \phi^*_x \) and \( \hat{\theta}^*_t, \hat{\phi}^*_t \) in (135), we have

\[
Q_{t+1}^{HR}(x) - h_{\beta_{t+1}}(\theta^*_t x, \phi^*_t) \geq 0.
\]

\[ \text{A.7. Proof of Theorem 3} \]

**Proof.** For each user \( t \), let \( \pi_t^{HR} = \{x_{t,1}, x_{t,2}, \ldots\} \) denote the action sequence under the HR-UCB policy. Under HR-UCB, \( \hat{\theta}_t \) and \( \hat{\phi}_t \) are updated only after the departure of each user. This fact implies that \( x_{t,i} = x_{t,j} \), for all \( i, j \). Therefore, we can use \( x_t \) to denote the action chosen by HR-UCB for the user \( t \), to simplify notation. Let \( T_t^{HR} \) denote the expected lifetime of user \( t \) under HR-UCB. Similar to (30), we have

\[
\overline{T}_t^{HR} = \left( \frac{\beta_t - \theta^*_x x_t}{\sqrt{f(\phi^*_x x_t)}} \right)^{-1} = h_{\beta_t}(\theta^*_x x_t, \phi^*_x x_t).
\]

Recall that \( \pi^{oracle} \) and \( x_t^* \) denote the oracle policy and the context of the action of the oracle policy for user \( t \), respectively. We compute the pseudo regret of HR-UCB as

\[
\text{Regret}_T = \sum_{t=1}^{T} R_t^{HR} - \overline{R}_t^{HR} \leq \sum_{t=1}^{T} h_{\beta_t}(\theta^*_x x_t, \phi^*_x x_t) - h_{\beta_t}(\theta^*_x x_t, \phi^*_x x_t).
\]

To simplify notation, we use \( w_t \) as a shorthand for \( h_{\beta_t}(\theta^*_x x_t, \phi^*_x x_t) - h_{\beta_t}(\theta^*_x x_t, \phi^*_x x_t) \). Given any \( \delta > 0 \), define an event \( E_\delta \) in which (12) and (17) hold under the given \( \delta \), for all \( t \in \mathbb{N} \). By Lemma 1 and Theorem 1, we know that the event \( E_\delta \) occurs with probability at least 1 - 3\( \delta \). Therefore, with probability at least 1 - 3\( \delta \), for all \( t \in \mathbb{N} \),

\[
w_t \leq Q_t^{HR}(x_t^*) - h_{\beta_t}(\theta^*_x x_t, \phi^*_x x_t) \leq Q_t^{HR}(x_t) - h_{\beta_t}(\theta^*_x x_t, \phi^*_x x_t) = h_{\beta_t}(\theta^*_x x_t, \phi^*_x x_t) + \xi_t(\delta) \|x_t\|_{V^{-1}} - h_{\beta_t}(\theta^*_x x_t, \phi^*_x x_t) \leq 2\xi_t(\delta) \|x_t\|_{V^{-1}}.
\]

where (141) and (143) follow directly from the definition of the UCB index, (142) follows from the design of HR-UCB algorithm, and (145) is a direct result under the event \( E_\delta \). Now, we are ready to conclude that with probability at least 1 - 3\( \delta \), we have

\[
\text{Regret}_T = \sum_{t=1}^{T} w_t \leq \sum_{t=1}^{T} w_t^2 \leq 4\xi_t^2(\delta)T \sum_{t=1}^{T} \min\{\|x_t\|_{V^{-1}}, 1\} \leq 8\xi_t^2(\delta)T \cdot d \log \left( \frac{S(T) + \lambda d}{\lambda d} \right),
\]

where (146) follows from the Cauchy-Schwarz inequality, (147) follows from the fact that \( \xi_t(\delta) \) is an increasing function in \( t \), and (148) follows from Lemma 10 and 11 in [Abbasi-Yadkori et al., 2011] and the fact that \( V_t = M_d + X_t^T X_t = \lambda I_d + \sum_{i=1}^{t} x_i x_i^T \). By substituting \( \xi_t(\delta) \) into (148) and using the fact that \( S(T) \leq \Gamma(T) \), we know

\[
\text{Regret}_T = O\left( \sqrt{T \log(\Gamma(T)) \cdot \left( \log(\Gamma(T)) + \log\left( \frac{1}{\delta} \right) \right)^2} \right).
\]

By choosing \( \Gamma(T) = KT \) for some constant \( K > 0 \), we thereby conclude that

\[
\text{Regret}_T = O\left( \sqrt{T \log T \cdot \left( \log T + \log\left( \frac{1}{\delta} \right) \right)^2} \right).
\]

The proof is complete. \( \square \)