Supplementary Material for Multiplicative Weights Updates as a Distributed Constrained Optimization Algorithm: Convergence to Second-order Stationary Points Almost Always

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A. Stable manifold theorem

Theorem A.1 (Center-stable manifold theorem, III.7 (Shub, 1987))**.** *Let x^* be a fixed point for the C^r local diffeomorphism $g : \mathcal{X} \to \mathcal{X}$. Suppose that $E = E_s \oplus E_u$, where E_s is the span of the eigenvectors corresponding to eigenvalues of magnitude less than or equal to one of $Dg(x^*)$, and E_u is the span of the eigenvectors corresponding to eigenvalues of magnitude greater than one of $Dg(x^*)$*[1]. *Then there exists a C^r embedded disk W^{cs}_{loc} of dimension $dim(E^s)$ that is tangent to E_s at x^* called the* local stable center manifold. *Moreover, there exists a neighborhood B of x^*, such that $g(W^{cs}_{loc}) \cap B \subset W^{cs}_{loc}$, and $\cap_{k=0}^{\infty} g^{-k}(B) \subset W^{cs}_{loc}$.*

B. Preliminaries on Topology

This section provides fundamentals used in the proof of Theorem 2.3. For more information on proper maps and the fundamental group, see (Ho, 1975) and (Hatcher, 2002).

Definition B.1. X is a *Hausdorff space* if for every pair of points $p, q \in X$, there are disjoint open subsets $U, V \subset X$ such that $p \in U$ and $q \in V$.

Definition B.2. Let X and Y are topological spaces. A map from X to Y, denoted $f : X \to Y$, is called *proper* if the inverse of each compact subset of Y is a compact subset of X.

Example B.3. Let X be a compact space and Y be a Hausdorff space. And suppose $f : X \to Y$ is continuous. Then f is a proper map. Furthermore, if D is compact subset of $\mathbb{R}^n$, then a continuous map $f : D \to D$ is proper.

Definition B.4. A topological space X is *connected* if there are no disjoint open subsets $U, V \subset X$, such that $U \cup V = X$.

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Proceedings of the 36^{th} International Conference on Machine Learning, Long Beach, California, PMLR 97, 2019.

[1]Jacobian of function g.

Definition B.5. A *path* from a point x to a point y in a topological space X is a continuous function $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$. The space X is said to be *path-connected* if there exists a path joining any two points in X. A *homotopy* of paths in X is a family $f_t : [0, 1] \to X$, $0 \le t \le 1$, such that

- The endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t.
- The associated map $F : [0, 1] \times [0, 1] \to X$ defined by $F(s, t) = f_t(s)$ is continuous.

When two paths f_0 and f_1 are connected in this way by a homotopy f_t, they are said to be *homotopic*. The notation for this is $f_0 \simeq f_1$.

Proposition B.6. *The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.*

The equivalence class of a path f under the equivalence relation of homotopy is denoted $[f]$ and called the *homotopy class* of f.
Given two paths $f, g : [0, 1] \to X$ such that $f(1) = g(0)$, there is a *product path* $f \cdot g$ that traverses first f and then g, defined by the formula

$$f \cdot g(s) = \begin{cases} f(2s), & [1 \le s \le \frac{1}{2}] \\ g(2s-1), & [\frac{1}{2} \le s \le 1] \end{cases}$$

Definition B.7. The paths with the same starting and ending point are called *loops*, and the common starting and ending point is called the *basepoint*. The set of all homotopy classes $[f]$ of loops $f : [0, 1] \to X$ at the basepoint x_0 is denoted $\pi_1(X, x_0)$.

Proposition B.8. *$\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [f \cdot g]$.*

And the group $\pi_1(X, x_0)$ is called the *fundamental group*.

Definition B.9. A topological space X is *simply connected* if it is path-connected and has trivial fundamental group.

Example B.10. The space D that is a product of simplexes is simply-connected.

The following theorem (used in the proof of Theorem 2.3) gives a sufficient and necessary condition under which a local homeomorphism $f : X \to Y$ becomes a global homeomorphism.

Theorem B.11 (Theorem 2, (Ho, 1975))**.** *Let X be path-connected and Y be simply-connected Hausdorff spaces. A local homeomorphism $f : X \to Y$ is a global homeomorphism of X to Y if and only if the map f is proper.*

References

Hatcher, A. *Algebraic Topology*. Cambridge University Press, 2002.

Ho, C. A note on proper maps. In *Proceedings of the American Mathematical Society*, 1975.

Shub, M. *Global Stability of Dynamical Systems*. Springer Science and Business Media, 1987.