Supplementary Material:
Spectral Approximate Inference

A. Proof of Claim 1

We first prove $f(\Omega) \subset B$. To this end we introduce the following inequalities for all $x \in \{-1, 1\}^n$:

$$
\langle u_j, x \rangle \leq \|u_j\|_1, \quad \|u_j\|_1 + c \cdot f_j(x) \leq c \cdot \frac{n+1}{2}
$$

which directly leads us to $|c \cdot f_j(x)| \leq \|u_j\|_1 + c \cdot (n+1)/2 \leq c \cdot b_j$, and therefore $f(\Omega) \subset B$. Here, the first inequality of (17) is trivial. The second inequality of (17) is from the fact that the error between each quantization, the second inequality of (17) holds.

The third equality of (18) is based on (9) which implies that for all $x \in S_i \setminus S_{i-1}$.

The fourth equality of (18) is from replacing the summation over $\sum_{x \in \mathcal{F}^{-1}(k) \cap (S_i \setminus S_{i-1})}$ by that over $\sum_{x \in \mathcal{F}^{-1}(k) \cap (S_i \setminus S_{i-1})}$.

Now we prove the bound of $|B|$. From the definition of $B$ and $b_j$, one can easily observe that the following bound on $|B|$ holds:

$$
|B| = \prod_{j=1}^r (2b_j + 1) = 2^r \prod_{j=1}^r \left( \frac{1}{c} \|u_j\|_1 + \frac{n}{2} + 1 \right)
$$

$$
\leq 2^r \prod_{j=1}^r \left( \frac{1}{c} \sqrt{\lambda_j} \|v_j\|_1 + \frac{n}{2} + 1 \right)
$$

where the inequality is from $\|v_j\|_1 \leq \sqrt{n} \|v_j\|_2 = \sqrt{n}$.

B. Proof of Claim 2

Claim 2 holds since

$$
t_i(k) = t_{i-1}(k) + \sum_{x \in \mathcal{F}^{-1}(k) \cap (S_i \setminus S_{i-1})} \exp \left( \langle \theta, x \rangle \right)
$$

$$
= t_{i-1}(k) + \sum_{g_i(x) \in \mathcal{G}_i \cap \mathcal{F}^{-1}(k) \cap (S_i \setminus S_{i-1})} \exp \left( \langle \theta, x \rangle \right)
$$

$$
= t_{i-1}(k) + \sum_{x' \in \mathcal{F}^{-1}(k) \cap (\hat{\Omega}_j, y_{j,1} = 1) \cap S_{i-1}} \exp \left( \langle \theta, g_i^{-1}(x') \rangle \right)
$$

$$
= t_{i-1}(k) + \sum_{x' \in \mathcal{F}^{-1}(k) \cap (\hat{\Omega}_j, y_{j,1} = 1) \cap S_{i-1}} \exp \left( 2\theta_i + \langle \theta, x' \rangle \right)
$$

$$
= t_{i-1}(k) + \exp(2\theta_i) \cdot t_{i-1}(k - \hat{\alpha}_j, y_{j,1} = 1).
$$

In the above, $g_i : S_i \setminus S_{i-1} \rightarrow S_{i-1}$ is a bijection defined by $g_i(x) = x'$ such that $x'_i = x_i$ except for $i = i$. The second equality of (18) is from replacing the summation over $\mathcal{F}^{-1}(k) \cap (S_i \setminus S_{i-1})$ by that over $\mathcal{G}_i$ $\mathcal{F}^{-1}(k) \cap (S_i \setminus S_{i-1})$. The third equality of (18) is based on (9) which implies that for all $x \in S_i \setminus S_{i-1}$, $x' = g_i(x)$ satisfies

$$
f(x) - \hat{\alpha}_j, y_{j,1} = f(x').
$$
Hence, (19) leads us to

\[ g_i(f^{-1}(k) \cap (S_i \setminus S_{i-1})) = g_i(\{x \in S_i \setminus S_{i-1} : f(x) = k\}) \]
\[ = \{x' \in S_{i-1} : f(x') = k - |\bar{a}_{ji}|_{j=1}^r\} \]
\[ = f^{-1}(k - |\bar{a}_{ji}|_{j=1}^r) \cap S_{i-1} \]

and the third equality of (18) follows. The fourth equality of (18) directly follows from the definition of \( g_i \) that \( x'_i = -1 \) and \( (g_i^{-1}(x'))_i = x_i = 1 \).

**C. Proof of Theorem 3**

We first prove the computational complexity of Algorithm 1. Since each \( t(k), t'(k) \) possesses a memory of \( O(|B|) \) and \( |B| \leq 2^r \prod_{j=1}^r (\sqrt{|\lambda_j|}n/c + n/2 + 1) \) from Claim 1, the space complexity of Algorithm 1 is \( O(2^r \prod_{j=1}^r (\sqrt{|\lambda_j|}n/c + n/2 + 1)) \). In addition, as the algorithm iterates \( n \) times while each iteration accesses to \( t(k) \) and \( t'(k) \), Algorithm 1 has \( O(n2^r \prod_{j=1}^r (\sqrt{|\lambda_j|}n/c + n/2 + 1)) \) computational complexity.

Now we provide the bound on the partition function approximation. First, we refer the following error bound introduced in the proof of Claim 1.

\[ |\langle u_j, x \rangle - c \cdot f_j(x)\rangle| \leq c \cdot \frac{n + 1}{2}. \tag{20} \]

Using (20), we provide a bound for \( |\langle u_j, x \rangle^2 - (c \cdot f_j(x))^2\rangle| \) as follows

\[ |\langle u_j, x \rangle^2 - (c \cdot f_j(x))^2\rangle| = |\langle u_j, x \rangle - c \cdot f_j(x)\rangle| |\langle u_j, x \rangle + c \cdot f_j(x)\rangle| \]
\[ \leq c \cdot \frac{n + 1}{2} \left( |2\langle u_j, x \rangle| + c \cdot \frac{n + 1}{2} \right) \]
\[ \leq \frac{1}{4} c^2(n + 1)^2 + c \sqrt{|\lambda_j|n(n + 1)} \tag{21} \]

where the first inequality is from (20) and the second inequality is from \( |\langle u_j, x \rangle| \leq \|u_j\|_1 \leq \sqrt{|\lambda_j|n}. \) From (21), the error bound can be derived as

\[ \frac{Z}{\hat{Z}} = \frac{\sum_{x \in \Omega} \exp \left( \langle \theta, x \rangle + \sum_{j=1}^r \text{sign}(\lambda_j)\langle u_j, x \rangle^2 \right)}{\sum_{x \in \Omega} \exp \left( \langle \theta, x \rangle + \sum_{j=1}^r \text{sign}(\lambda_j)(c \cdot f_j(x))^2 \right)} \]
\[ \leq \max_{x \in \Omega} \frac{\exp \left( \langle \theta, x \rangle + \sum_{j=1}^r \text{sign}(\lambda_j)\langle u_j, x \rangle^2 \right)}{\exp \left( \langle \theta, x \rangle + \sum_{j=1}^r \text{sign}(\lambda_j)(c \cdot f_j(x))^2 \right)} \]
\[ \leq \max_{x \in \Omega} \exp \left( \sum_{j=1}^r |\langle u_j, x \rangle^2 - (c \cdot f_j(x))^2| \right) \]
\[ \leq \exp \left( \frac{1}{4} c^2(n + 1)^2 + c \sqrt{n(n + 1)} \sum_{j=1}^r \sqrt{|\lambda_j|} \right) \]

where the last inequality follows from (21). One can obtain a same bound for \( \hat{Z}/Z \) and this completes the proof of Theorem 3.
D. Proof of Claim 4

The result of Claim 4 directly follows from the following inequality:

\[
\text{KL}(P_Y \| \prod_{j=1}^{r} q_j(y_j)) = - \sum_{y \in Y} P_Y(y) \log \prod_{j=1}^{r} q_j(y_j) - H(P_Y(y)) \\
= - \sum_{y \in Y} P_Y(y) \sum_{j=1}^{r} \log q_j(y_j) - H(P_Y(y)) \\
= - \sum_{j=1}^{r} \sum_{y_j: y \in Y} P_Y(y_j) \log q_j(y_j) - H(P_Y(y)) \\
\geq - \sum_{j=1}^{r} \sum_{y_j: y \in Y} P_Y(y_j) \log P_Y(y_j) - H(P_Y(y))
\]

where the last inequality follows from the source coding theorem (Shannon, 1948).