Hiring Under Uncertainty: Supplementary Material

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1. Gap between optimal adaptive and value-ordered strategies

The following example shows a gap between the optimal adaptive strategy and the any value-ordered strategy in the sequential setting.

$$(p_1, v_1) = (1, 1) \quad (p_2, v_2) = (q, 1)$$
$$(p_3, v_3) = (q, 1) \quad (p_4, v_4) = (q(1-q)/(v-q), v)$$

Here, we set $q = 0.63667$, and take the limit as v goes to ∞.

The optimal value-ordered strategy is make offers to 1, 2, and possibly 3 if 2 rejects. This yields value $1 + 2q - q^2$. The optimal strategy is shown in Figure 1 and yields value $1 + 2q - q^2 + (1-q)q^2(v-1)/(v-q)$. As $v \to \infty$, the approximation ratio approaches

$$\frac{1 + 2q - q^3}{1 + 2q - q^2} \approx 1.0788$$

Moreover, this example demonstrates that simple greedy algorithms are suboptimal – in particular, making offers greedily by decreasing p_i, v_i, and $p_i v_i$ all yield suboptimal value.

2. Lower bound for LP-based stochastic matching

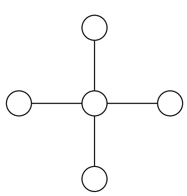

Figure 1. Lower bound for integrality gap

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Consider the star graph as shown in Figure 1 with $n+1$ vertices, with n leaves and 1 vertex in the middle. Each edge has $p_e = \frac{1}{n}$ and value 1. Let the number of probes be $t = n$. The value of the LP (3) is 1, assigning $x_e = 1$ to all edges. Since all edges are identical, any strategy is an optimal probing strategy, yielding expected value

$$\begin{aligned}
\sum_{i=1}^{n} \left(\frac{1}{n}\right)\left(1 - \frac{1}{n}\right)^{i-1} &= \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{1}{n}\right)^{i-1} \\
&= \frac{1}{n} \cdot \frac{1 - \left(1 - \frac{1}{n}\right)^n}{\frac{1}{n}} \\
&= 1 - \left(1 - \frac{1}{n}\right)^n
\end{aligned}$$

In the limit, this is $1 - 1/e$, so no probing strategy can be better than an $\frac{e}{e-1} \approx 1.581$-approximation.

3. Deferred Proofs

Proof of Claim 3. For any set A of at most t items, $\mathsf{size}_2(A) = \sum_{i \in A} \mathsf{size}_2(i) = |A|/t \leq 1$. Further, by Markov's inequality, we have $Pr[\mathsf{size}_1(A) \geq 1] \leq \mathbb{E}\left[\min\{\mathsf{size}_1(A), 1\}\right] \leq \mathbb{E}\left[\sum_{i \in A} \min\{s_i, 1\}\right] = \mu_1(A)$. Consequently, we have $Pr(\|\mathsf{size}(A)\|_\infty < 1) \geq 1 - \mu_1(A)$. $\square$

Claim 1. $\mathbb{E}\left[v_J\right] \geq \frac{1}{2}\mathbb{E}\left[v_I\right]$

Proof. Let W be the random set of elements on this segment, up to and including I, and let $W_x \subseteq W$ be the subset of those elements with value at least x. For ease of notation, we define $q_i = 1 - p_i$. Then, we can write

$$\mathbb{E}\left[v_I\right] = \int_0^\infty \Pr[v_I \geq x]\, dx = \int_0^\infty \sum_{i \in W_x} p_i \prod_{j < i} q_j \, dx. \tag{1}$$

Let A be the random set of "active" candidates who will accept an offer if they receive one. Then, we have

$$\begin{aligned}\mathbb{E}[v_J] &= \int_0^\infty \Pr[v_J \geq x]\, dx \\ &= \int_0^\infty \Pr[A \cap W_x \neq \emptyset]\, dx \\ &= \int_0^\infty \sum_{i \in W_x} p_i \cdot \Pr[I \geq i] \cdot \Pr\left[\bigcap_{j<i, j \in W_x} j \notin A\right] dx \\ &= \int_0^\infty \sum_{i \in W_x} p_i \left(\prod_{j<i} q_j\right)\left(\prod_{j<i, j\in W_x} q_j\right) dx \\ &= \int_0^\infty \sum_{i \in W_x} p_i \left(\prod_{j<i, j\in W_x} q_j^2\right)\left(\prod_{j<i, j\notin W_x} q_j\right) dx \end{aligned} \tag{2}$$

We can write (1) as

$$\mathbb{E}\left[\sum_{i \in W_x} p_i \left(\prod_{j<i, j\in W_x} q_j\right)\left(\prod_{j<i, j\notin W_x} \mathbf{1}_{q_j}\right)\right], \tag{3}$$

where the expectation is taken over the indicators $\mathbf{1}_{q_j}$ for $j < i, j \notin W_x$. Similarly, we write (2) as

$$\mathbb{E}\left[\sum_{i \in W_x} p_i \left(\prod_{j<i, j\in W_x} q_j^2\right)\left(\prod_{j<i, j\notin W_x} \mathbf{1}_{q_j}\right)\right], \tag{4}$$

Conditioning on the realizations of these indicators, it is sufficient to show that

$$\sum_{i \in W_x} p_i \left(\prod_{j<i, j\in W_x} q_j^2\right) \geq \frac{1}{2} \sum_{i \in W_x} p_i \left(\prod_{j<i, j\in W_x} q_j\right), \tag{5}$$

which is true by Claim 2. □

Claim 2 ((Gupta et al., 2017), Claim 3.4)**.** *For any ordered set A of probabilities* $\{a_1, a_2, \ldots, a_{|A|}\}$*, let* b_j *denote* $1 - a_j$ *for* $j \in [1, |A|]$*. Then,*

$$\sum_i a_i \left(\prod_{j<i} b_j\right)^2 \geq \frac{1}{2} \sum_i a_i \prod_{j<i} b_i$$

3.1. Equivalence to Stochastic Knapsack

We must show that the the optimal solution remains unchanged whether values are received stochastically or deterministically.

It is easy to verify that the vector item sizes and knapsack capacities capture the budget and deadline requirements of the knapsack hiring problem. However, in the reduction, item i deterministically yields a value of $p_i v_i$ instead of value v_i when i is active (happens with probability p_i) and value 0 otherwise.

To account for this, observe that the optimal item to probe next depends only on the subset of remaining items, the number of probes left, and the capacity of the knapsack – the value accumulated thus far has no bearing on the next action. Let $\mathsf{opt}(S, t, b)$ be the optimal value achievable with items (candidates) S, number of probes t, and budget b remaining. The optimal strategy is then given by an exponential sized dynamic program, with the following recurrence

$$\begin{aligned}\mathsf{opt}(S,t,b) = \max_{i \in S} \Big\{ & p_i(v_i + \mathsf{opt}(S\backslash\{i\}, t-1, b - s_i)) \\ & + (1-p_i)\mathsf{opt}(S\backslash\{i\}, t-1, b)\Big\}. \end{aligned} \tag{6}$$

Assuming inductively that $\mathsf{opt}(S', t, b)$ is unchanged whether i contributes value v_i with probability p_i or deterministic value $p_i v_i$ for all smaller sets S', we see that (6) is optimized by the same i in both cases. Thus, the optimal strategy is unchanged in the deterministic and random cases and our reduction is complete.

References

Gupta, A., Nagarajan, V., and Singla, S. Adaptivity gaps for stochastic probing: Submodular and XOS functions. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pp. 1688–1702. SIAM, 2017.