Appendix for Locally Private Bayesian Inference for Count Models

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1. Supplementary results

1.1. Community detection $C \in \{5, 10, 20\}$ results

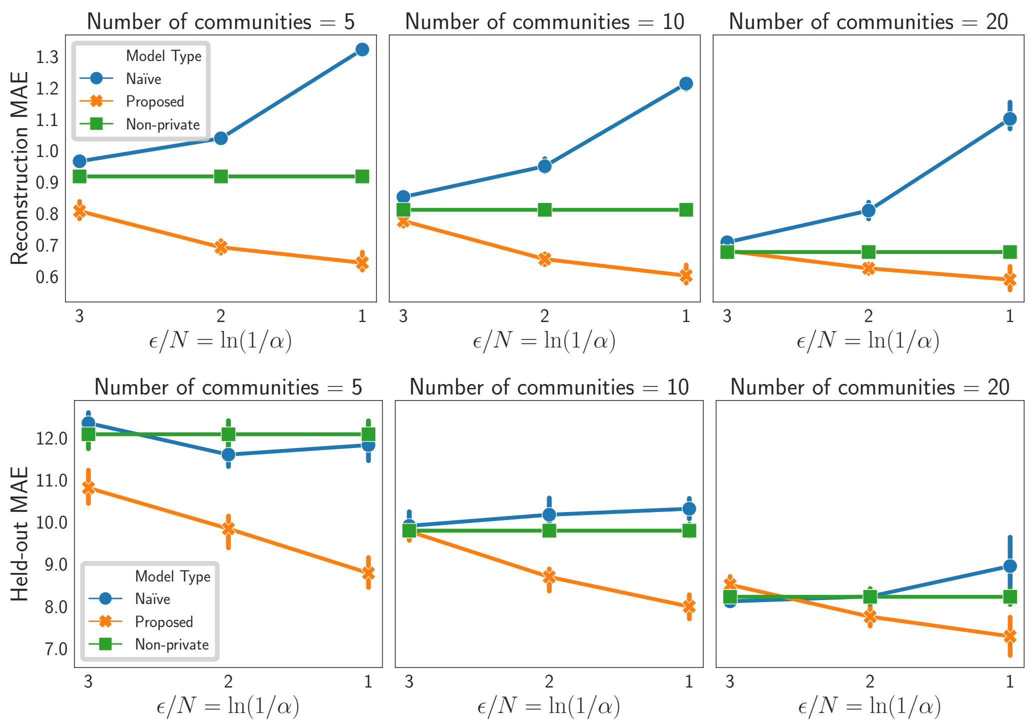

Figure 1. Each subplot compares our method to the two reference methods for increasing levels of privacy. The top subplots depict reconstruction error (lower values are better); the bottom subplots depict held-out reconstruction error (lower values are better).

2. Proof of geometric mechanism as LPLP mechanism

Theorem 1. *Let randomized response method $\mathcal{R}(\cdot)$ be the geometric mechanism with parameter α. Then for any positive integer N, and any pair of observations $y, y' \in \mathcal{Y}$ such that $\|y - y'\|_1 \leq N$, $\mathcal{R}(\cdot)$ satisfies*

$$P\left(\mathcal{R}(y) \in \mathcal{S}\right) \leq e^{\epsilon} P\left(\mathcal{R}(y') \in \mathcal{S}\right) \tag{1}$$

for all subsets $\mathcal{S}$ in the range of $\mathcal{R}(\cdot)$, where

$$\epsilon = N \ln\left(\frac{1}{\alpha}\right). \tag{2}$$

Therefore, for any positive integer N, the geometric mechanism with parameter α is an (N, ϵ)-private randomized response method with $\epsilon = N \ln\left(\frac{1}{\alpha}\right)$. If $\frac{\epsilon'}{N'} = \frac{\epsilon}{N}$, then the geometric mechanism with parameter α is also (N', ϵ')-private.

Proof. It suffices to show that for any integer-valued vector $o \in \mathbb{Z}^d$, the following inequality holds for any pair of observations $y, y' \in \mathcal{Y} \subseteq \mathbb{Z}^d$ such that $\|y - y'\|_1 \leq N$:

$$\exp(-\epsilon) \leq \frac{P\left(\mathcal{R}(y) = o\right)}{P\left(\mathcal{R}(y') = o\right)} \leq \exp(\epsilon), \tag{3}$$

where $\epsilon = N \ln\left(\frac{1}{\alpha}\right)$.

Let ν denote a d-dimensional noise vector with elements drawn independently from $2\text{Geo}(\alpha)$. Then,

$$\begin{aligned}
\frac{P\left(\mathcal{R}(y) = o\right)}{P\left(\mathcal{R}(y') = o\right)} &= \frac{P(\nu = o - y)}{P(\nu = o - y')} &(4)\\
&= \frac{\prod_{i=1}^d \frac{1-\alpha}{1+\alpha}\, \alpha^{|o_i - y_i|}}{\prod_{i=1}^d \frac{1-\alpha}{1+\alpha}\, \alpha^{|o_i - y'_i|}} &(5)\\
&= \alpha^{\left(\sum_{i=1}^d |o_i - y_i| - |o_i - y'_i|\right)}. &(6)
\end{aligned}$$

By the triangle inequality, we also know that for each i,

$$-\,|y_i - y'_i| \leq |o_i - y_i| - |o_i - y'_i| \leq |y_i - y'_i|. \tag{7}$$

Therefore,

$$-\,\|y - y'\|_1 \leq \sum_{i=1}^d \left(|o_i - y_i| - |o_i - y'_i|\right) \leq \|y - y'\|_1. \tag{8}$$

It follows that

$$\alpha^{-N} \leq \frac{P\left(\mathcal{R}(y) = o\right)}{P\left(\mathcal{R}(y') = o\right)} \leq \alpha^N. \tag{9}$$

If $\epsilon = N \ln\left(\frac{1}{\alpha}\right)$, then we recover the bound in equation 3. $\square$

3. Proof of two-sided geometric noise as exponentially-randomized Skellam noise

Theorem 2. *A two-sided geometric random variable $\tau \sim$ 2Geo(α) can be generated as follows:*

$$\tau \sim \textit{Skel}(\lambda^{(+)}, \lambda^{(-)}), \quad \lambda^{(*)} \sim \textit{Exp}(\tfrac{\alpha}{1-\alpha}), \tag{10}$$

where Exp$(\cdot)$ and Skel$(\cdot)$ are the exponential and Skellam distributions. The latter is the marginal distribution of the difference $\tau := g^{(+)} - g^{(-)}$ of two independent Poisson random variables $g^{(*)} \sim$ Pois$(\lambda^{(*)})$, where $* \in \{+, -\}$.

Proof. A two-sided geometric random variable $\tau \sim 2\text{Geo}(\alpha)$ can be generated by taking the difference of two independent and identically distributed geometric random variables:[1]

$$g^{(+)} \sim \text{Geo}(\alpha),\;\; g^{(-)} \sim \text{Geo}(\alpha),\;\; \tau := g^{(+)} - g^{(-)}. \tag{11}$$

The geometric distribution is a special case of the negative binomial distribution, with shape parameter equal to one (Johnson et al., 2005). Furthermore, the negative binomial distribution can be represented as a mixture of Poisson distributions with a gamma mixing distribution. We can therefore re-express equation 11 as follows:

$$\lambda^{(+)} \sim \text{Gam}(1, \tfrac{\alpha}{1-\alpha}),\;\; \lambda^{(-)} \sim \text{Gam}(1, \tfrac{\alpha}{1-\alpha}),\;\; g^{(+)} \sim \text{Pois}(\lambda^{(+)}),\;\; g^{(-)} \sim \text{Pois}(\lambda^{(-)}),\;\; \tau := g^{(+)} - g^{(-)}. \tag{12}$$

Finally, a gamma distribution with shape parameter equal to one is an exponential distribution, while the signed difference of two independent Poisson random variables is marginally a Skellam random variable (Skellam, 1946). We therefore recover the generative process in equation 10. We visualize the Skellam and two-sided geometric distributions in figure 2. $\square$

[1] A video of Unnikrishna Pillai deriving this is available at `https://www.youtube.com/watch?v=V1EyqL1cqTE`.

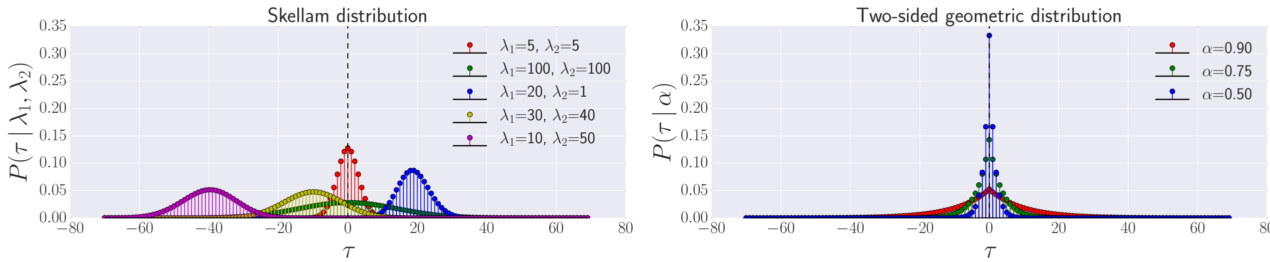

Figure 2. The two-sided geometric distribution (right) can be obtained by randomizing the parameters of the Skellam distribution (left). With fixed parameters, the Skellam distribution can be asymmetric and centered at a value other than zero; however, the two-sided geometric distribution is symmetric and centered at zero. It is also heavy tailed and the discrete analog of the Laplace distribution.

4. Proof of relationship between the Bessel and Skellam distributions

Theorem 3. *Consider two Poisson random variables $y_1 \sim \textit{Pois}(\lambda^{(+)})$ and $y_2 \sim \textit{Pois}(\lambda^{(-)})$. Their minimum $m := \textit{min}\{y_1, y_2\}$ and their difference $\tau := y_1 - y_2$ are deterministic functions of y_1 and y_2. However, if not conditioned on y_1 and y_2, the random variables m and τ can be marginally generated as follows:*

$$\tau \sim \textit{Skel}(\lambda^{(+)}, \lambda^{(-)}),\, m \sim \textit{Bes}\left(|\tau|, 2\sqrt{\lambda^{(+)}\lambda^{(-)}}\right). \tag{13}$$

Proof. We begin by writing out the raw joint probability of y_1 and y_2:

$$\begin{aligned}
P(y_1, y_2) &= \text{Pois}(y_1; \lambda^{(+)})\, \text{Pois}(y_2; \lambda^{(-)}) &(14)\\
&= \frac{(\lambda^{(+)})^{y_1}}{y_1!}\, e^{-\lambda^{(+)}}\, \frac{(\lambda^{(-)})^{y_2}}{y_2!}\, e^{-\lambda^{(-)}} &(15)\\
&= \frac{(\sqrt{\lambda^{(+)}\lambda^{(-)}})^{y_1+y_2}}{y_1!\, y_2!}\, e^{-(\lambda^{(+)}+\lambda^{(-)})} \left(\frac{\lambda^{(+)}}{\lambda^{(-)}}\right)^{(y_1-y_2)\,/\,2}. &(16)
\end{aligned}$$

If $y_1 \geq y_2$, then

$$\begin{aligned}
P(y_1, y_2) &= \frac{(\sqrt{\lambda^{(+)}\lambda^{(-)}})^{y_1+y_2}}{I_{y_1-y_2}(2\sqrt{\lambda^{(+)}\lambda^{(-)}})\, y_1!\, y_2!}\, e^{-(\lambda^{(+)}+\lambda^{(-)})} \left(\frac{\lambda^{(+)}}{\lambda^{(-)}}\right)^{(y_1-y_2)\,/\,2} I_{y_1-y_2}(2\sqrt{\lambda^{(+)}\lambda^{(-)}}) &(17)\\
&= \text{Bes}\left(y_2;\, y_1 - y_2, 2\sqrt{\lambda^{(+)}\lambda^{(-)}}\right) \text{Skel}(y_1 - y_2; \lambda^{(+)}, \lambda^{(-)}); &(18)
\end{aligned}$$

otherwise

$$\begin{aligned}
P(y_1, y_2) &= \frac{(\sqrt{\lambda^{(+)}\lambda^{(-)}})^{y_1+y_2}}{I_{y_2-y_1}(2\sqrt{\lambda^{(+)}\lambda^{(-)}})\, y_1!\, y_2!}\, e^{-(\lambda^{(+)}+\lambda^{(-)})} \left(\frac{\lambda^{(-)}}{\lambda^{(+)}}\right)^{(y_2-y_1)\,/\,2} I_{y_2-y_1}(2\sqrt{\lambda^{(+)}\lambda^{(-)}}) &(19)\\
&= \text{Bes}\left(y_1;\, y_2 - y_1, 2\sqrt{\lambda^{(+)}\lambda^{(-)}}\right) \text{Skel}(y_2 - y_1; \lambda^{(-)}, \lambda^{(+)}) &\\
&= \text{Bes}\left(y_1; -(y_1 - y_2), 2\sqrt{\lambda^{(+)}\lambda^{(-)}}\right) \text{Skel}(y_1 - y_2; \lambda^{(+)}, \lambda^{(-)}). &(20)
\end{aligned}$$

If

$$m := \text{min}\{y_1, y_2\}, \;\; \tau := y_1 - y_2, \tag{21}$$

then we can change variables using the deterministic relationships

$$\begin{aligned}
y_2 = m, \;\; y_1 = m + \tau \;\; &\text{if } \tau \geq 0 &(22)\\
y_1 = m, \;\; y_2 = m - \tau \;\; &\text{otherwise.} &(23)
\end{aligned}$$

The Jacobean can be computed as

$$\begin{vmatrix} \frac{\partial y_1}{\partial m} & \frac{\partial y_1}{\partial \tau} \\ \frac{\partial y_2}{\partial m} & \frac{\partial y_2}{\partial \tau} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}^{\tau \geq 0} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}^{\tau < 0} = 1, \tag{24}$$

so

$$\begin{aligned} P(m, \tau) &= P(y_1, y_2) \begin{vmatrix} \frac{\partial y_1}{\partial m} & \frac{\partial y_1}{\partial \tau} \\ \frac{\partial y_2}{\partial m} & \frac{\partial y_2}{\partial \tau} \end{vmatrix} \\ &= \text{Bes}\left(m; |\tau|, 2\sqrt{\lambda^{(+)}\lambda^{(-)}}\right) \text{Skel}(\tau; \lambda^{(+)}, \lambda^{(-)}). \end{aligned} \tag{25}$$

$\square$

References

Johnson, N. L., Kemp, A. W., and Kotz, S. *Univariate discrete distributions*. 2005.

Skellam, J. G. The frequency distribution of the difference between two Poisson variates belonging to different populations. *Journal of the Royal Statistical Society, Series A (General)*, 109:296, 1946.