1. Supplementary results

1.1. Community detection $C \in \{5, 10, 20\}$ results

![Graph showing community detection results for different numbers of communities and privacy levels.](image)

Figure 1. Each subplot compares our method to the two reference methods for increasing levels of privacy. The top subplots depict reconstruction error (lower values are better); the bottom subplots depict held-out reconstruction error (lower values are better).

2. Proof of geometric mechanism as LPLP mechanism

**Theorem 1.** Let randomized response method $R(\cdot)$ be the geometric mechanism with parameter $\alpha$. Then for any positive integer $N$, and any pair of observations $y, y' \in \mathcal{Y}$ such that $\|y - y'\|_1 \leq N$, $R(\cdot)$ satisfies

$$P(R(y) \in S) \leq e^\epsilon P(R(y') \in S)$$

for all subsets $S$ in the range of $R(\cdot)$, where

$$\epsilon = N \ln \left( \frac{1}{\alpha} \right).$$

Therefore, for any positive integer $N$, the geometric mechanism with parameter $\alpha$ is an $(N, \epsilon)$-private randomized response method with $\epsilon = N \ln \left( \frac{1}{\alpha} \right)$. If $\frac{\epsilon'}{N'} = \frac{\epsilon}{N}$, then the geometric mechanism with parameter $\alpha$ is also $(N', \epsilon')$-private.

**Proof.** It suffices to show that for any integer-valued vector $o \in \mathbb{Z}^d$, the following inequality holds for any pair of
Appendix for Locally Private Bayesian Inference for Count Models

observations $y, y' \in \mathcal{Y} \subseteq \mathbb{Z}^d$ such that $\|y - y'\|_1 \leq N$:

$$\exp(-\epsilon) \leq \frac{P(\mathcal{R}(y) = o)}{P(\mathcal{R}(y') = o)} \leq \exp(\epsilon),$$

where $\epsilon = N \ln \left( \frac{1}{\alpha} \right)$.

Let $\nu$ denote a $d$-dimensional noise vector with elements drawn independently from $2\text{Geo}(\alpha)$. Then,

$$\frac{P(\mathcal{R}(y) = o)}{P(\mathcal{R}(y') = o)} = \frac{P(\nu = o - y)}{P(\nu = o - y')}$$

$$= \frac{\prod_{i=1}^{d} \frac{1-\alpha}{1+\alpha} \alpha^{\mid o_i - y_i \mid}}{\prod_{i=1}^{d} \frac{1-\alpha}{1+\alpha} \alpha^{\mid o_i - y'_i \mid}}$$

$$= \alpha^{\sum_{i=1}^{d} \mid o_i - y_i \mid - \mid o_i - y'_i \mid}.$$ (4)

By the triangle inequality, we also know that for each $i$,

$$-\mid y_i - y'_i \mid \leq \mid o_i - y_i \mid - \mid o_i - y'_i \mid \leq \mid y_i - y'_i \mid.$$ (5)

Therefore,

$$-\|y - y'\|_1 \leq \sum_{i=1}^{d} \mid o_i - y_i \mid - \mid o_i - y'_i \mid \leq \|y - y'\|_1.$$ (6)

It follows that

$$\alpha^{-N} \leq \frac{P(\mathcal{R}(y) = o)}{P(\mathcal{R}(y') = o)} \leq \alpha^{N}.$$ (7)

If $\epsilon = N \ln \left( \frac{1}{\alpha} \right)$, then we recover the bound in equation 3.

3. Proof of two-sided geometric noise as exponentially-randomized Skellam noise

**Theorem 2.** A two-sided geometric random variable $\tau \sim 2\text{Geo}(\alpha)$ can be generated as follows:

$$\tau \sim \text{Skel}(\lambda^{(+)}, \lambda^{(-)}), \quad \lambda^{(*)} \sim \text{Exp}(\frac{\alpha}{1-\alpha}),$$

where $\text{Exp}(\cdot)$ and $\text{Skel}(\cdot)$ are the exponential and Skellam distributions. The latter is the marginal distribution of the difference $\tau := g^{(+) - g^{(-)}$ of two independent Poisson random variables $g^{(*)} \sim \text{Pois}(\lambda^{(*)})$, where $* \in \{+, -, \}$.

**Proof.** A two-sided geometric random variable $\tau \sim 2\text{Geo}(\alpha)$ can be generated by taking the difference of two independent and identically distributed geometric random variables:\n
$$g^{(+)} \sim \text{Geo}(\alpha), \quad g^{(-)} \sim \text{Geo}(\alpha), \quad \tau := g^{(+) - g^{(-)}.$$ (8)

The geometric distribution is a special case of the negative binomial distribution, with shape parameter equal to one (Johnson et al., 2005). Furthermore, the negative binomial distribution can be represented as a mixture of Poisson distributions with a gamma mixing distribution. We can therefore re-express equation 11 as follows:

$$\lambda^{(+)} \sim \text{Gam}(1, \frac{\alpha}{1-\alpha}), \quad \lambda^{(-)} \sim \text{Gam}(1, \frac{\alpha}{1-\alpha}), \quad g^{(+)} \sim \text{Pois}(\lambda^{(+)}), \quad g^{(-)} \sim \text{Pois}(\lambda^{(-)}), \quad \tau := g^{(+) - g^{(-)}.$$ (9)

Finally, a gamma distribution with shape parameter equal to one is an exponential distribution, while the signed difference of two independent Poisson random variables is marginally a Skellam random variable (Skellam, 1946). We therefore recover the generative process in equation 10. We visualize the Skellam and two-sided geometric distributions in figure 2. \hfill \Box

---

1A video of Unnikrishna Pillai deriving this is available at [https://www.youtube.com/watch?v=V1EyqL1cqTE](https://www.youtube.com/watch?v=V1EyqL1cqTE).
Theorem 3. Consider two Poisson random variables \( y_1 \sim \text{Pois}(\lambda^+) \) and \( y_2 \sim \text{Pois}(\lambda^-) \). Their minimum \( m := \min\{y_1, y_2\} \) and their difference \( \tau := y_1 - y_2 \) are deterministic functions of \( y_1 \) and \( y_2 \). However, if not conditioned on \( y_1 \) and \( y_2 \), the random variables \( m \) and \( \tau \) can be marginally generated as follows:

\[
\tau \sim \text{Skell}(\lambda^+, \lambda^-), \quad m \sim \text{Bes}\left(\frac{\tau}{2\sqrt{\lambda^+\lambda^-}}\right).
\]

Proof. We begin by writing out the raw joint probability of \( y_1 \) and \( y_2 \):

\[
P(y_1, y_2) = \text{Pois}(y_1; \lambda^+) \text{Pois}(y_2; \lambda^-) = \frac{(\lambda^+)^{y_1}}{y_1!} e^{-\lambda^+} \frac{(\lambda^-)^{y_2}}{y_2!} e^{-\lambda^-}
\]

\[
= \frac{(\sqrt{\lambda^+\lambda^-})^{y_1+y_2}}{y_1!y_2!} e^{-(\lambda^+ + \lambda^-)} \left(\frac{\lambda^+}{\lambda^-}\right)^{(y_1-y_2)/2} I_{y_1-y_2}(2\sqrt{\lambda^+\lambda^-})
\]

If \( y_1 \geq y_2 \), then

\[
P(y_1, y_2) = \frac{\left(\sqrt{\lambda^+\lambda^-}\right)^{y_1+y_2}}{I_{y_1-y_2}(2\sqrt{\lambda^+\lambda^-})} \frac{e^{-(\lambda^+ + \lambda^-)} \left(\frac{\lambda^+}{\lambda^-}\right)^{(y_1-y_2)/2}}{y_1!y_2!} I_{y_1-y_2}(2\sqrt{\lambda^+\lambda^-})
\]

\[
= \text{Bes}\left(y_2; y_1 - y_2, 2\sqrt{\lambda^+\lambda^-}\right) \text{Skell}(y_1 - y_2; \lambda^+, \lambda^-);
\]

otherwise

\[
P(y_1, y_2) = \frac{\left(\sqrt{\lambda^+\lambda^-}\right)^{y_1+y_2}}{I_{y_2-y_1}(2\sqrt{\lambda^+\lambda^-})} \frac{e^{-(\lambda^+ + \lambda^-)} \left(\frac{\lambda^-}{\lambda^+}\right)^{(y_2-y_1)/2}}{y_1!y_2!} I_{y_2-y_1}(2\sqrt{\lambda^+\lambda^-})
\]

\[
= \text{Bes}\left(y_1; y_2 - y_1, 2\sqrt{\lambda^+\lambda^-}\right) \text{Skell}(y_2 - y_1; \lambda^-, \lambda^+)
\]

\[
= \text{Bes}\left(y_1; -(y_1 - y_2), 2\sqrt{\lambda^+\lambda^-}\right) \text{Skell}(y_1 - y_2; \lambda^+, \lambda^-);
\]

If

\[
m := \min\{y_1, y_2\}, \quad \tau := y_1 - y_2,
\]

then we can change variables using the deterministic relationships

\[
y_2 = m, \quad y_1 = m + \tau \quad \text{if} \quad \tau \geq 0
\]

\[
y_1 = m, \quad y_2 = m - \tau \quad \text{otherwise}.
\]
The Jacobian can be computed as
\[
\begin{vmatrix}
\frac{\partial y_1}{\partial m} & \frac{\partial y_1}{\partial \tau} \\
\frac{\partial y_2}{\partial m} & \frac{\partial y_2}{\partial \tau}
\end{vmatrix} = \begin{vmatrix}
1 & 1 & \tau \geq 0 \\
1 & 0 & \tau < 0
\end{vmatrix} = 1,
\]
so
\[
P(m, \tau) = P(y_1, y_2) \begin{vmatrix}
\frac{\partial y_1}{\partial m} & \frac{\partial y_1}{\partial \tau} \\
\frac{\partial y_2}{\partial m} & \frac{\partial y_2}{\partial \tau}
\end{vmatrix} = \text{Bes}(m; |\tau|, 2\sqrt{\lambda^+\lambda^-}) \text{Skel}(\tau; \lambda^+, \lambda^-).
\]

References

Skellam, J. G. The frequency distribution of the difference between two Poisson variates belonging to different populations. *Journal of the Royal Statistical Society, Series A (General)*, 109:296, 1946.