A. Proof of Proposition 1

From the definition of $C_{\preceq}(y)$, $x \in C_{\preceq}(y)$ iff $x \preceq y$. Then, we will show that $x \in C_{\preceq}(y) \Leftrightarrow C_{\preceq}(x) \subseteq C_{\preceq}(y)$.

($\Leftarrow$) This is obvious because $x \in C_{\preceq}(x)$ holds.

($\Rightarrow$) For arbitrary $z \in C_{\preceq}(x)$, $z \preceq x$ follows the definition of $C_{\preceq}(x)$. Likewise, $x \preceq y$ follows $x \in C_{\preceq}(y)$. Then, $z \preceq y$ holds because of the transitivity, which implies that $C_{\preceq}(x) \subseteq C_{\preceq}(y)$. $\square$

B. Proof of Proposition 2

B.1. Non-negativity

We will demonstrate this proposition by contradiction. Assume $d_W(\boldsymbol{x}, \boldsymbol{y}) < 0$; then, $s_j := \boldsymbol{w}_j^\top(\boldsymbol{x} - \boldsymbol{y}) < 0$ holds for all $j = 1, \cdots, m$. From the assumption $\mathrm{coni}(W) = \mathbb{R}^n$, there exists $a_1, \cdots, a_m \geq 0$ such that $\boldsymbol{x} - \boldsymbol{y} = \sum_{j=1}^m a_j \boldsymbol{w}$. Therefore,

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{y}\|^2 &= (\boldsymbol{x} - \boldsymbol{y})^\top(\boldsymbol{x} - \boldsymbol{y}) \\ &= \sum_{j=1}^m a_j \boldsymbol{w}_j^\top(\boldsymbol{x} - \boldsymbol{y}) = \sum_{j=1}^m a_j s_j. \end{aligned} \tag{B.1}$$

Considering $a_j \geq 0$ and $s_j < 0$, $\|\boldsymbol{x} - \boldsymbol{y}\|^2 < 0$ leads to a contradiction.

B.2. Identity of indiscernibles

If $d_W(x, y) = 0$, $s_j \leq 0$ holds for all $j = 1, \cdots, m$. Considering $a_j \geq 0$ and $s_j \leq 0$ in (B.1), we obtain $\|\boldsymbol{x} - \boldsymbol{y}\|^2 \leq 0$; then, $\boldsymbol{x} = \boldsymbol{y}$.

B.3. Subadditivity

$$\begin{aligned} d_W(\boldsymbol{x}, \boldsymbol{y}) &= \max_j\{\boldsymbol{w}_j^\top(\boldsymbol{x} - \boldsymbol{y})\} \\ &= \max_j\{\boldsymbol{w}_j^\top(\boldsymbol{x} - \boldsymbol{z}) + \boldsymbol{w}_j^\top(\boldsymbol{z} - \boldsymbol{y})\} \\ &\leq \max_j\{\boldsymbol{w}_j^\top(\boldsymbol{x} - \boldsymbol{z})\} + \max_j\{\boldsymbol{w}_j^\top(\boldsymbol{z} - \boldsymbol{y})\} \\ &= d_W(\boldsymbol{x}, \boldsymbol{z}) + d_W(\boldsymbol{z}, \boldsymbol{y}). \quad \square \end{aligned}$$

C. Proof of Theorem 1

Condition (16) is equivalent to $\max_k\{x_k - y_k\} \leq 0$. Thus, we will show that $\max_k\{x_k - y_k\} = d(\boldsymbol{x}', \boldsymbol{y}') - r_x + r_y$ if $\phi_{\mathrm{ord}}(\boldsymbol{x}) = (\boldsymbol{x}', r_x)$, $\phi_{\mathrm{ord}}(\boldsymbol{y}) = (\boldsymbol{y}', r_y)$.

Let $P_\perp = I - P = \frac{1}{n}\mathbf{1}\mathbf{1}^\top$; then,

$$\begin{aligned} \max_k\{x_k - y_k\} &= \max_k\{\boldsymbol{e}_k^\top(\boldsymbol{x} - \boldsymbol{y})\} \\ &= \max_k\{\boldsymbol{e}_k^\top(P + P_\perp)(\boldsymbol{x} - \boldsymbol{y})\}. \end{aligned}$$

Here, considering $P_\perp \boldsymbol{e}_k = \frac{1}{n}\mathbf{1},\ P\boldsymbol{e}_k = \boldsymbol{w}_k,\ P^2 = P$, we find

$$\begin{aligned} \max_k\{x_k - y_k\} &= \max_k\{\boldsymbol{w}_k P(\boldsymbol{x} - \boldsymbol{y})\} + \frac{\mathbf{1}^\top \boldsymbol{x}}{n} - \frac{\mathbf{1}^\top \boldsymbol{y}}{n} \\ &= \max_k\{\boldsymbol{w}_k(\boldsymbol{x}' - \boldsymbol{y}')\} + (a - r_x) - (a - r_y) \\ &= d_W(\boldsymbol{x}', \boldsymbol{y}') - r_x + r_y. \quad \square \end{aligned} \tag{C.2}$$

D. Proof of Theorem 2

By using a uniform norm in (18) instead of a Euclidean norm,

$$\begin{aligned}\|h_+(\boldsymbol{x}-\boldsymbol{y})\|_\infty &= \max_k \{|h_+(x_k - y_k)|\} \\ &= h_+\left(\max_k \{x_k - y_k\}\right) \\ &= h_+\left(d_W\left(\boldsymbol{x}', \boldsymbol{y}'\right) - r_x + r_y\right) \\ &= h_+\left(l_{xy}\right), \end{aligned} \tag{D.3}$$

where $l_{xy} = l\left(\boldsymbol{x}', r_x; \boldsymbol{y}', r_y\right)$ and h_+ is applied element-wise. We used (C.2) for the third equation of (D.3).

From the inequality between the uniform norm and the Euclidean norm $\|\boldsymbol{x}\| \geq \|\boldsymbol{x}\|_\infty$, we find

$$E^{\text{ord}}(\boldsymbol{x}, \boldsymbol{y}) = \|h_+(\boldsymbol{x}-\boldsymbol{y})\|^2 \geq \|h_+(\boldsymbol{x}-\boldsymbol{y})\|_\infty^2 = h_+\left(l_{xy}\right)^2.$$

The equality holds iff

$$\|h_+(\boldsymbol{x}-\boldsymbol{y})\| = \|h_+(\boldsymbol{x}-\boldsymbol{y})\|_\infty,$$

i.e.,

$$|\{k | h_+(x_k - y_k) \neq 0\}| \leq 1. \ \square$$

E. Proof of Theorem 3

We first prove Theorem 4 and then use our results to prove Theorem 3. Thus, see Sec. F first.

By eliminating d_x from (F.5) and (F.11), we obtain

$$\sin(r_x + \theta_0) = \frac{1 + \|\boldsymbol{x}\|^2}{2\|\boldsymbol{x}\|} \sin\theta_0,$$

which is followed by (21).

The equivalence of ordering (3) and (20) is directly derived from Theorem 4 since

$$E_{ij}^{\text{hyp}} \leq 0 \Leftrightarrow l_{ij} \leq 0. \ \square \tag{E.4}$$

F. Proof of Theorem 4

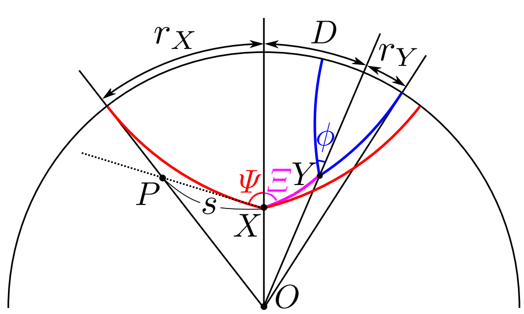

Figure 5. Hyperbolic Entailment cones.

To obtain (23), we present $\psi - \Xi$ in Figure 5 as a function of r_X, r_Y, and D. Let d_x, d_y, and d_{xy} be

$$\begin{aligned} d_x &= d_{\mathbb{D}}(O, X) \\ d_y &= d_{\mathbb{D}}(O, Y) \\ d_{xy} &= d_{\mathbb{D}}(X, Y) \end{aligned}$$

and x and y be

$$x = \|\boldsymbol{x}\| = \tanh \frac{d_x}{2}. \tag{F.5}$$

$$y = \|\boldsymbol{y}\| = \tanh \frac{d_y}{2}. \tag{F.6}$$

Assume that $XP = s$ in the Euclidean triangle $\triangle OPX$; then, $OP = 1 - s$. Thus,

$$(1-s)\sin r_X = s \sin\psi. \tag{F.7}$$

By applying the law of cosines to $\angle POX$, it is shown that

$$s^2 = x^2 + (1-s)^2 - 2x(1-s)\cos r_X. \tag{F.8}$$

By removing s from (F.7) and (F.8) and substituting (F.5), we have

$$\sin\psi = \frac{\sin r_X}{\cosh d_x - \sinh d_x \cos r_X}. \tag{F.9}$$

In addition, from the assumption of Hyperbolic Cones (Ganea et al., 2018),

$$\sin\psi = K\frac{1-x^2}{x} = \frac{2K}{\sinh d_x}. \tag{F.10}$$

Comparing the right-hand side of equations (F.9) and (F.10), we have

$$\coth d_x = \cos r_X + \frac{1}{2K}\sin r_X = \frac{\sin(r_X + \theta_0)}{\sin\theta_0} \tag{F.11}$$

where $\theta_0 = \arctan 2K$.

In the same manner, we have

$$\coth d_y = \frac{\sin(r_Y + \theta_0)}{\sin\theta_0}. \tag{F.12}$$

By substituting (F.11) into (F.10),

$$\begin{aligned}\sin\psi &= 2K\sqrt{\coth^2 d_x - 1} \\ &= \tan\theta_0\sqrt{\frac{\sin^2(r_X - \theta_0)}{\sin^2\theta_0} - 1} \\ &= \frac{\sqrt{\sin(r_X)\sin(r_X + 2\phi_0)}}{\cos\theta_0}.\end{aligned} \tag{F.13}$$

Applying the law of sines and the law of cosines to the hyperbolic triangle $\triangle OXY$, we have

$$\sinh d_y \sin D = \sinh d_{xy} \sin\Xi, \tag{F.14}$$

$$\cos D = \frac{\cosh d_x \cosh d_y - \cosh d_{xy}}{\sinh d_x \sinh d_y}. \tag{F.15}$$

By eliminating d_{xy} from (F.14) and (F.15), and substituting (F.11) and (F.12), it is finally shown that

$$\sin(\Xi - \psi) = 2\sin\left(\frac{r_X - r_Y - D}{2}\right)\frac{\cos\left(\frac{r_X + r_Y - D}{2} + \theta_0\right)}{\cos\theta_0\ \sin\theta_0}\sqrt{\frac{\sin r_X \sin(r_X + 2\theta_0)}{s_X^2 + s_Y^2 - 2s_X s_Y \cos D - \sin^2 D}}, \tag{F.16}$$

where

$$s_X = \frac{\sin(r_X + \theta_0)}{\sin\theta_0}, s_Y = \frac{\sin(r_Y + \theta_0)}{\sin\theta_0}.$$

G. Euclidean Entailment Cones

Similar to Hyperbolic Cones (Ganea et al., 2018), Euclidean Cones are also considered as Disk Embeddings. Here, we will show that $\psi - \Xi$ in Euclidean entailment cones is also represented by Rx, Ry and D.

Let d_x, d_y, and d_{xy} be

$$\begin{aligned} d_x &= d(O, X) = x, \\ d_y &= d(O, Y) = y, \\ d_{xy} &= d(x, y). \end{aligned}$$

(G.17), (G.18), and (G.19) are determined by applying the law of sines to $\triangle OAB$ and $\triangle OXY$:

$$\frac{\sin(\psi - R_x)}{x} = \sin\psi, \tag{G.17}$$

$$\frac{\sin(\phi - R_y)}{y} = \sin\phi, \tag{G.18}$$

$$\frac{\sin\Xi}{y} = \frac{\sin D}{d_{xy}}. \tag{G.19}$$

Moreover, for Euclid entailment cones,

$$\sin\psi = \frac{K}{x}, \tag{G.20}$$

$$\sin\phi = \frac{K}{y}. \tag{G.21}$$

$$\tag{G.22}$$

By applying the law of cosines to $\triangle OXY$, we obtain d_{xy}:

$${d_{xy}}^2 = x^2 + y^2 - 2xy\cos D. \tag{G.23}$$

We represent $\psi - \Xi$ as r_X, r_Y, d_{xy}, and K. By eliminating x, y, d_{xy}, and ϕ from (G.17) to (G.23), it is finally shown that

$$\sin(\psi - \Xi) = \frac{2\sigma_X \cos\left(\frac{r_X - r_Y - D}{2}\right) \sin\left(\frac{r_X + r_Y - D}{2} + \xi_0\right)}{\sqrt{\sigma_X^2 + \sigma_Y^2 - 2\sigma_X\sigma_Y\cos D}}, \tag{G.24}$$

where $\sigma_X = \sin(r_X + \xi_0), \sigma_Y = \sin(r_Y + \xi_0)$ and $\xi_0 = \arcsin K$.

H. Loss functions for Hyperbolic Entailment Cones in Disk Embedding format

In Figure 6, we illustrate values of energy function (23) for l_{ij} with fixed r_i, r_j.

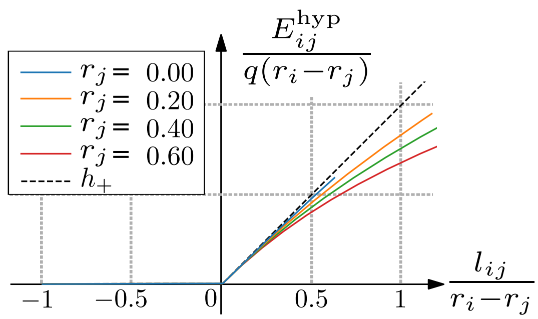

Figure 6. Values of E_{ij}^{hyp} for l_{ij} with fixed r_i, r_j.