I. RESTATEMENT OF THE MAIN RESULT AND ITS PROOF

We restate Theorem 1 below which now includes the pde formulation as well.

**Theorem 1.** Consider the variational problem (11)-(14).

(i) For the probabilistic form (11) of the variational problem, the optimal control $U_t^* = e^{\alpha t} - \gamma Y_t$, where the optimal trajectory $\{(X_t, Y_t)\}_{t \geq 0}$ evolves according to the Hamilton's odes:

\[
\frac{dX_t}{dt} = U_t^* = e^{\alpha t} - \gamma Y_t, \quad X_0 \sim \rho_0
\]

\[
\frac{dY_t}{dt} = -e^{\alpha t + \beta} \nabla \rho F(\rho_t)(X_t), \quad Y_0 = \nabla \phi_0(X_0)
\]

where $\phi_0$ is a convex function, and $\rho_t = \text{Law}(X_t)$.

(ii) For the pde form (14) of the variational problem, the optimal control is $u_t^* = e^{\alpha t} - \gamma \nabla \phi_t(x)$, where the optimal trajectory $\{({\rho_t}, \phi_t)\}_{t \geq 0}$ evolves according to the Hamilton's pdes:

\[
\frac{\partial \rho_t}{\partial t} = -\nabla \cdot (\rho_t e^{\alpha t - \gamma} \nabla \phi_t), \quad \text{initial condn. } \rho_0
\]

\[
\frac{\partial \phi_t}{\partial t} = -e^{\alpha t - \gamma} \frac{\vert \nabla \phi_t \vert^2}{2} - e^{\alpha t + \beta} \nabla \rho F(\rho)
\]

(iii) The solutions of the two forms are equivalent in the following sense:

Law$(X_t) = \rho_t, \quad U_t = u_t(X_t), \quad Y_t = \nabla \phi_t(X_t)$

(iv) Suppose additionally that the functional $F$ is displacement convex and $\rho_\infty$ is its minimizer. Define

\[
V(t) = \frac{1}{2} E(|X_t + e^{-\gamma} Y_t - T_{\rho_t}^{\rho_\infty}(X_t)|^2) + e^{\beta} (F(\rho) - F(\rho_\infty))
\]

where the map $T_{\rho_t}^{\rho_\infty} : \mathbb{R}^d \to \mathbb{R}^d$ is the optimal transport map from $\rho_t$ to $\rho_\infty$. Assume the dimension $d = 1$. Consequently, the following rate of convergence is obtained along the optimal trajectory

$F(\rho_t) - F(\rho_\infty) \leq O(e^{-\beta t}), \quad \forall t \geq 0$

**Proof.** (i) The Hamiltonian function defined in (12) is equal to

\[
H(t,x,\rho,y,u) = y \cdot u - e^{\gamma - \alpha} \frac{1}{2} |u|^2 + e^{\alpha + \gamma \beta} F(\rho, x)
\]
after inserting the formula for the Lagrangian. According to the maximum principle in probabilistic form for (mean-field) optimal control problems (see [1, Sec. 6.2.3]), the optimal control law

\[ U^*_t = \arg \min_v \mathcal{H}(t, X_t, \rho_t, Y_t, v) = e^{\alpha_t - \gamma} Y_t \]

and the Hamilton’s equations are

\[
\frac{dX_t}{dt} = + \nabla_x H(t, X_t, \rho_t, Y_t, U^*_t) = U^*_t = e^{\alpha_t - \gamma} Y_t
\]

\[
\frac{dY_t}{dt} = - \nabla_x H(t, X_t, \rho_t, Y_t, U^*_t) - \tilde{E}[\nabla_\rho H(t, \tilde{X}_t, \rho_t, \tilde{Y}_t, \tilde{U}^*_t)(X_t)]
\]

where \( \tilde{X}_t, \tilde{Y}_t, \tilde{U}^*_t \) are independent copies of \( X_t, Y_t, U^*_t \). The derivatives

\[
\nabla_x H(t, x, \rho, y, u) = e^{\alpha_t + \beta_t + \gamma} \nabla_x \tilde{F}(\rho, x)
\]

\[
\nabla_\rho H(t, x, \rho, y, u) = e^{\alpha_t + \beta_t + \gamma} \nabla_\rho \tilde{F}(\rho, x)
\]

It follows that

\[
\frac{dY_t}{dt} = - e^{\alpha_t + \beta_t + \gamma} \left( \nabla_x \tilde{F}(\rho_t, X_t) + \tilde{E}[\nabla_\rho \tilde{F}(\rho_t, \tilde{X}_t)](X_t) \right) = - e^{\alpha_t + \beta_t + \gamma} \nabla_\rho F(\rho)(X_t)
\]

where we used the definition \( F(\rho) = \int \tilde{F}(x, \rho) \rho(x) \, dx \) and the identity [1, Sec. 5.2.2 Example 3]

\[
\nabla_\rho F(\rho)(x) = \nabla_x \tilde{F}(\rho, x) + \int \nabla_\rho \tilde{F}(\rho, \tilde{x})(x) \rho(\tilde{x}) \, d\tilde{x}
\]

(ii) The Hamiltonian function defined in (15) is equal to

\[
\mathcal{H}(t, \rho, \phi, u) = \int \left[ \nabla \phi(x) \cdot u(x) - \frac{1}{2} e^{\gamma - \alpha_t} |u(x)|^2 \right] \rho(x) \, dx + e^{\alpha_t + \gamma + \beta_t} F(\rho)
\]

after inserting the formula for the Lagrangian. According to the maximum principle for pde formulation of mean-field optimal control problems (see [1, Sec. 6.2.4]) the optimal control vector field is

\[
U^*_t = \arg \min_v \mathcal{H}(t, \rho_t, \phi_t, v) = e^{\alpha_t - \gamma} \nabla \phi_t
\]

and the Hamilton’s equations are:

\[
\frac{\partial \rho_t}{\partial t} = + \frac{\partial \mathcal{H}}{\partial \rho}(t, \rho_t, \phi_t, u_t) = - \nabla \cdot (\rho_t \nabla u^*_t)
\]

\[
\frac{\partial \phi_t}{\partial t} = - \frac{\partial \mathcal{H}}{\partial \rho}(t, \rho_t, \phi_t, u_t) = - (\nabla \phi \cdot u^* - e^{\gamma - \alpha_t} 1/2 |u^*_t|^2) + e^{\alpha_t + \gamma + \beta_t} \frac{\partial F}{\partial \rho}(\rho_t)
\]

inserting the formula \( u^*_t = e^{\alpha_t - \gamma} \nabla \phi_t \) concludes the result.
(iii) Consider the \((\rho_t, \phi_t)\) defined from (I.2). The distribution \(\rho_t\) is identified with a stochastic process \(\tilde{X}_t\) such that \(\frac{d\tilde{X}_t}{dt} = e^{\alpha - \gamma} \nabla \phi_t(\tilde{X}_t)\) and Law\((\tilde{X}_t) = \rho_t\). Then define \(\tilde{Y}_t = \nabla \phi_t(\tilde{X}_t)\). Taking the time derivative shows that
\[
\frac{d\tilde{Y}_t}{dt} = \frac{d}{dt} \nabla \phi_t(\tilde{X}_t) = \nabla^2 \phi_t(\tilde{X}_t) \frac{d\tilde{X}_t}{dt} + \nabla \frac{\partial \phi_t}{\partial t}(X_t)
\]
\[
= e^{\alpha - \gamma} \nabla^2 \phi_t(\tilde{X}_t) \nabla \phi_t(\tilde{X}_t) - e^{\alpha - \gamma} \nabla^2 \phi_t(\tilde{X}_t) \nabla \phi_t(X_t) - e^{\alpha + \beta + \gamma} \nabla \frac{\partial F}{\partial \rho}(\rho_t)(\tilde{X}_t)
\]
\[
= -e^{\alpha + \beta + \gamma} \nabla \rho F(\rho_t)(\tilde{X}_t)
\]
with the initial condition \(\tilde{Y}_0 = \nabla \phi_0(\tilde{X}_0)\), where we used the identity \(\nabla x \frac{\partial F}{\partial \rho}(\rho) = \nabla \rho F(\rho)\) [1, Prop. 5.48]. Therefore the equations for \(\tilde{X}_t\) and \(\tilde{Y}_t\) are identical. Hence one can identify \((X_t, Y_t)\) with \((\tilde{X}_t, \tilde{Y}_t)\).

(iv) The energy functional
\[
V(t) = \frac{1}{2} \mathbb{E} \left[ \left| X_t + e^{-\gamma} Y_t - T_{\rho_t}^{\rho_{\infty}}(X_t) \right|^2 \right] + e^{\beta}(F(\rho) - F(\rho_{\infty}))
\]
Then the derivative of the first term is
\[
\mathbb{E} \left[ \left( X_t + e^{-\gamma} Y_t - T_{\rho_t}^{\rho_{\infty}}(X_t) \right) \cdot (e^{\alpha - \gamma} Y_t - \gamma e^{-\gamma} Y_t - e^{\alpha + \beta} \nabla \rho F(\rho_t)(X_t) + \xi(T_{\rho_t}^{\rho_{\infty}}(X_t))) \right]
\]
where \(\xi(T_{\rho_t}^{\rho_{\infty}}(X_t)) := \frac{d}{dt} T_{\rho_t}^{\rho_{\infty}}(X_t)\). Using the scaling condition \(\gamma_t = e^{\alpha_t}\) the derivative of the first term simplifies to
\[
\mathbb{E} \left[ \left( X_t + e^{-\gamma} Y_t - T_{\rho_t}^{\rho_{\infty}}(X_t) \right) \cdot (-e^{\alpha + \beta} \nabla \rho F(\rho_t)(X_t) + \xi(T_{\rho_t}^{\rho_{\infty}}(X_t))) \right]
\]
We claim that when the dimension \(d = 1\), the expectation
\[
\mathbb{E}[(X_t + e^{-\gamma} Y_t - T_{\rho_t}^{\rho_{\infty}}(X_t)) \cdot \xi(T_{\rho_t}^{\rho_{\infty}}(X_t))] = 0 \quad (I.4)
\]
We present the proof for the claim at the end. Assuming that the claim is true, the derivative of the first term simplifies to
\[
\mathbb{E} \left[ \left( X_t + e^{-\gamma} Y_t - T_{\rho_t}^{\rho_{\infty}}(X_t) \right) \cdot (-e^{\alpha + \beta} \nabla \rho F(\rho_t)(X_t)) \right]
\]
The derivative of the second term is
\[
\frac{d}{dt}(\text{second term}) = \dot{\beta} e^\beta (F(\rho_t) - F(\rho_{\infty})) + e^\beta \frac{d}{dt} F(\rho_t)
\]
\[
= e^{\alpha + \beta} (F(\rho_t) - F(\rho_{\infty})) + e^\beta \mathbb{E}[\nabla \rho F(\rho_t)(X_t) e^{\alpha - \gamma} Y_t]
\]
where we used the scaling condition $\dot{\beta}_t = e^{\alpha t}$ and the chain-rule for the Wasserstein gradient [2, Ch. 10, E. Chain rule]. Adding the derivative of the first and second term yields:

$$\frac{dV}{dt}(t) = e^{\alpha t} + \beta_t (F(\rho_t) - F(\rho_\infty) - E \left[ (X_t - T_{\rho_t}^{\rho_\infty}(X_t)) \cdot \nabla \rho F(\rho_t)(X_t) \right])$$

which is negative by variational inequality characterization of the displacement convex function $F(\rho)$ [2, Eq. 10.1.7].

We now present the proof of the claim (I.4) under the assumption that $d = 1$. According to Brenier theorem [3], there exists a convex function $\psi_t$ such that $T_{\rho_t}^{\rho_\infty}(x) = \nabla \psi_t(x)$ and $T_{\rho_\infty}^{\rho_t}(x) = \nabla \psi_t^*(x)$ where $\psi_t^*$ is the convex conjugate of $\psi_t$. Because $\rho_\infty$ is the push-forward of $\rho_t$ under the map $\nabla \psi_t$, we have

$$E[g(\nabla \psi_t(X_t))] = \int g(x) \rho_\infty(x) \, dx,$$

for all measurable functions $g$. Upon taking the derivative with respect to time,

$$\frac{d}{dt} E[g(\nabla \psi_t(X_t))] = \frac{d}{dt} \int g(x) \rho_\infty(x) \, dx = 0$$

Hence by application of the dominated convergence theorem (DCT) and interchanging the expectation and the derivative,

$$E[\frac{d}{dt} g(\nabla \psi_t(X_t))] = E[\nabla g(\nabla \psi_t(X_t)) \cdot \xi(\nabla \psi_t(X_t))] = 0 \quad \text{(I.5)}$$

Letting $g(x) = \psi_t^*(x) - e^{-\gamma t} \int_{-\infty}^x \nabla \phi_t(\nabla \psi_t^*(z)) \, dz - \frac{1}{2} |x|^2$ where $\phi_t$ is defined in part-(ii) of the theorem 1 concludes

$$0 = E[\nabla g(\nabla \psi_t(X_t)) \cdot \xi(\nabla \psi_t(X_t))] = E[X_t - e^{-\gamma t} \nabla \phi_t(X_t) - \nabla \psi_t(X_t)) \cdot \xi(\nabla \psi_t(X_t)))]
= E[X_t - e^{-\gamma t} Y_t - \nabla \psi_t(X_t)) \cdot \xi(\nabla \psi_t(X_t)))]$$

where we used $Y_t = \nabla \phi_t(X_t)$ from part-(iii) of Theorem 1. This concludes the proof of the claim. Note that the application of DCT in (I.5) follows from smoothness of $g(x)$ and assuming $T_{\rho_t}^{\rho_\infty}(x)$ is differentiable with respect to time. Showing $T_{\rho_t}^{\rho_\infty}(x)$ is differentiable with respect to time is technical out of the scope of this work.
II. Wasserstein gradient and Gâteaux derivative

This section contains definitions of the Wasserstein gradient and Gâteaux derivative [2], [1].

Let $F : \mathcal{P}_{ac,2}(\mathbb{R}^d) \to \mathbb{R}$ be a (smooth) functional on the space of probability distributions.

**Gâteaux derivative:** The Gâteaux derivative of $F$ at $\rho \in \mathcal{P}_{ac,2}(\mathbb{R}^d)$ is a real-valued function on $\mathbb{R}^d$ denoted as $\frac{\partial F}{\partial \rho}(\rho) : \mathbb{R}^d \to \mathbb{R}$. It is defined as a function that satisfies the identity

$$
\frac{d}{dt} F(\rho_t) \bigg|_{t=0} = \int_{\mathbb{R}^d} \frac{\partial F}{\partial \rho}(\rho)(x)(-\nabla \cdot (\rho(x)u(x))) \, dx
$$

for all path $\rho_t$ in $\mathcal{P}_{ac,2}(\mathbb{R}^d)$ such that $\frac{\partial \rho_t}{\partial t} = -\nabla \cdot (\rho_t u)$ with $\rho_0 = \rho \in \mathcal{P}_{ac,2}(\mathbb{R}^d)$.

**Wasserstein gradient:** The Wasserstein gradient of $F$ at $\rho$ is a vector-field on $\mathbb{R}^d$ denoted as $\nabla_{\rho} F(\rho) : \mathbb{R}^d \to \mathbb{R}^d$. It is defined as a vector-field that satisfies the identity

$$
\frac{d}{dt} F(\rho_t) \bigg|_{t=0} = \int_{\mathbb{R}^d} \nabla_{\rho} F(\rho)(x) \cdot u(x) \, \rho(x) \, dx
$$

for all path $\rho_t$ in $\mathcal{P}_{ac,2}(\mathbb{R}^d)$ such that $\frac{\partial \rho_t}{\partial t} = -\nabla \cdot (\rho_t u)$ with $\rho_0 = \rho \in \mathcal{P}_{ac,2}(\mathbb{R}^d)$.

The two definitions imply the following relationship [1, Prop. 5.48]:

$$
\nabla_{\rho} F(\rho)(\cdot) = \nabla_{x} \frac{\partial F}{\partial \rho}(\rho)(\cdot)
$$

**Example:** Let $F(\rho) = \int \log \left( \frac{\rho(x)}{\rho_\infty(x)} \right) \rho(x) \, dx$ be the relative entropy functional. Consider a path $\rho_t$ in $\mathcal{P}_{ac,2}(\mathbb{R}^d)$ such that $\frac{\partial \rho_t}{\partial t} = -\nabla \cdot (\rho_t u)$ with $\rho_0 = \rho \in \mathcal{P}_{ac,2}(\mathbb{R}^d)$. Then

$$
\frac{d}{dt} F(\rho_t) = \int \log \left( \frac{\rho_t(x)}{\rho_\infty(x)} \right) \frac{\partial \rho_t}{\partial t}(x) \, dx + \int \frac{\partial \rho_t}{\partial t}(x) \, dx
$$

$$
= -\int \log \left( \frac{\rho_t(x)}{\rho_\infty(x)} \right) \nabla \cdot (\rho_t(x)u(x)) \, dx
$$

$$
= \int \nabla_x \log \left( \frac{\rho_t(x)}{\rho_\infty(x)} \right) \cdot u(x) \, \rho_t(x) \, dx
$$

where the divergence theorem is used in the last step. The definitions of the Gâteaux derivative and Wasserstein gradient imply

$$
\frac{\partial F}{\partial \rho}(\rho)(x) = \log \left( \frac{\rho(x)}{\rho_\infty(x)} \right)
$$

$$
\nabla_{\rho} F(\rho)(x) = \nabla_x \log \left( \frac{\rho(x)}{\rho_\infty(x)} \right)
$$
III. RELATIONSHIP WITH THE UNDER-DAMPED LANGEVIN EQUATION

A basic form of the under-damped (or second order) Langevin equation is given in [4]

\[ dX_t = v_t dt \]
\[ dv_t = -\gamma v_t dt - \nabla f(X_t) dt + \sqrt{2} dB_t \]  

(III.1)

where \( \{B_t\}_{t \geq 0} \) is the standard Brownian motion.

Consider next, the the accelerated flow (19). Denote \( v_t := e^{\alpha t - \gamma t} Y_t \). Then, with an appropriate choice of scaling parameters (e.g. \( \alpha_t = 0, \beta_t = 0 \) and \( \gamma_t = -\gamma t \) ):

\[ dX_t = v_t dt \]
\[ dv_t = -\gamma v_t dt - \nabla f(X_t) dt - \nabla \log(\rho_t(X_t)) \]  

(III.2)

The scaling parameters are chosen here for the sake of comparison and do not satisfy the ideal scaling conditions of [5].

The sdes (III.1) and (III.2) are similar except that the stochastic term \( \sqrt{2} dB_t \) in (III.1) is replaced with a deterministic term \( -\nabla \log(\rho_t(X_t)) \) in (III.2). Because of this difference, the resulting distributions are different. Let \( p_t(x,v) \) denote the joint distribution on \( (X_t,v_t) \) of (III.1) and let \( q_t(x,v) \) denote the joint distribution on \( (X_t,v_t) \) of (III.2). Then the corresponding Fokker-Planck equations are:

\[ \frac{\partial p}{\partial t}(x,v) = -\nabla_x \cdot (p_t(x,v)v) + \nabla_v \cdot (p_t(x,v)(\gamma v + \nabla f(x))) + \Delta_v p_t(x,v) \]
\[ \frac{\partial q}{\partial t}(x,v) = -\nabla_x \cdot (q_t(x,v)v) + \nabla_v \cdot (q_t(x,v)(\gamma v + \nabla f(x))) + \nabla_v \cdot (q_t(x,y)\nabla_x \log(\rho_t(x))) \]

where \( \rho_t(x) = \int q_t(x,v) dv \) is the marginal of \( q_t(x,v) \) on \( x \). The final term in the Fokker-Planck equations are clearly different. The joint distributions are different as well.

The situation is in contrast to the first order Langevin equation, where the stochastic term \( \sqrt{2} dB_t \) and \( -\nabla \log(\rho_t(X_t)) \) are equivalent, in the sense that the resulting distributions have the same marginal distribution as a function of time. To illustrate this point, consider the following two forms of the Langevin equation:

\[ dX_t = -\nabla f(X_t) dt + \sqrt{2} dB_t \]  

(III.3)
\[ dX_t = -\nabla f(X_t) dt - \nabla \log(\rho_t(X_t)) \]  

(III.4)
Let \( p_t(x) \) denote the distribution of \( X_t \) of (III.3) and let \( q_t(x) \) denote the distribution of \( X_t \) of (III.4). The corresponding Fokker-Planck equations are as follows

\[
\frac{\partial p}{\partial t}(x) = -\nabla \cdot (p_t(x) \nabla f(x)) + \Delta p_t(x)
\]
\[
\frac{\partial q}{\partial t}(x) = -\nabla \cdot (q_t(x) \nabla f(x)) + \nabla \cdot (q_t(x) \nabla \log(\rho_t(x)))
\]
\[
= -\nabla \cdot (q_t(x) \nabla f(x)) + \nabla \cdot (q_t(x) \nabla \log(q_t(x)))
\]
\[
= -\nabla \cdot (q_t(x) \nabla f(x)) + \Delta q_t(x)
\]

where we used \( \rho_t(x) = q_t(x) \). In particular, this implies that the marginal probability distribution of the stochastic process \( X_t \) are the same for first order Langevin sde (III.3) and (III.4).

REFERENCES