Supplementary Material
Analytic expressions for $\nabla h_X(M)$

Abstract
This document contains the main technical arguments for the computation of the gradient expressions $\nabla h_X(M)$ introduced as Propositions 1–8 in the main article. Further details on similar derivations can be found in (Couillet et al., 2018).

1. Derivation of the gradient for the examples provided in the article
Our main concern is to find an analytical expression of the gradient defined as:

$$\nabla h_X(M) = -\frac{\delta(M, X)}{\pi p} \oint_{\hat{\Gamma}} g\left(-m_{\tilde{\mu}_p}(z; M)\right) \text{sym}\left(\hat{C}(M^{-1}\hat{C} - zI_p)^{-2}\right) dz.$$  \hspace{1cm} (1)

Equation (1) can be written as:

$$\nabla h_X(M) = -\frac{\delta(M, X)}{\pi p} \text{sym}\left(\hat{C}U\left(\oint_{\hat{\Gamma}} g\left(-m_{\tilde{\mu}_p}(z; M)\right)(\Lambda - zI_p)^{-2}dz\right)U^T\right).$$ \hspace{1cm} (2)

where $M^{-1}\hat{C} = U\Lambda U^T$ in its spectral decomposition. Our main focus is on the diagonal matrix

$$A = \frac{1}{2i\pi} \oint_{\hat{\Gamma}} g\left(-m_{\tilde{\mu}_p}(z; M)\right)(\Lambda - zI_p)^{-2}dz$$

and particularly on its $k$-th diagonal element

$$A_{kk} = \frac{1}{2i\pi} \oint_{\hat{\Gamma}} g\left(-m_{\tilde{\mu}_p}(z; M)\right)\frac{1}{(\lambda_k - z)^2} dz \hspace{1cm} (3)$$

with $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p)$.

To solve (3), we use elementary properties of the rational function $m_{\tilde{\mu}_p}(z; M)$ that we recall is defined as

$$m_{\tilde{\mu}_p}(z; M) = \frac{c}{p} \sum_{i=1}^{p} \frac{1}{\lambda_i - z} + \frac{1-c}{z}.$$  \hspace{1cm} (4)

Remarking that the poles of $m_{\tilde{\mu}_p}(z; M)$ are $\{\lambda_i\}_{i=1}^{p}$ and 0, and that $\{\xi_i\}_{i=1}^{p}$ are the zeros of $m_{\tilde{\mu}_p}(z; M)$ (see (Couillet et al., 2018) for details), we have:

$$m_{\tilde{\mu}_p}(z; M) = \frac{\prod_{i=1}^{p} (z - \xi_i)}{z \prod_{i=1}^{p} (z - \lambda_i)}.$$ \hspace{1cm} (5)

With these ingredients, we can evaluate $A_{kk}$ for various functions $f$ (recall that $g(z) = f(1/z)$).
1.1. Case $f(t) = t$

For this case, $g(t) = \frac{1}{t}$, and thus the integrand of $A_{kk}$ is

$$I = \frac{z \prod_{i=1}^{p} (z - \lambda_i)}{(z - \lambda_k) \prod_{i=1}^{p} (z - \xi_i)}.$$ 

Under this rational form, $A_{kk}$ is easy to evaluate since it only requires to evaluate the residue for each pole of $I$:

- the first order pole $\lambda_k$ for which the residue $R_1$ is given by

$$R_1 = \lim_{z \to \lambda_k} \frac{z \prod_{i=1}^{p} (z - \lambda_i)}{\prod_{i=1}^{p} (z - \xi_i)} = \frac{1}{m_{\tilde{p}}(\lambda_k)} = -\frac{p}{c}$$

- and the first order poles $\xi_j, j \in \{1, \ldots, p\}$ for which the residue $R_2$ is given by:

$$R_2 = \sum_{j=1}^{p} \lim_{z \to \xi_j} \frac{z \prod_{i=1}^{p} (z - \lambda_i)}{(z - \lambda_k)^2 \prod_{i \neq j}^{p} (z - \xi_i)} = \sum_{j=1}^{p} \frac{1}{(\xi_j - \lambda_k)^2} \lim_{z \to \xi_j} \frac{z - \xi_j}{m_{\tilde{p}}'(\xi_j)} = \sum_{j=1}^{p} \frac{1}{(\xi_j - \lambda_k)^2 m_{\tilde{p}}'(\xi_j)}$$

Putting the $p + 1$ residues together then yields:

$$A_{kk} = -\frac{p}{c} + \sum_{j=1}^{p} \frac{1}{(\xi_j - \lambda_k)^2 m_{\tilde{p}}'(\xi_j)}.$$ 

1.2. Case $f(t) = \log(t)$

For this case, $g(t) = -\log(t)$ and therefore the integrand of $A_{kk}$ becomes

$$I = -\frac{\log \left( \frac{\prod_{i=1}^{p} (z - \xi_i)}{\prod_{i=1}^{p} (z - \lambda_i)} \right)}{(\lambda_k - z)^2}.$$ 

Elementary functional analysis allows us to find the discontinuities of this multi-valued function (the $z$'s for which the argument of the logarithm function is negative). This set of points, or branch cuts, are exactly the segments $[\xi_i, \lambda_i]$, $i = 1, \ldots, p$. These segments lie inside the integration contour $\Gamma$, that needs be modified for proper integration; the new contour, denoted $\Gamma_n$, is depicted in Figure 1.

Under $\Gamma_n$, $A_{kk}$ is the sum of several integrals, subdivided in four types:

- integrals over the circles surrounding $\{\xi_j\}_{j=1}^{p}$ which, thanks to the variable change $z = \xi_j + e^{i\theta}$, reduce to

$$\lim_{\epsilon \to 0^-} -\int_{\epsilon}^{2\pi - \epsilon} \log \left( \frac{e^{i\theta} \prod_{i \neq j}^{p} (\xi_j + e^{i\theta} - \xi_i)}{(\xi_j + e^{i\theta}) \prod_{i=1}^{p} (\xi_j + e^{i\theta} - \lambda_i)} \right) \frac{e^{i\theta}}{\lambda_k - \xi_j - e^{i\theta}} \frac{1}{(\lambda_k - \xi_j - e^{i\theta})^2} d\epsilon e^{i\theta} = 0.$$
integrals over the circles surrounding \( \{ \lambda_i \}_{i=1}^{p} \) which are null following the same line of reasoning as for \( \xi_i \).

real integrals over the segments \([\xi_i, \lambda_i]\) which can be computed by remarking that the log function has a discontinuity of \(2\pi i\) at the branch cut.

\[
\frac{1}{2\pi i} \sum_{j=1}^{p} \int_{\xi_j + \epsilon}^{\xi_j - \epsilon} \log |m_{\mu_p}(\xi_j; M)| + i\pi - \log |m_{\mu_p}(\lambda_k - \xi_j)| + i\pi - \log |m_{\mu_p}(\lambda_k - \xi_j)| + i\pi - \log |m_{\mu_p}(\lambda_k - \xi_j)| + i\pi - \log |m_{\mu_p}(\lambda_k - \xi_j)|
\]

the integral over the circle surrounding \( \lambda_k \) computed by remarking that \( \lambda_k \) is a second order pole

\[
\lim_{z \to \lambda_k} \frac{\partial}{\partial z} \left( \log \left( -m_{\mu_p}(z; M) \right) \right)
\]

\[
= \lim_{z \to \lambda_k} \sum_{j=1}^{p} \frac{1}{z - \xi_j} - \frac{1}{z - \lambda_k} - \sum_{j=1}^{p} \frac{1}{z - \lambda_j}
\]

\[
= \sum_{j=1}^{p} \frac{1}{\lambda_k - \xi_j} - \frac{1}{\lambda_k} - \sum_{j=1}^{p} \frac{1}{\lambda_k - \lambda_j} - \lim_{z \to \lambda_k} \frac{1}{z - \lambda_k}
\]

where the second line is obtained remarking that:

\[
\frac{m_{\mu_p}(z; M)}{m_{\mu_p}(0; M)} = \sum_{j=1}^{p} \frac{1}{z - \xi_j} - \frac{1}{z - \lambda_k} - \sum_{j=1}^{p} \frac{1}{z - \lambda_j}.
\]
Combining these integrals then yields to the solution of the integral:

\[ A_{kk} = -\frac{1}{\lambda_k}. \]

1.3. Case \( f(t) = \log(1 + st) \)

For this case, the integrand of \( A_{kk} \) can be derived similarly as in the case of the logarithm by noting that the argument of the logarithm \( (1 - s/m \tilde{\mu}_p(z)) \) is a polynomial for which the poles are \( \lambda_i \) and 0. The zeros are in number \( p + 1 \) and denoted \( \kappa_i, i = 0, \ldots, p \) with \( \kappa_0 < 0 < \kappa_1 < \ldots < \kappa_p \); in particular, only \( \kappa_1, \ldots, \kappa_p \) are inside the integration contour (see (Couillet et al., 2018) for details). Therefore, the integrand is written similarly as for the log function as:

\[ I = -\log \left( \frac{\prod_{i=1}^{p}(z-\kappa_i)}{\prod_{i=1}^{p}(z-\lambda_i)} \right) \left( \frac{\lambda_k}{z-\lambda_k} \right)^2. \]

The integration contour can be deformed as for the log function. Using similar integration techniques, the calculus then yields to the solution derived in the article.

1.4. Case \( f(t) = \log^2(t) \)

Here \( g(t) = f(t) = \log^2(t) \) and the integrand for this case is simply

\[ I = -\log^2 \left( \frac{\prod_{i=1}^{p}(z-\xi_i)}{\prod_{i=1}^{p}(z-\lambda_i)} \right) \left( \frac{\lambda_k}{z-\lambda_k} \right)^2. \]

Again, we use here exactly the same line of work performed on the \( \log(t) \) and \( \log(1 + st) \) functions. Technical difficulties however arise when addressing the real integrals which involve products of logarithms and rational functions. These difficulties are mostly cumbersome calculus which are addressed similar to (Couillet et al., 2018).

The treatment of the complex integrals resulting from the estimation of \( C^{-1} \) is performed similarly.

References