# Exploiting equality constraints in causal inference

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## Abstract

Assumptions about equality of effects are commonly made in causal inference tasks. For example, the well-known "difference-in-differences" method assumes that confounding remains constant across time periods. Similarly, it is not unreasonable to assume that causal effects apply equally to units undergoing interference. Finally, sensitivity analysis often hypothesizes equality among existing and unaccounted for confounders. Despite the ubiquity of these "equality constraints," modern identification methods have not leveraged their presence in a systematic way. In this paper, we develop a novel graphical criterion that extends the well-known method of generalized instrumental sets to exploit such additional constraints for causal identification in linear models. We further demonstrate how it solves many diverse problems found in the literature in a general way, including difference-in-differences, interference, as well as benchmarking in sensitivity analysis.

## 1 Introduction

The assumption that certain causal effects are equal is present in a number of diverse causal inference problems. For example, many popular identification strategies, including the widely known "difference-in-differences" technique, rely on the assumption that confounding mechanisms remain invariant over different time periods (Angrist and Pischke, 2009; Kim and Steiner, 2019). Similarly, while interference among subjects can complicate identification, sometimes it may be defensible to assume that certain causal effects apply equally to each subject. Lastly, a common practice in sensitivity analysis is to benchmark the strength of a potential, unaccounted for confounder to the strength of observed confounders (Cinelli and Hazlett, 2020). In each of these cases, equality constraints between structural parameters play an important role to identify or bound causal effects.

However, despite the ubiquity of these "equality constraints," currently there is no known efficient algorithm that is able to systematically exploit them for identification.[1] While in the past few decades significant progress has been made in developing efficient identification algorithms for linear causal models (Brito and Pearl, 2012; Foygel et al., 2012; Chen et al., 2017; Weihs et al., 2018; Kumor et al., 2019, 2020), such techniques can only systematically handle two types of assumptions encoded in a causal diagram: (i) the absence of a direct effect between certain variables; and (ii) the absence of association between error terms.

As a result, the current literature handling equality constraints has mostly worked with *ad-hoc* structures on a case-by-case basis. For example, Kim and Steiner (2019) discuss the gain-score method for solving certain models; Chalak (2013, 2019) provides a more general method in which difference-in-differences is a special case, but still restricted to few cases; and while Cinelli et al. (2018) demonstrate that benchmarking in sensitivity analysis can be reduced to an identification problem with equality constraints, they only do so for specific model structures.

In this paper, we develop a novel graphical criterion that extends generalized instrumental sets (Brito and Pearl, 2012) to exploit external equality constraints for causal identification in linear models. We prove

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[1]One could use methods from computer algebra (García-Puente et al., 2010), but these are often computationally intractable, making it practically infeasible for models larger than 4 or 5 nodes.

the soundness of our method and demonstrate that it generalizes various known identification strategies such as difference-in-differences, improves our ability to identify effects under interference, and refines existing results for benchmarking in sensitivity analysis. Moreover, since most modern linear identification algorithms build upon generalized instrumental sets (Kumor et al., 2019, 2020), our results may be used towards an algorithmic approach to exploit equality constraints in linear causal identification.

## 2 Preliminaries

**Linear Structural Causal Models.** In this paper, we restrict our attention to acyclic semi-Markovian linear structural causal models (SCMs) (Wright, 1921; Pearl, 2009). Formally, linear SCMs are represented by a system of linear equations $X = \Lambda^T X + \epsilon$ where $X$ is a vector of observed variables, $\epsilon$ is a vector of latent variables, and $\Lambda$ is an upper triangular matrix of direct effects, whose $ij$th element, $\lambda_{v_i v_j}$ gives the magnitude of the direct causal effect of $v_i$ on $v_j$. Without loss of generality, we assume variables have been standardized to have mean 0 and variance 1. In linear models, the error term $\epsilon$ is commonly assumed to be normally distributed with covariance matrix $\mathcal{E}$. This means that the covariance matrix of the observed data $\Sigma := XX^T$ fully characterizes the observational distribution. This matrix can be linked to the underlying structural parameters through the system of polynomial equations $\Sigma = XX^T = (I-\Lambda)^{-T}\mathcal{E}(I-\Lambda)^{-1}$, and the problem of identification reduces then to finding the elements of $\Lambda$ that are uniquely determined by the above system. In this paper, we deal with the problem of identification "almost everywhere" (Brito and Pearl, 2012).

**Causal Graph.** The causal graph of an SCM is defined as a triple $G = (V, D, B)$, representing the nodes, directed, and bidirected edges, respectively. We will use the terms *node* and *variable* interchangeably, where they can denote the variables in the SCM or the nodes in the causal graph. There is a directed edge from $v_i$ to $v_j$ for each non-zero $\lambda_{v_i v_j} \in \Lambda$, and a bidirected edge between $v_i$ and $v_j$ for each non-zero $\varepsilon_{v_i v_j} \in \mathcal{E}$. For convenience, we will often refer to a structural parameter using its associated edge, or, conversely, use the structural parameters $\lambda_{v_i v_j}$ and $\varepsilon_{v_i v_j}$ to refer to the corresponding directed and bidirected edges between $v_i$ and $v_j$ in the graph. We will use $\theta$ to denote a generic structural parameter of the model (either a directed edge or a bidirected edge). If there are unobserved variables in the graph, we will usually work with its latent projection (Verma, 1993; Pearl, 2009), where we use bidirected edges to denote unobserved confounders. We also borrow the following definitions from Brito and Pearl (2012) and Chen et al. (2017): $Inc(y)$ for some variable $y$ denotes the set of edges that have an arrowhead at $y$; $G_{E-}$ where $E$ is a set of edges denotes the graph $G$ with the edges $E$ removed; and $p[v_1 \sim v_2]$ where $p$ is a path and $v_1, v_2$ are two variables on $p$ denotes the subpath between $v_1$ and $v_2$ (both $v_1$ and $v_2$ included).

**Wright's Rules.** We use $\rho_{v_i v_j \cdot W}$ to denote the partial correlation between two variables, $v_i$ and $v_j$, given a set of variables, $W$. We will extensively use Wright's path tracing rules. Wright's rules (Wright, 1921) allows us to equate the model-implied correlation $\rho_{v_i v_j}$ between any pair of variables, $v_i$ and $v_j$, to the sum of products of parameters along unblocked paths between $v_i$ and $v_j$. For example, in Figure 1, Wright's rules gives us $\rho_{xy} = \varepsilon_{xy} + \lambda_{xm}\lambda_{my}$, $\rho_{my} = \lambda_{xm}\varepsilon_{xy} + \lambda_{my}$, and $\rho_{xm} = \lambda_{xm}$.

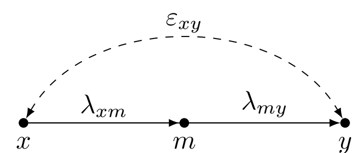



Figure 1: DAG illustrating Wright's rules.

**Instrumental Variables and Instrumental Sets.** A useful method for causal identification in linear models is the instrumental variable (IV) (Bowden and Turkington, 1990). An example is given in Figure 2. Note that $\lambda_{xy}$ is not identified without $z$, due to the unobserved confounder between $x$ and $y$. However, if $z$ is observed, $\lambda_{xy}$ can be identified by solving the linear equation, $\rho_{zy} = \rho_{zx}\lambda_{xy}$.

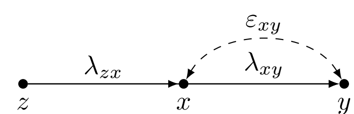



Figure 2: $z$ is an IV for $\lambda_{xy}$.

A generalization of the traditional IV is the method of generalized instrumental sets (IV sets) proposed by Brito and Pearl (2012). IV sets allow the simultaneous use of multiple IVs to construct a full-rank system of linear equations, and the causal parameters are identified by solving the system. An example is given in Figure 8(a), which we discuss in detail in Section 5. Our method extends generalized instrumental sets, by relaxing the need of finding a full-rank system of equations, which is then supplemented by external equality constraints.

# 3 Motivating Examples and Related Work

## 3.1 Equiconfounding

A number of identification techniques use the assumption of equiconfounding, where observed variables are equally affected by an unobserved confounder. Chalak (2013) discusses some special types of equiconfounding where point-identification is possible, including when two joint responses are equiconfounded, and when two causes and one response are equiconfounded.

The most widely applied special case of equiconfounding is "difference-in-differences" (Angrist and Pischke, 2009; Kim and Steiner, 2019), which assumes two joint responses are equally affected by unobserved confounding. A commonly cited example involves estimating the effect of raising the minimum wage on unemployment. In this case, the change in employment after minimum wage was increased in New Jersey (NJ) was compared to the change in employment in Pennsylvania (PA) over the same time period, where minimum wage was not changed (Card and Krueger, 1993). The usual structure can be depicted as in Figure 3, where $x$ represents minimum wage, $y$ represents unemployment after the change in minimum wage, $w$ represents unemployment before the change in minimum wage, and $u$ represents the unobserved confounder. The equality constraint of this model is that $\lambda_{uw} = \lambda_{uy}$ (DAG on the left), without which the causal effect is not identifiable. In the latent projection (Pearl, 2009) (DAG on the right) the equality constraint becomes $\varepsilon_{xw} = \varepsilon_{xy}$.

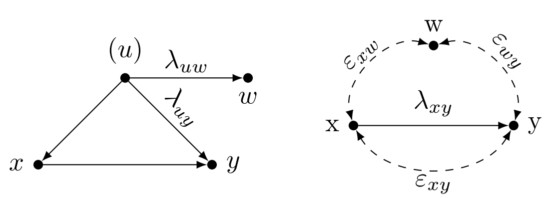



Figure 3: When two joint responses are equiconfounded ($\lambda_{uy} = \lambda_{uw}$) this can aid in identification. Left: Latent variable DAG. Right: Latent projection.

As discussed in (Chalak, 2013, Chapter 4), another common case of equiconfounding happens when two joint causes and one response are affected by the unobserved confounder by the same or proportional magnitude. For example, in Figure 4 (left), we have an equality constraint on three edges, $\lambda_{ux_1} = \lambda_{ux_2} = \lambda_{uy}$ (in the latent projection (right), this translates to the equality constraint $\varepsilon_{x_1x_2} = \varepsilon_{yx_1} = \varepsilon_{x_2y}$). The causal effect $\lambda_{x_1y}$ is not identifiable without this constraint.

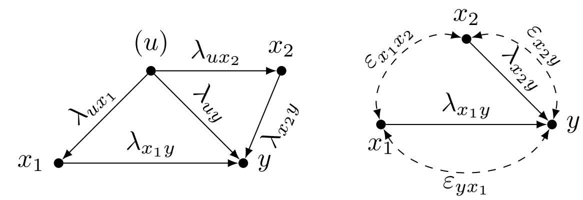



Figure 4: If two joint causes and one response are equiconfounded ($\lambda_{ux_1} = \lambda_{ux_2} = \lambda_{uy}$) this enables identification. Left: Latent variable DAG. Right: Latent projection.

## 3.2 Interference

In causal analysis, interference is often used to describe settings in which one subject's exposure may affect another subject's outcome. A common assumption when dealing with interference is that it occurs only within subgroups or *blocks* of subjects, such as, for instance, a household (Sobel, 2006; Rosenbaum, 2007; Hudgens and Halloran, 2008; Tchetgen and VanderWeele, 2012). Ogburn et al. (2014) demonstrate how interference in such cases can be represented and solved graphically.

However, these approaches usually do not handle unobserved confounders, which complicates identification and, in some cases, makes it impossible. Luckily, if equality constraints can be defended, they can help identification even under the presence of confounding. For instance, perhaps one could argue that the effect of the treatment on the outcome should be equivalent for subjects within the block. Alternatively, one could also surmise that effects of one subject on another subject (i.e. the interference) is similar within the block.

Figure 5 graphically depicts the interference structure within a block (Ogburn et al., 2014), where three subjects are interfering with one another. In this case, $x_1$,

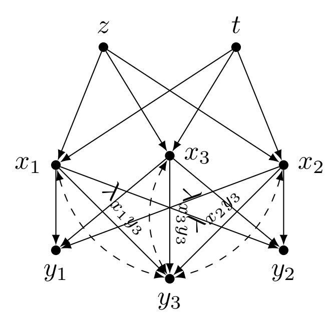



Figure 5: The assumption that $x_1$ and $x_2$ have equal effects on $y_3$ allows the identification of $\lambda_{x_1y_3}$, $\lambda_{x_2y_3}$, and $\lambda_{x_3y_3}$. Bidrected edges between other $x_i$ and $y_j$ omitted for clarity.

$x_2$, and $x_3$ represent treatments for different subjects, and $y_1$, $y_2$, and $y_3$ represent their outcomes; $z$ and $t$ are two instrumental variables (e.g, randomized incentive for taking the treatment) applied to the whole block (e.g, the household). Here, if one posits that the effects of $x_1$ and $x_2$ on $y_3$ are the same, this enables the identification of $\lambda_{x_1y_3}$, $\lambda_{x_2y_3}$, and $\lambda_{x_3y_3}$. Of course, such strict equality may not always be assumed. In these cases, one could relax the degree of equality, and obtain bounds on the causal effects instead of point identification.

### 3.3 Benchmarking in Sensitivity Analysis

Causal inference requires knowledge or assumptions about the data generating process, and sensitivity analysis aims to understand the extent of bias when these assumptions are violated (Rosenbaum, 2010, 2017). Often, these violations render the causal effect of interest unidentifiable, and, therefore, additional constraints are needed to identify the causal effect and derive the bias (Cinelli et al., 2019).

A common practice is to "benchmark" the extent to which the assumption is violated (Cinelli and Hazlett, 2020). For example, if we want to assess the sensitivity of our estimate to omitted variable bias, we might ask what the bias would be if the missing confounder were as strong as an observable confounder. One could then argue that, as long as the strongest confounders have been accounted for, this value represents an upper bound on the potential bias due to a missing variable.

Solving this problem again reduces to identification in the presence of an equality constraint. For example, suppose that we wanted to determine the bias if an unobserved confounder, depicted by the bidirected edge in Figure 6 right, were $k$ times as strong as strong as the observed confounder, $z$, for some known constant $k$. In this case, we posit that $\varepsilon_{xy} = k\lambda_{zx}\lambda_{zy}$. This equality constraint permits the identification of $\lambda_{xy}$, enabling us to compute the bias under this hypothesized relative strength of confounding.

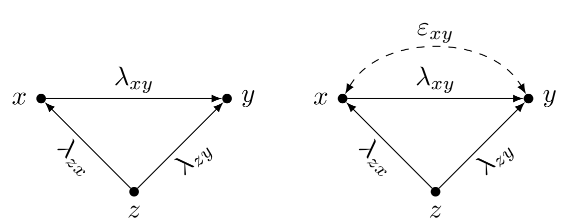



Figure 6: Left: Original DAG. Right: Potential violation with unobserved confounders $\varepsilon_{xy}$. The assumption that $\varepsilon_{xy} = k\lambda_{zx}\lambda_{zy}$ allows identifying $\lambda_{xy}$.

## 4 Problem Setup

The three types of problems presented in the previous section have one commonality: the identification of causal effects of interest only becomes possible when equality amongst certain structural parameters is known a priori. In this section, we formally define the problem of identification using equality constraints.

We first define $C$-identifiability, denoting identifiability of model parameters of a linear SCM, $M$, given a set of external constraints $C$, beyond those already induced by the causal graph $G$.

**Definition 1** ($C$-identifiability)**.** *Let $M$ be a linear SCM (as specified by $G$) and let $C$ be a set of additional constraints on the parameters of $M$. A causal quantity $\theta$ is said to be $C$-identifiable if $\theta$ is uniquely computable from $C$ and the covariance matrix of $M$.*

In this paper, we consider the problem of $C$-identifiability specifically when $C$ is composed of equality constraints on two structural parameters where one parameter is a multiple of the other. We restrict our attention to two edges because this is the type of equality constraint of interest in the applications cited, and also the main focus of our results in Section 6.

Formally, we have the following definition.

**Definition 2** (External Equality Constraint)**.** *An* external equality constraint *for a model $M$ is a constraint of the form*

$$c\theta_1 + \theta_2 = 0, \tag{1}$$

*where $c$ is a constant, and $\theta_1$ and $\theta_2$ are structural parameters of $M$.*

Here, the two structural parameters $\theta_1$ and $\theta_2$ can be two directed edges, two bidirected edges, or one directed edge and another bidirected edge. In fact, benchmarking in sensitivity analysis involves constraints where directed edges are equal to bidirected edges. We discuss that in detail in Section 7.

We use an example to illustrate the idea of $C$-identification. Suppose we are given the SCM of Figure 7. If we do not know the value of any of the edges, and we are given only the graph as well as the correlations among the three variables, then neither $\lambda_{ac}$ nor $\lambda_{bc}$ can be identified. To demonstrate, $\lambda_{ac}$, $\lambda_{bc}$, $\varepsilon_{ab}$, $\varepsilon_{bc}$ could be $0.4, 0.25, 0.4, 0.41$ respectively, and this model implies the same correlations as those of the SCM of Figure 7 (this can be easily checked using Wright's rules). However, if we know the equality constraint between $\lambda_{ac}$, $\lambda_{bc}$, i.e., $-5/3\lambda_{ac} + \lambda_{bc} = 0$, then we can uniquely solve for $\lambda_{ac}$ and $\lambda_{bc}$. Thus, $\lambda_{ac}$

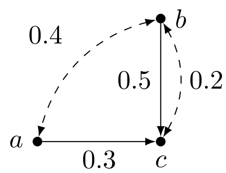



Figure 7: Numerical example of $C$-identifiability. In this model, $\lambda_{ac}$ and $\lambda_{bc}$ are not identified just with the constraints provided by the DAG. However, they become identified if the constraint $c\lambda_{ac} + \lambda_{bc} = 0$ is added.

and $\lambda_{bc}$ are not identifiable but are both $C$-identifiable with $C$ being $-5/3\lambda_{ac} + \lambda_{bc} = 0$.

As we see, our goal is to find cases where the provided external equality constraints can supplement the limited information we have from the graph $G$ alone, and thus help with the identification of more structural parameters of the model. As we discuss next, we tackle this problem by finding the linear constraints induced by the graph $G$ and combining them with the external equality constraints $C$. This allows the construction of a system of linear equations that can solve for the parameters of interest.

## 5 Searching for Graph-Induced Linear Constraints

Given a DAG $G$ and the covariance matrix of the modeled variables, some relationships between structural parameters can be deduced. Here we are interested in finding linear equations among the structural parameters, since these equations can be used to solve for the structural parameters using linear algebra. In this section, we provide graphical conditions to find such linear constraints on the graph.

We first formally define this type of linear relationship, which we name *graph-induced linear constraint*.

**Definition 3.** *Let $\theta_1, \ldots, \theta_p$ be structural parameters of a linear model $M$. If the graph $G = (V, D, B)$ induces a linear equation of the type,*

$$l_{\theta_1,\ldots,\theta_p} := a_1\theta_1 + a_2\theta_2 + \cdots + a_p\theta_p = c$$

*where $a_1 \ldots a_p$ and $c$ are functions of $\Sigma$, then we say $l_{\theta_1,\ldots,\theta_p}$ is a graph-induced linear constraint on $\theta_1 \ldots, \theta_p$ from $G$.*

One way to search for graph-induced linear constraints is through searching for generalized instrumental sets (IV sets) of Brito and Pearl (2012). If an IV set can be found in the graph, one can then use them to construct a full-rank system of linear equations on certain structural parameters. Instead of aiming for a full-rank system that guarantees point identification, the basic idea of our method is simply to search for such linear relationships among edges, even if we cannot have as many equations as there are unknowns (here including directed and bidirected edges).

For example, in Figure 5, if we search for a generalized instrumental set on the edges $\lambda_{x_1y_3}, \lambda_{x_2y_3}, \lambda_{x_3y_3}$, we will not be able to find one, since there are only two possible instruments, $z$ and $t$, while all other variables violate the requirements for a generalized instrumental set. However, although it is not possible to identify any of the three edges, we can still construct two linear constraints on these three edges:

$$\rho_{zy_3} = \rho_{zx_1}\lambda_{x_1y_3} + \rho_{zx_3}\lambda_{x_3y_3} + \rho_{zx_2}\lambda_{x_2y_3} \quad (2)$$
$$\rho_{ty_3} = \rho_{tx_1}\lambda_{x_1y_3} + \rho_{tx_3}\lambda_{x_3y_3} + \rho_{tx_2}\lambda_{x_2y_3} \quad (3)$$

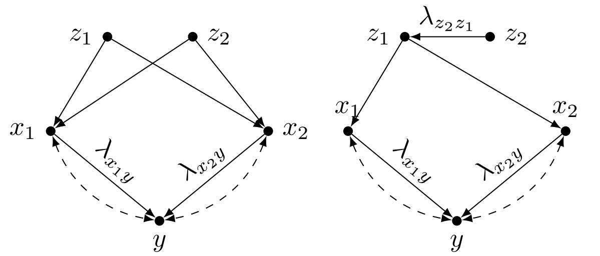
Now note that those linear constraints can still be used to identify the three edges, provided we have a third external equality constraint to supplement the missing information.

Below we define *partial-instrumental sets*, which relaxes the traditional definition of generalized instrumental sets of Brito and Pearl (2012), by allowing the inclusion of a larger set of directed and bidirected edges.

**Definition 4** (Partial-Instrumental Set)**.** *In a graph $G = (V, D, B)$, let $y$ be a variable in $V$ and let $E$ be a set of $n$ edges where $E \subseteq Inc(y)$. Given a set of $n'$ edges, $E' = \{e_1, e_2, \ldots, e_{n'}\}$ where $E' \subseteq E$, and a set of $n'$ variables, $Z = \{z_1, z_2, \ldots, z_{n'}\}$, $Z$ is a* partial-instrumental set *for $E$ on $E'$ if there exists triples $(z_1, W_1, p_1), \ldots, (z_{n'}, W_{n'}, p_{n'})$ such that:*

1. *For $i = 1, \ldots, n'$, the elements of $W_i$ are non-descendants of $y$, and either:*
   - *(a) $(z_i \perp\!\!\!\perp y | W_i)_{G_{(E \cap D)-}}$, or*
   - *(b) if there exists a bidirected edge between $z_i$ and $y$: $\varepsilon_i$, and $\varepsilon_i \in E$, $W_i$ are non-descendants of $z_i$, and $(z_i \perp\!\!\!\perp y | W_i)_{G_{(E \cap D) \cup \{\varepsilon_i\}-}}$.*
2. *for $i = 1, \ldots, n'$, $p_i$ is a path between $z_i$ and $y$ that is not blocked by $W_i$ and passes through $e_i$, and*
3. *for $1 \le i < j \le n'$, variable $z_j$ does not appear in path $p_i$, and if paths $p_i$ and $p_j$ have a common variable $v$, then both $p_i[v \sim y]$ and $p_j[z_j \sim v]$ point to $v$.*

In this definition, the set of edges, $E$, contains the edges we are interested in solving for and might not be able to be removed from consideration by conditioning. Note $|E'|$ number of linear constraints on $E$ can be




(a) Distinct constraints (b) Redundant constraints

Figure 8: Different numbers of independent linear constraints can be constructed in different graphs.

generated if such a partial-instrumental set exists. The set of edges, $E'$, is considered a "critical set" for the constraints generated, where each constraint matches to an edge in $E'$. For each $i$ in $1, \ldots, n'$, we say that the constraint $l_i$ generated from $z_i$ "matches to" the edge $e_i$. The constraint $l_i$ matching to $e_i$ indicates that $l_i$ has additional information about $e_i$, which cannot be deduced from other constraints.

We explain the matching between constraints and edges using the example in Figure 8. Starting with Figure 8(a), using Wright's rules, we can find two linear equations on the edges, $\lambda_{x_1y}$ and $\lambda_{x_2y}$. They are

$$l_1 : \rho_{z_1x_1}\lambda_{x_1y} + \rho_{z_1x_2}\lambda_{x_2y} = \rho_{z_1y} \tag{4}$$

$$l_2 : \rho_{z_2x_1}\lambda_{x_1y} + \rho_{z_2x_2}\lambda_{x_2y} = \rho_{z_2y} \tag{5}$$

When we have two graph-induced linear constraints on two parameters, we have a system of linear equations to solve for both parameters. However, now moving to Figure 8(b), note that here the two equations are in fact "equivalent," since the coefficients ($\rho_{z_1x_1}, \rho_{z_1x_2}$, and $\rho_{z_1y}$) in Eq. (4) multiplied with $\lambda_{z_2z_1}$ are equal to the corresponding coefficients ($\rho_{z_2x_1}, \rho_{z_2x_2}$, and $\rho_{z_2y}$) in Eq. (5). The reason behind this is, given $z_1$, there is no additional information $z_2$ can provide on $\lambda_{x_1y}$ or $\lambda_{x_2y}$, because $z_2$ is connected to $\lambda_{x_1y}$ or $\lambda_{x_2y}$ only through $z_1$. Hence, $l_2$ cannot be "matched to" $\lambda_{x_1y}$ or $\lambda_{x_2y}$, which makes $z_2$ an invalid candidate instrument when $z_1$ is present. Condition 3 in Definition 4 is used to guarantee that each constraint generated will have unique information on one edge in $E'$, since it disallows the path for one instrument to subsume the path for another instrument.

Nevertheless, Figure 8(b) is still an example of partial-instrumental set. One possible choice of $Z$, $E$, $E'$ is $Z = \{z_2\}$, $E = \{\lambda_{x_1y}, \lambda_{x_2y}\}$, $E' = \{\lambda_{x_1y}\}$, so that $Z$ is a partial-instrumental set for $E$ on $E'$. In this case, $W_1 = \emptyset$ and $p_1$ is $z_2 \to z_1 \to x_1 \to y$. In other words, although we cannot solve the system, we can still extract one non-redundant linear equation on the two parameters. This may still be useful, as such equation may be combined with an external equality constraint on those parameters to build a full-rank system of equations.

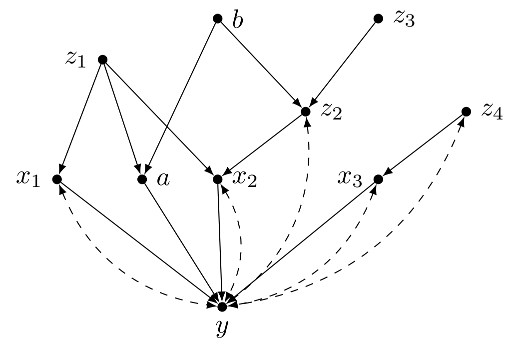



Figure 9: It is possible to construct 4 linear equations on 5 edges, $E = \{\lambda_{x_1y}, \lambda_{x_2y}, \lambda_{x_3y}, \varepsilon_{z_2y}, \varepsilon_{z_4y}\}$.

Another example is given in Figure 9. If we define $Z = \{z_1, z_2, z_3, z_4\}$, $E = \{\lambda_{x_1y}, \lambda_{x_2y}, \lambda_{x_3y}, \varepsilon_{z_2y}, \varepsilon_{z_4y}\}$, and $E' = \{\lambda_{x_1y}, \varepsilon_{z_2y}, \lambda_{x_2y}, \lambda_{x_3y}\}$, then $Z$ is a partial-instrumental set for $E$ on $E'$. The constraints generated from $z_1, z_2, z_3, z_4$ are matched to $\lambda_{x_1y}, \varepsilon_{z_2y}, \lambda_{x_2y}, \lambda_{x_3y}$, respectively, with conditioning sets $W = \{\{a, b\}, \{b\}, \emptyset, \emptyset\}$, and the paths $P = \{z_1 \to x_1 \to y, z_2 \leftrightarrow y, z_3 \to z_2 \to x_2 \to y, z_4 \to x_3 \to y\}$.

Note that when $E' = E$ and $E$ contains only directed edges, Definition 4 degenerates to the traditional generalized instrumental set. Lemma 1 below states that we can construct graph-induced linear constraints on edges in $E$, which might contain both directed and bidirected edges. The number of constraints constructed, $|E'|$, might be fewer than the number of edges involved in the equations, when $E'$ is a strict subset of $E$. For example, for the DAG in Figure 9, we can construct 4 linear equations on 5 edges.

**Lemma 1.** *For an SCM $M$ with graph $G = (V, D, B)$, if there exists a partial-instrumental set $Z = \{z_1, \ldots, z_{n'}\}$ for $E = \{\theta_1, \ldots, \theta_n\}$ on $E'$ where $|E| = n$ and $|Z| = |E'| = n'$, then there exists a set of $n'$ graph-induced linear constraints on $E$. Specifically, given the triples in Definition 4 as $(z_1, W_1, p_1), \ldots, (z_{n'}, W_{n'}, p_{n'})$, for each $i = 1, \ldots, n'$, we have a constraint,*

$$l_i : \rho_{z_iy \cdot W_i} = c_{i1}\theta_1 + \cdots + c_{in}\theta_n, \tag{6}$$

*where $c_{ij}$ is a function on the correlations of variables in $M$ for all $j = 1, \ldots, n$.*

See Brito and Pearl (2012) for how to compute the coefficients $c_{i1}, \ldots, c_{in}$.

## 6 Incorporating External Equality Constraints

Given a set of linear constraints, it is important to check for the uniqueness of such constraints given the model $M$—is a newly found constraint equivalent to a previously found one, or can it be deduced from several previously found ones? In other words, what are the criteria for a set of constraints to be "full-rank?" This question becomes harder when external equality constraints and known edges are provided, since it is not trivial to decide whether one constraint can be a linear combination of several other constraints of any type. As discussed, each constraint of an instrumental set can be "matched to" an edge. The same idea applies to partial-instrumental sets, where more edges of both types are involved. We now show that we can also apply this simple strategy when combining graph-induced linear constraints with external equality constraints and known edges.

To begin with, we have the following lemma.

**Lemma 2.** *Given only $n'$ constraints constructed in Lemma 1 from the partial-instrumental set $Z$ for $E$ on $E'$, no edge in $E \setminus E'$ can be solved.*

The correctness of this lemma is evident for the reason that, if an edge is not "matched to" by any constraint constructed from a partial-instrumental set, then it cannot be solved given those constraints. In other words, the value of any variable in $E \setminus E'$ cannot be deduced from $\mathcal{L}$. Hence, we can combine external information on the edges $E \setminus E'$ with the constraints of $\mathcal{L}$, without worrying about such external constraints being redundant.

Building on top of this, we have the main theorem of this paper. Theorem 1 provides a sufficient condition that, when satisfied, guarantees a full-rank set system of linear equations can be constructed by combining a set of graph-induced linear constraints, external equality constraints, and the values of known edges.

**Theorem 1.** *For an SCM $M$ with graph $G = (V, D, B)$, let $y$ be a variable in $V$ and let $E$ be a set of $n$ edges where $E \subseteq Inc(y)$. Suppose there exists a partial-instrumental set, $Z$, for $E$ on $E'$ where $|Z| = |E'| = n'$, and we are given the following external information:*

1. *a set of $n_k$ edges, $E_k \subseteq E$, whose coefficients are known, and*
2. *a set of $n_e$ linearly independent external equality constraints, $L_e$, on edges $E_e$, where $E_e \subseteq E$.*

*If $E_k \cap E' = \emptyset$, and there exists a way to simultaneously select one edge from each constraint $l \in L_e$ such that the selected edges 1) are not repetitive, 2) do not contain any edge in $E' \cup E_k$, then there exists a full-rank set of $n' + n_k + n_e$ linear constraints on $E$.*

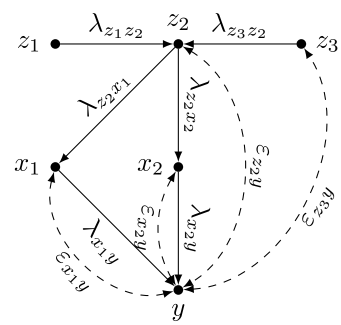



Figure 10: Variables $z_1$, $z_2$, $z_3$ form a partial instrument set for $E = \{\lambda_{x_1y}, \lambda_{x_2y}, \varepsilon_{x_2y}, \varepsilon_{z_3y}\}$ on $E' = \{\lambda_{x_1y}, \varepsilon_{x_2y}, \varepsilon_{z_3y}\}$.

The intuition behind Theorem 1 is that a full-rank set of constraints can be constructed if we can find an edge for each constraint, where that constraint contains some unique information on that edge. Specifically, the constraints are given by: $n'$ constraints constructed from the partial-instrumental set as in Lemma 1, $n_k$ constraints in the form of $\theta_i = c_i$ where $\theta_i \in E_k$ and $c_i$ is the known value of $\theta_i$ for $i = 1, \ldots, n_k$, and $n_e$ external equality constraints. A special case of Theorem 1 is when there exists no partial-instrumental set—we can still construct a full-rank constraint set from known edges and equality constraints only. For example, given $\theta_1 = \theta_2$ and $\theta_1 = k$, they form a full-rank set and we immediately have $\theta_2 = k$.

An immediate result from Theorem 1 is that when $n' + n_k + n_e = |E|$, i.e., the number of linear constraints we can find is equal to the number of structural parameters $E$ that those constraints are on, then we can solve for all the structural parameters in $E$. We use Figure 10 to show how to apply Theorem 1. The set of variables $Z = \{z_1, z_2, z_3\}$ is a partial-instrumental set for $E = \{\lambda_{x_1y}, \lambda_{x_2y}, \varepsilon_{x_2y}, \varepsilon_{z_3y}\}$ on $E' = \{\lambda_{x_1y}, \varepsilon_{x_2y}, \varepsilon_{z_3y}\}$. We can construct three constraints using the instruments $z_1, z_2, z_3$, and those constraints are matched to $\lambda_{x_1y}, \varepsilon_{x_2y}, \varepsilon_{z_3y}$, with the paths $z_1 \to z_2 \to x_1 \to y$, $z_2 \leftrightarrow y$, $z_3 \leftrightarrow y$, respectively.

Now we analyze different possible types of external information given. Let $k, l, m$ denote constants:

1. the constraint $\varepsilon_{x_2y} = k\varepsilon_{x_3y}$ cannot be combined with our graph-induced linear constraints, since both $\varepsilon_{x_2y}$ and $\varepsilon_{z_3y}$ are in $E'$, and there is no way to select an edge from this equality constraint that is not in $E' \cup E_k$;

2. the constraint $\lambda_{x_1y} = l\lambda_{x_2y}$ can be combined with our graph-induced linear constraints, since $\lambda_{x_2y}$ is not in $E'$, so we can select the edge $\lambda_{x_2y}$ from this equality constraint that is not in $E' \cup E_k$;
3. the constraint $\lambda_{x_2y} = m$ (either from previous identification or prior knowledge) can be combined with our graph-induced linear constraints, since $\lambda_{x_2y}$ is not in $E'$, so $E_k \cap E' = \emptyset$.

The three graph-induced linear constraints are:

$$\rho_{z_1y} = \rho_{z_1x_1}\lambda_{x_1y} + \rho_{z_1x_2}\lambda_{x_2y} \tag{7}$$

$$\begin{aligned}\rho_{z_2y\cdot\{z_3\}} = & \frac{\rho_{z_2x_1} + \rho_{z_3x_1}}{(1-\rho^2_{z_2z_3})^{1/2}(1-\rho^2_{z_3y})^{1/2}}\lambda_{x_1y} \\ & + \frac{\rho_{z_2x_2} + \rho_{z_3x_2}}{(1-\rho^2_{z_2z_3})^{1/2}(1-\rho^2_{z_3y})^{1/2}}\lambda_{x_2y} \\ & + \frac{1}{(1-\rho^2_{z_2z_3})^{1/2}(1-\rho^2_{z_3y})^{1/2}}\varepsilon_{x_2y}\end{aligned} \tag{8}$$

$$\rho_{z_3y} = \rho_{z_3x_1}\lambda_{x_1y} + \rho_{z_3x_2}\lambda_{x_2y} + \varepsilon_{z_3y} \tag{9}$$

By Wright's rules, all three equations above have the equal ratio of the coefficient for $\lambda_{x_1y}$ to the coefficient for $\lambda_{x_2y}$. Hence, $\lambda_{x_1y}$ and $\lambda_{x_2y}$ can be eliminated together, and $\varepsilon_{x_2y}$ and $\varepsilon_{z_3y}$ can thus both be solved. This again explains why we cannot combine the external information $\varepsilon_{x_2y} = k\varepsilon_{z_3y}$ with the three graph-induced linear constraints: this external constraint can be deduced from the three graph-induced linear constraints. On the other hand, if the external information is $\lambda_{x_1y} = k\lambda_{x_2y}$, since neither edge can be solved from the graph-induced linear constraints, the equality constraint cannot be deduced from the system, and is therefore not redundant.

Though in this paper we present our method based on generalized instrumental sets, we conjecture that this approach can be generalized to combine with most of existing linear causal identification methods. This is due to the nature of identification methods for linear models, most of which construct a system of linear equations to solve for a set of structural parameters, $E$. We hence believe that we can match each equation to one parameter in $E$ as required by Theorem 1. Proving this conjecture is beyond the scope of this paper and we leave it for future work.

# 7 Case Studies

In this section, we revisit the applications in Section 3 and show how our method can be used to solve them.

## 7.1 Equiconfounding

The first example we showed is Figure 3, the well-known "difference-in-differences" graph, or the case when two joint responses are equiconfounded (Chalak, 2013). First, we can see that the bidirected edge, $\varepsilon_{xw}$ is identified in this latent projection DAG and is equal to $\rho_{xw}$, since $x \leftrightarrow w$ is the only unblocked path between $x$ and $w$. So we can plug it in to the equality constraint, $\varepsilon_{xw} = \varepsilon_{xy}$ and get $\varepsilon_{xy} = \rho_{xw}$. Next, we see that $Z = \{x\}$ is a partial-instrumental set for $E = \{\lambda_{xy}, \varepsilon_{xy}\}$ on $E' = \{\lambda_{xy}\}$, so we have the graph-induced linear constraint $\lambda_{xy} + \varepsilon_{xy} = \rho_{yx}$. Together with the known edge constraint $\varepsilon_{xy} = \rho_{xw}$, we have a full-rank set of two constraints on two variables, and $\lambda_{xy}$ can be solved, which gives $\lambda_{xy} = \rho_{yx} - \rho_{xw}$.

The second example is when two joint causes and one response are equiconfounded, as in Figure 4. This case is similar to the previous one. $\varepsilon_{x_1x_2}$ can be identified ($\varepsilon_{x_1x_2} = \rho_{x_1x_2}$), and plugging into the equality constraint identifies $\varepsilon_{x_2y}$ and $\varepsilon_{yx_1}$. Next, we observe that $Z = \{x_1, x_2\}$ is a partial-instrumental set for $E = \{\varepsilon_{x_2y}, \lambda_{x_1y}, \lambda_{x_2y}, \varepsilon_{yx_1}\}$ on $E' = \{\lambda_{x_1y}, \lambda_{x_2y}\}$. As a result, we have a full-rank set of four constraints, including two graph-induced constraints and two constant (known edge) constraints, and we can thus solve for all the four edges in $E$.

Another more complex example, (Chalak, 2013, Graph 2), can be solved similarly using our method, and we skip the discussion of that. Our method can solve all the cases where point identification is possible in Chalak (2013). We can also solve other simple generic cases of equiconfounding which have not been discussed in Chalak (2013). For instance, by replacing the equality constraint with $\lambda_{ux} = \lambda_{uw}$ or $\lambda_{ux} = \lambda_{uy}$ in Figure 3 left, we have two different examples that we can both solve. We leave the discussion to the appendix.

## 7.2 Interference

For the interference example in Figure 5, there is an equality constraint $\lambda_{x_1y_3} = \lambda_{x_2y_3}$. $Z = \{z, t\}$ is a partial-instrumental set for $E = \{\lambda_{x_1y_3}, \lambda_{x_2y_3}, \lambda_{x_3y_3}\}$ on $E' = \{\lambda_{x_1y_3}, \lambda_{x_3y_3}\}$. Note that here we can either choose $E'$ to be $\{\lambda_{x_1y_3}, \lambda_{x_3y_3}\}$ or $\{\lambda_{x_2y_3}, \lambda_{x_3y_3}\}$ but not $\{\lambda_{x_1y_3}, \lambda_{x_2y_3}\}$. Otherwise, the equality constraint has no edge to select from for it to match to, so the condition in Theorem 1 will fail. If we choose $E' = \{\lambda_{x_1y_3}, \lambda_{x_3y_3}\}$, we have a full-rank set of three equations on $\lambda_{x_1y_3}, \lambda_{x_2y_3}, \lambda_{x_3y_3}$, with two graph-induced linear constraints generated from $z$ and $t$, and one external equality constraint.

## 7.3 Benchmarking in Sensitivity Analysis

For the example of benchmarking in sensitivity analysis of Figure 6, we have the external information that $\varepsilon_{xy} = k\lambda_{zx}\lambda_{zy}$. First notice that the edge, $\lambda_{zx}$

can be identified using $z$ as an instrument to itself, and we get $\lambda_{zx} = \rho_{zx}$. Hence, the equality constraint reduces to $\varepsilon_{xy} = k\rho_{zx}\lambda_{zy}$, which is now in the form of $\theta_1 = k'\theta_2$ that our method can handle. We next examine the DAG and see the set $Z = \{x, z\}$ is a partial-instrumental set for $E = \{\varepsilon_{xy}, \lambda_{xy}, \lambda_{zy}\}$ on $E' = \{\lambda_{xy}, \lambda_{zy}\}$. We can thus construct two graph-induced linear constraints, as follows.

$$\varepsilon_{xy} + \lambda_{xy} + \rho_{zx}\lambda_{zy} = \rho_{xy} \tag{10}$$

$$\rho_{zx}\lambda_{xy} + \lambda_{zy} = \rho_{zy} \tag{11}$$

Together with the equality constraint $\varepsilon_{xy} = k\rho_{zx}\lambda_{zy}$, we have a full-rank set of linear constraints from Theorem 1, where the equality constraint is matched to the edge $\varepsilon_{xy}$. Note that this is just one possible choice of $E'$, and we can also choose $E' = \{\varepsilon_{xy}, \lambda_{xy}\}$, where the equality constraint will be matched to $\lambda_{zy}$. Either way, we have three equations on three unknowns, and all of them are solved. Specifically,

$$\lambda_{xy} = \frac{\rho_{zx}\rho_{zy}(k+1) - \rho_{xy}}{(k+1)\rho_{zx}^2 - 1} \tag{12}$$

$$\lambda_{zy} = \frac{\rho_{xy}\rho_{zx} - \rho_{zy}}{(k+1)\rho_{zx}^2 - 1} \tag{13}$$

$$\varepsilon_{xy} = k\rho_{zx}\frac{\rho_{xy}\rho_{zx} - \rho_{zy}}{(k+1)\rho_{zx}^2 - 1}. \tag{14}$$

As we see, those parameters are point-identified if we know the value of $k$, which is how strong the unobserved confounder is compared to an observed confounder, $z$. If one does not know the exact value of $k$, but only its plausible range (for instance, $k \leq 2$), it is still possible to use this result to bound the target parameters.

## 8 Conclusion

We developed a novel graphical criterion that allows researchers to leverage equality constraints for identification in linear systems. We showed how several apparently diverse problems in the literature can be reduced to identification with equality constraints, consisting of special cases handled by our method. We hope the results of this paper can be used towards the construction of a systematic, algorithmic approach to exploit equality constraints in causal inference. Extensions to more general forms of equality constraints, and incorporating such results into state-of-the art linear identification algorithms are promising directions for future work.

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