Random Effect Bandits

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Abstract

This paper studies regret minimization in a multi-armed bandit. It is well known that side information, such as the prior distribution of arm means in Thompson sampling, can improve the statistical efficiency of the bandit algorithm. While the prior is a blessing when correctly specified, it is a curse when misspecified. To address this issue, we introduce the assumption of a random-effect model to bandits. In this model, the mean arm rewards are drawn independently from an unknown distribution, which we estimate. We derive a random-effect estimator of the arm means, analyze its uncertainty, and design a UCB algorithm ReUCB that uses it. We analyze ReUCB and derive an upper bound on its $n$-round Bayes regret, which improves upon not using the random-effect structure. Our experiments show that ReUCB can outperform Thompson sampling, without knowing the prior distribution of arm means.

1 INTRODUCTION

We study stochastic multi-armed bandits (Lai and Robbins, 1985; Auer et al., 2002; Lattimore and Szepesvari, 2019), where the learning agent sequentially takes actions in order to maximize its cumulative reward. As the agent learns through experience, it faces a trade-off between exploration and exploitation: exploiting actions that maximize immediate rewards, as estimated by its current model; or improving its future rewards by exploring and learning a better model. Side information, such as the prior distribution of arm means in Thompson sampling (TS) (Thompson, 1933; Chapelle and Li, 2011; Agrawal and Goyal, 2012, 2013; Russo and Van Roy, 2014; Abeille and Lazaric, 2017), can improve the statistical efficiency of the bandit algorithm and make it more practical.

While the prior is a blessing when correctly specified, a misspecified prior is a curse. Take online advertising as an example. It is well known that click probabilities of ads are low. Therefore, when estimating the click probability of a cold-start ad, it is important to model this structure. One approach would be Bayesian modeling, where the prior distribution is beta with a low mean. The shortcoming of this approach is that the prior needs to be specified, and is potentially misspecified. Therefore, design of bandit algorithms that depend less on exact priors is an important direction.

To address this issue, we study random-effect models (Henderson, 1975; Robinson, 1991) in the bandit setting, and refer to the setting as a random-effect bandit. Random-effect models were developed in statistics and econometrics (Diggle et al., 2013; Wooldridge, 2001), and are frequentist counterparts of hierarchical Bayesian models (Carlin and Louis, 2000). In our model, the arm means are sampled i.i.d. from a fixed unknown distribution. The estimator of arm means is a weighted sum of two terms. The first term is the average of observed rewards of the arm. The second term estimates the common mean from all observations. The weights are chosen adaptively based on data, and balance the common mean estimate with that of the specific arm. Due to this structure, the resulting estimator of arm means is more statistically efficient than in the classical setting.

Our proposed bandit algorithm uses upper confidence bounds (UCBs), which is a popular approach to exploration with guarantees (Lai and Robbins, 1985; Auer et al., 2002; Audibert et al., 2009; Garivier and Cappe, 2011). In round $t \in [n]$, it pulls the arm with the highest UCB, observes its reward, and then updates its estimated arm means and their high-probability confidence intervals. The main difference from the classical algorithms is that all estimates are based on the random-effect model. Our method is essentially a

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The work started while being at Google Research.
random-effect UCB1 (Auer et al., 2002), and thus we call it ReUCB.

Since our arm means are stochastic, ReUCB is related to both TS and Bayes-UCB (Kaufmann et al., 2012), which rely on posterior distributions. TS is popular in practice, but the assumption of knowing the prior exactly is rarely satisfied. In ReUCB, we do not require that the prior is fully specified, and thus we relax this assumption.

We make the following contributions. First, we introduce the assumption of random-effect models to multi-armed bandits, and properly formulate the corresponding bandit problem. Second, we propose a UCB-like algorithm for this problem, which we call ReUCB. ReUCB estimates arm means using the best linear unbiased predictor (BLUP) (Henderson, 1975; Robinson, 1991), a method of estimating random effects without assumptions on distributions. The BLUP estimates leverage the structure of our problem and yield tighter assumptions on distributions. The BLUP estimates are smaller. We improve over traditional bandit designs (Auer et al., 2002) by using the stochasticity of \( \mu_k \). Unlike in Thompson sampling (Thompson, 1933; Chapelle and Li, 2011; Russo and Van Roy, 2014) or Bayes-UCB (Kaufmann et al., 2012), we do not assume that the prior of arm means is conjugate or fully specified. We only require that \( P(\mu_k | 0, \sigma_k^2) \) has a finite second-order moment.

The reward of arm \( k \) after the \( j \)-th pull is denoted by \( r_{k,j} \) and we assume that it is generated i.i.d. as

\[
  r_{k,j} \sim P(\mu_k, \sigma^2),
\]

where \( P(\mu_k, \sigma^2) \) is a distribution with mean \( \mu_k \) and variance \( \sigma^2 \). Similarly to \( P(\mu_k | 0, \sigma_k^2) \) in (1), we only require that its second-order moment is finite.

Our bandit has \( K \) arms and a horizon of \( n \) rounds. Before the first round, the mean reward of each arm is generated according to (1). In round \( t \in [n] \), the agent pulls arm \( I_t \in [K] \) and observes its stochastic reward, drawn according to (2). For any arm \( k \) and round \( t \), we denote by \( n_{k,t} \) the number of pulls of arm \( k \) up to round \( t \), and by \( r_{k,1}, \ldots, r_{k,n_{k,t}} \) the sequence of associated rewards. We call this problem a random-effect bandit.

### 3 MODEL ESTIMATION

This section describes our estimators of arms. In Section 3.1, we estimate \( \mu_k \) under the assumption that \( \mu_0 \) is known. In Section 3.2, we provide an estimator for \( \mu_k \) when \( \mu_0 \) is unknown. Additionally, we show how to estimate the variance parameters \( \sigma_0^2 \) and \( \sigma^2 \) in Appendix D. Because this section is devoted to estimating means and their variances at a fixed round \( t \), we drop subindexing by \( t \) to reduce clutter.

#### 3.1 Estimating \( \mu_k \) When \( \mu_0 \) Is Known

We estimate \( \mu_k \) using the best linear unbiased prediction (BLUP), which is a common method for estimating random effects (Henderson, 1975; Robinson, 1991). The BLUP estimator of \( \mu_k \) minimizes the mean squared error among the class of linear unbiased estimators that do not depend on the distribution of model error.

We call the sample mean of arm \( k \) its direct estimator, and define it as \( \bar{r}_k = k^{-1} \sum_{j=1}^{n_k} r_{k,j} \). From (2), we get that \( \text{Var}(\bar{r}_k) = n_k^{-1} \sigma_0^2 \). We improve upon this estimator with a class of linear unbiased estimators of form

\[
  \hat{\mu}_k := \mu_0 + a(\bar{r}_k - \mu_0),
\]

where \( a \in \mathbb{R} \) is a to-be-optimized coefficient. Since \( \mu_k \) is random rather than fixed, BLUP minimizes the...
From (4), we have that
\[ \mathbb{E}[(\hat{\mu}_k - \mu_k)^2] = \mathbb{E}[(a(\bar{r}_k - \mu_k) + (1-a)(\mu_0 - \mu_k))^2] = a^2 \sigma_0^2 + (1-a)^2 \sigma_0^2, \] (3)
where the last equality is from (1) and (2), and that the reward noise is independent of \( \delta_k \). When (3) is minimized with respect to \( a \), the optimal value of \( a \) is
\[ w_k = \sigma_0^2/(\sigma_0^2 + n_k^{-1} \sigma^2) = 1/(1 + n_k^{-1} \sigma^2 / \sigma_0^2). \] (4)
Thus, if \( \mu_0, \sigma_0^2, \) and \( \sigma^2 \) were known; and we plugged our derived \( w_k \) into the definition of \( \hat{\mu}_k \), we would get the following BLUP estimator of \( \mu_k \)
\[ \hat{\mu}_k = \mu_0 + w_k(\bar{r}_k - \mu_0) = (1 - w_k) \mu_0 + w_k \bar{r}_k. \] (5)
From (4), we have that \( \sigma^{-2} \sigma_0^2 \leq w_k < 1 \) for \( n_k \geq 1 \), and that \( w_k \rightarrow 1 \) as \( n_k \) increases. We also have
\[ w_k n_k^{-1} \sigma^2 = (1 - w_k) \sigma_0^2. \] (6)
These properties are important in our analysis.

The estimator \( \hat{\mu}_k \) in (5) is biased. The degree of this bias depends on both \( n_k \) and \( \sigma^2/\sigma_0^2 \) in (4). If the arm has not been pulled enough, \( w_k \) is low and \( \hat{\mu}_k \) is biased towards \( \mu_0 \). So we are not as aggressive in exploring as if \( w_k = 1 \). As the arm is pulled more, \( w_k \rightarrow 1 \) and the bias reduces to zero. When \( \sigma_0^2 \) decreases, the gaps among the arms decrease, and the effect of \( \mu_0 \) increases. Similarly, as \( \sigma^2 \) increases, the uncertainty in the direct estimator \( \bar{r}_k \) increases, and so does the effect of \( \mu_0 \).

Now we set \( a = w_k \) in (3) and get
\[ \mathbb{E}[(\hat{\mu}_k - \mu_k)^2] = w_k^2 \sigma_0^2 + (1 - w_k)^2 \sigma^2 = w_k \sigma_0^2, \] (7)
where the last step is from (6). As \( \text{Var}(\bar{r}_k) = n_k^{-1} \sigma^2 \), (7) shows that \( \hat{\mu}_k \) is a better estimator of \( \mu_k \) than \( \bar{r}_k \), since \( w_k < 1 \).

### 3.2 Estimating \( \mu_k \) When \( \mu_0 \) Is Unknown

When \( \sigma_0^2 \) and \( \sigma^2 \) are known, the mean of arm means \( \mu_0 \) can be estimated by the generalized least squares estimator (Rao, 2001). That estimator is
\[ \hat{\mu}_0 = \left[ \sum_{k=1}^{K} (1 - w_k) n_k \right]^{-1} \sum_{k=1}^{K} (1 - w_k) n_k \bar{r}_k \] (8)
and we derive it in Appendix A. The estimator is more statistically efficient than the least squares estimator because it weights the mean estimates of individual arms by their heteroscedasticity. Since \( 1 - w_k \) equals to that \( \hat{\mu}_k \) and \( \bar{r}_k \), since \( w_k < 1 \).

### Algorithm 1 ReUCB for random-effect bandits.

1: for \( t = 1, \ldots, n \) do
2: for \( k = 1, \ldots, K \) do
3: \( U_{k,t} \leftarrow \hat{\mu}_k + U_k \)
4: end for
5: if \( t \leq K \) then \( I_t \leftarrow t \)
6: else \( I_t \leftarrow \arg \max_{k \in [K]} U_{k,t} \)
7: Pull arm \( I_t \) and observe its reward \( r_{I_t,n_{I_t,t}+1} \)
8: Update all statistics
9: end for

### Proposition 1.

For any arm \( k \in [K] \), and any \( \sigma^2 > 0 \) and \( n_k \geq 1 \), we have \( \bar{\tau}_k^2 < \sigma^2/\bar{n}_k \).

The proof is in Appendix C. Proposition 1 shows that \( \bar{\tau}_k^2 \) is always lower than \( \sigma^2/\bar{n}_k \) when \( \sigma^2 > 0 \), where the latter is the variance estimate in the classical bandit setting. In the worst case, for \( \sigma^2 = 0 \), we get \( \bar{\tau}_k^2 = \sigma^2/\bar{n}_k \), implying that the variance of \( \hat{\mu}_k \) equals to that of \( \bar{r}_k \). Thus, by using the synthetic estimator \( \hat{\mu}_k \), we can be less optimistic than UCB1.

### 4 ALGORITHM

We propose a UCB algorithm for random-effect bandits. The key idea in UCB algorithms (Auer et al., 2002; Audibert et al., 2009) is to pull the arm with the highest sum of its mean reward estimate and a weighted standard deviation of that estimate. In the setting of Section 3.2, the estimated mean reward of arm \( k \) is \( \hat{\mu}_k \) in (9) and its variance is \( \bar{\tau}_k^2 \) in (10). Due to space constraints, we do not present the algorithm for the setting in Section 3.1. In this case, \( \hat{\mu}_k \) would be replaced by \( \hat{\mu}_k \) and \( \bar{\tau}_k^2 \) would be replaced by \( \bar{\tau}_k^2 \).
Our algorithm is presented in Algorithm 1 and we call it ReUCB, which stands for random-effect UCB. We subindex all statistics in Section 3 with an additional t, to make clear that we refer to round t. As an example, \(\hat{\mu}_{k,t}\) and \(\sigma^2_{k,t}\) are the respective values of (9) and (10) at the beginning of round t. ReUCB works as follows. It is initialized by pulling each arm once. The upper confidence bound (UCB) of arm k in round t is

\[ U_{k,t} = \hat{\mu}_{k,t} + c_{k,t}, \]

where \(c_{k,t} = \sqrt{a\sigma^2_{k,t}\log t}\) is its uncertainty bonus and \(a > 0\) is a tunable parameter. In Section 5, we prove regret bounds for \(a \geq 1\). In round t, ReUCB pulls the arm with the highest UCB \(I_t = \arg\max_{k\in[K]} U_{k,t}\). To break ties, any fixed rule can be used.

4.1 Related Algorithm Designs

ReUCB extends UCB1 to a better BLUP estimator. For \(w_{k,t} = 1\) and \(a = 1\), ReUCB has a similar UCB to UCB1, \(U_{k,t} = \hat{r}_{k,t} + \sqrt{n^{-1}_{k,t}\sigma^2\log t}\). We call this algorithm ReUCB\(^\infty\) and evaluate it empirically in Figure 5 in Appendix I. Our results show that ReUCB\(^\infty\) is comparable to TS, but worse than ReUCB. This shows the benefit of our model. Specifically, the estimate of \(\mu_k\) in ReUCB borrows information from other arms. This increases its statistical efficiency (Proposition 1), since the confidence interval of \(\mu_k\) in ReUCB can be narrower than in the classical setting. Note that ReUCB with \(a = 1\) reduces to UCB1 only if all weights \(w_{k,t}\) are one. This could happen only if all arms were pulled infinitely often. So ReUCB with \(a = 1\) does not behave like UCB1.

Due to assuming random arm means, ReUCB is related to both TS (Thompson, 1933; Chapelle and Li, 2011; Russo and Van Roy, 2014) and Bayes-UCB (Kaufmann et al., 2012). Both Bayes-UCB and TS maintain posterior distributions. The computation of the posteriors requires that the mean of the prior \(\mu_0\) is known. ReUCB employs an alternative random-effect estimator that does not need it.

Li et al. (2011) proposed a hybrid model, where some coefficients are shared by all arms. However, this model is still traditional in the sense that the coefficients that are not shared are estimated separately in each arm. Gupta et al. (2021) recently proposed correlated multi-armed bandits, where the learning agent knows an upper bound on the mean reward of each arm given the mean reward of any other single arm. Such side information could be derived in our setting. However, it is also clearly not as powerful as using the observations of all arms jointly, as in (9) and (10).

5 REGRET ANALYSIS

We derive an upper bound on the n-round regret of ReUCB. In our setting, \(\mu_k\) are random variables. Under the assumption that \(r_{k,j} \sim \mathcal{N}(\mu_k, \sigma^2)\) and \(\mu_k \sim \mathcal{N}(\mu_0, \sigma_0^2)\), which is used in one of our analyses, \(\mu_k\) is the maximum a posteriori (MAP) estimate of \(\mu_k\) given history, meaning that \(\mu_{k,t}\) can be viewed as a Bayesian estimator. Because of that, we adopt the Bayes regret (Russo and Van Roy, 2014) to analyze ReUCB. The main novelty in our analysis is addressing the unknown mean of the prior.

Let \(H_t = (I_t, r_{I_t,n_{I_t,t+1}})^t\) be the history at the beginning of round t and \(I_t\) be the pulled arm in round t. The regret is the difference between the rewards we would have obtained by pulling the optimal arm \(I_t = \arg\max_{k\in[K]} \mu_k\) and the rewards that we did obtain in n rounds. Our goal is to bound the Bayes regret of ReUCB is

\[ R_n = \mathbb{E}\left[\sum_{t=1}^n \mu_{I_t} - \mu_{I_t}\right], \]

where the expectation is over stochastic rewards and random \(a_1, \ldots, a_K\). Our main result is stated below.

**Theorem 2.** Consider a K-armed Gaussian bandit with rewards \(r_{k,j} \sim \mathcal{N}(\mu_k, \sigma^2)\) and \(\mu_k \sim \mathcal{N}(\mu_0, \sigma_0^2)\). Let ReUCB use \(\sigma_0^2\) and \(\sigma^2\). Then (1) for any \(a \geq 1\), the n-round Bayes regret of ReUCB is

\[ R_n \leq 2\sqrt{\frac{a\log(1 + \sigma^{-2}\sigma_0^2 n)}{\log(1 + \sigma^{-2}\sigma_0^2)}} \left(1 + \frac{\sigma^2}{K\sigma_0^2}\right)\sigma_0^2 Kn \log n \]

\[ + \frac{K\sigma_0^2 + \sigma^2}{\sigma_0^2} \sqrt{\frac{8n\sigma_0^2\sigma^2}{\pi(\sigma_0^2 + \sigma^2)}}. \]

(2) for any \(a \geq 2\), the n-round Bayes regret of ReUCB is obtained by replacing the last term above with \((1 + \log n)(K + \sigma^2\sigma_0^{-2})\sqrt{2\sigma_0^2\sigma^2/(\pi(\sigma_0^2 + \sigma^2))})\).

5.1 Discussion

Up to logarithmic factors, Theorem 2 shows that the n-round Bayes regret of ReUCB is \(O(K\sqrt{n})\) for \(a \in [1, 2]\) and \(O(\sqrt{n})\) for \(a \geq 2\). So the regret is sublinear in n for any \(a \geq 1\). Since both bounds increase in a, we suggest using \(a = 1\), which performs extremely well in practice. Also note that the mean reward estimate in (9) is a weighted sum of the estimate of \(\mu_0\) (Term 1) and the per-arm reward mean (Term 2). The variance of the former is linear in \(\sigma_0^2\), which gives rise to the linear dependence on \(\sigma_0\) in Theorem 2. Note that this dependence is standard in Bayes regret analyses (Lu and Van Roy, 2019; Basu et al., 2021), and it is due to using similar techniques in our proofs.

A Bayes regret lower bound exists for a K-armed bandit (Lai, 1987). However, it has not been generalized to structured problems yet, including in seminal works on Bayes regret minimization (Russo and Van Roy,
Theorem 2 is proved under the assumption that \( \text{ReUCB} \) estimates \( \mu_0 \). Now consider a variant of \( \text{ReUCB} \) where \( \mu_0 \) is known. This agent with more information can be analyzed similarly to \( \text{ReUCB} \). In this analysis, \( \hat{\mu}_{k,t} \) and \( \tau_{k,t}^2 \) would be replaced by \( \hat{\mu}_{k,t} \) in (5) and \( \tilde{\tau}_{k,t}^2 \) in (7), respectively. The resulting regret bound would be the same as in Theorem 2, except for the extra factor of \( 1 + \sigma^2/(K\sigma_0^2) \). Therefore, this factor can be viewed as the price for learning \( \mu_0 \). As it is \( O(1 + 1/K) \), its impact on the Bayes regret of \( \text{ReUCB} \) is small when \( K \) is large.

Now suppose that \( \mu_0 \sim \mathcal{N}(0, \sigma_0^2) \). However, the structure that \( \mu_0 \) is the same for all arms is not modeled. This problem is equivalent to a Bayesian bandit with a per-arm prior \( \mathcal{N}(0, \sigma_q^2 + \sigma_0^2) \) and \( \text{ReUCB} \) with known \( \mu_0 = 0 \) can solve it. When analyzed, the leading term in Theorem 2 would be

\[
2 \sqrt{\frac{a \log(1 + \sigma^{-2}(\sigma_0^2 + \sigma_q^2)n)}{\log(1 + \sigma^{-2}(\sigma_q^2 + \sigma_0^2))}} (\sigma_q^2 + \sigma_0^2)Kn \log n.
\]

Thus, up to logarithmic factors, our regret bound is lower whenever \( (1 + \sigma^2/(K\sigma_0^2))\sigma_0^2 \leq \sigma_q^2 + \sigma_0^2 \), and it is beneficial to learn the common \( \mu_0 \) in this case. For any \( \sigma_q > 0 \), this is guaranteed as \( K \) increases.

Theorem 2 can be extended in several ways. First, we generalize the model in (2) to arm-dependent reward noise. Specifically, the reward of arm \( k \) after the \( j \)-th pull is drawn i.i.d. as \( r_{k,j} \sim \mathcal{N}(\mu_k, \sigma_k^2) \), where the variance \( \sigma_k^2 \) may depend on \( k \). In Appendix E, we show that the Bayes regret bound in Theorem 2 still holds for \( \sigma^2 = \max_{k \in [K]} \sigma_k^2 \). Second, the Gaussian assumption in Theorem 2 is replaced with bounded sub-Gaussianity in Appendix G.

Finally, we would like to point out the limitations of our results. First, our proofs rely on well-behaved posterior distributions, either Gaussian or bounded sub-Gaussian. This is due to limitations of existing Bayes regret analyses, which use it to bound tail events conditioned on history (Russo and Van Roy, 2014). We observe that it is not needed for good practical performance and believe that better analyses will be possible in the future. Second, our proofs are under the assumption that \( \sigma_0^2 \) and \( \sigma^2 \) are known. This is akin to existing Bayes regret analyses. We experiment with estimating \( \sigma_0^2 \) and \( \sigma^2 \) in Section 6.

Now we are ready to prove Theorem 2.
where the first inequality is from the definition of $c_{k,t}$ and $\log t \leq \log n$, and we used the Cauchy-Schwarz inequality in the second one.

Note that $x/\log(1+x) \leq m/\log(1+m)$ for $x \in [0,m]$, because $x = \log(1+x)$ at $x = 0$ and $x$ grows faster than $\log(1+x)$ on $[0,m]$. Now we apply this bound for $x = \sigma^2/(\sigma_0^2 + \sigma^2 n_{t,t})$ and $m = \sigma^2\sigma_0^2$, and get

$$\frac{1}{\sigma_0^2 + \sigma^2 n_{t,t}} \leq \frac{\gamma}{\sigma_0^2 + \sigma^2 n_{t,t}} \leq \frac{\log(1 + \sigma_0^2/\sigma^2 n_{t,t})}{\log(1 + \sigma_0^2/\sigma^2)} = \log\frac{\sigma_0^2 + \sigma^2 n_{t,t} + 1}{\sigma_0^2 + \sigma^2},$$

where $\gamma = \sigma_0^2/\log(1 + \sigma^2)$. The above leads to telescoping and

$$\sum_{t=1}^n \frac{1}{\sigma_0^2 + \sigma^2 n_{t,t}} \leq \gamma K [\log(\sigma_0^2 + \sigma^2 n) - \log(\sigma_0^2)] = \gamma K \log(1 + \sigma^2_0 n),$$

where we used that any arm is pulled at most $n$ times. Now we put everything together and get

$$\sum_{t=1}^n c_{t,t} \leq \frac{a \log(1 + \sigma_0^2 n)}{\log(1 + \sigma_0^2)} \beta \sigma^2 \sigma_0^2 K n \log n. \quad (14)$$

The next step is the second term in (13). To bound it, we show in Lemma 4 of Appendix F that $\mu_k|H_t \sim \mathcal{N}((\mu_k, t \tau_k^2), t)$, under the assumption of $r_{k,t} \sim \mathcal{N}(\mu_k, \sigma^2)$ and $\mu_k \sim \mathcal{N}(\mu_0, \sigma_0^2)$. By using this property,

$$\mathbb{E}[(\hat{\mu}_{k,t} - \mu_k) I\{E_{R(t)}\} | H_t]$$

$$\leq \sum_{k=1}^K \frac{1}{\sqrt{2\pi \tau_k^2}} \int_{x > 2\tau_k^2} x^2 \exp\left(-\frac{x^2}{2\tau_k^2}\right) dx \leq \frac{\delta_t}{\sqrt{2\pi}} \sum_{k=1}^K \tau_k,$$

where the last inequality is from (11). It follows that for $a \geq 1$,

$$\mathbb{E} \left[ \sum_{t=1}^n \mathbb{E}[(\hat{\mu}_{t,t} - \mu_k) I\{E_{R(t)}\} | H_t] \right]$$

$$\leq \frac{1}{\sqrt{2\pi}} \sum_{t=1}^n t^{-1/2} \beta K \sqrt{\frac{\sigma^2}{1 + \sigma^2 \sigma_0^2}}$$

$$\leq \frac{1}{\sqrt{2\pi}} \beta K \sqrt{\frac{2n\sigma^2}{1 + \sigma^2 \sigma_0^2}}, \quad (15)$$

where the first inequality follows from $\delta_t \leq t^{-1/2}$ for $a \geq 1$ (Lemma 3 of Appendix F), and the last one is from $\sum_{t=1}^n t^{-1/2} \leq 2\sqrt{n}$. Similarly, when $a \geq 2$,

$$\mathbb{E} \left[ \sum_{t=1}^n \mathbb{E}[(\hat{\mu}_{t,t} - \mu_k) I\{E_{R(t)}\} | H_t] \right]$$

$$\leq \frac{\beta K (1 + \log n)}{\sqrt{2\pi}} \sqrt{\frac{\sigma^2}{1 + \sigma^2 \sigma_0^2}}.$$

where the inequality is from $\delta_t \leq t^{-1}$ for $a \geq 2$ and $\sum_{t=1}^n t^{-1} \leq 1 + \log n$.

At last, we study the third term in (13). The result in (15) or (16) holds for the term. Therefore, by combing (13), (14), (15), and (16), the theorem is proved.

### 6 SYNTHETIC EXPERIMENTS

We study two bandit settings: Gaussian (Section 6.1) and Bernoulli (Section 6.2). Moreover, in Section 6.3, we study misspecified priors. ReUCB is compared to UCB1 (Auer et al., 2002) and TS (Thompson, 1933). TS is chosen because it uses the same structure as ReUCB, that arm means are random. However, it needs more knowledge, the prior distribution of $\mu_k$. Since ReUCB is a UCB algorithm, it is natural to compare it to other UCB algorithms. We focus on UCB1 due to its simplicity and popularity, but also compare to Bayes-UCB (Kaufmann et al., 2012) and KL-UCB (Garivier and Cappe, 2011) in Figure 5 of Appendix I. Both Bayes-UCB and KL-UCB improve over UCB1, but are not better than TS. This is consistent with other reported results in the literature (Kveton et al., 2019). There are many other potential baselines, such as Giro (Kveton et al., 2018) and PHE (Kveton et al., 2019). Our ReUCB is fundamentally different from these methods, since our arm means are random. Also, when compared to these methods, TS is typically a strong baseline (Kveton et al., 2019). Therefore, to make our empirical studies clean and focused, we compare to UCB1 and TS.

We evaluate two variants of ReUCB: (1) ReUCB*, where $\mu_0$ is estimated, and $\sigma_0^2$ and $\sigma^2$ are known; and (2) ReUCB, where all of $\mu_0$, $\sigma_0^2$, and $\sigma^2$ are estimated. The variance estimators $\hat{\sigma}_0^2$ and $\hat{\sigma}^2$ are provided in (22) and (23), respectively, of Appendix D. Unless specified, the default priors in Gaussian and Bernoulli TS are $\mathcal{N}(\mu_0, \sigma_0^2)$ and Beta(1, 1), respectively. The upper confidence bound in UCB1 is $\hat{\mu}_{k,t} + \sqrt{8n_{k,t}^{-1}} \sigma^2 \log t$. This is a generalization of the original algorithm to $\sigma^2$-sub-Gaussian rewards. In the original algorithm, $\sigma^2 = 1/4$. In Gaussian bandits, we set $\sigma$ to Gaussian noise. In Bernoulli bandits, this reduces to the UCB1 index since $\sigma^2 = 1/4$. All simulations are averaged over 1000 independent runs.

#### 6.1 Gaussian Bandits

Our first experiment is on $K$-armed Gaussian bandits. The reward distribution of arm $k$ is $\mathcal{N}(\mu_k, \sigma^2)$ where $\sigma = 0.5$. We generate $\mu_k$ in two ways. First, $\mu_k$ are drawn independently from Gaussian prior $\mathcal{N}(\mu_0, \sigma_0^2)$, where we study two settings of $(\mu_0, \sigma_0^2)$: (1, 0.04) (low coefficient of variation 0.2) and (1, 1) (high coefficient
of variation 1). Second, $\mu_k$ are drawn from uniform distribution $U[1,2]$. The number of arms is $K = 50$. The horizon is $n = 10^4$ rounds.

Figures 1(a) and 1(b) report results for Gaussian priors, while Figure 1(c) shows results for the uniform prior. We observe that TS works well and outperforms UCB1. ReUCB has a much lower regret than both UCB1 and TS. Besides good average performance, the distribution of the regret in the final round (lower row in Figure 1) shows good stability. The good performance of ReUCB in Figure 1(c) indicates that ReUCB works for various priors. ReUCB performs well empirically because high-confidence intervals are narrower than in the classical setting (Proposition 1). It outperforms TS with more information in Gaussian bandits because its confidence interval widths $\tau_{k,t}$ are narrower than the posterior widths of TS.

Now we compare the regret of ReUCB$^*$ and ReUCB in Figure 1. Clearly the estimation of $\sigma^2$ and $\sigma_0^2$ does not have a major impact on the regret of ReUCB. In fact, the regret slightly decreases. We believe that this is due to the additional randomness in our method-of-moments estimators of $\sigma^2$ and $\sigma_0^2$. These results suggest that one limitation of our analysis, that $\sigma^2$ and $\sigma_0^2$ are known, is not a limitation in practice.

Finally, we report the run times of all algorithms. All experiments are conducted in R, on a PC with 3GHz Intel i7 CPU, 8GB RAM, and OS X operating system. In Figure 1(a), a single run of ReUCB, UCB1, and TS takes on average 1.27s, 1.16s, and 3.19s, respectively. So ReUCB is slightly slower than UCB1 but much faster than TS, which is slower due to posterior sampling.

### 6.2 Bernoulli Bandits

The second experiment is conducted on $K$-armed Bernoulli bandits, where the reward distribution of arm $k$ is $\text{Bern}(\mu_k)$. The arm means $\mu_k$ are drawn i.i.d. from uniform distribution $U[0.2,0.5]$. We experiment with three settings for the number of arms $K \in \{20,50,100\}$, to show that ReUCB performs well across all of them. The horizon is $n = 10^4$ rounds.

As in Figure 1, we observe in Figure 2 that ReUCB has a much lower regret than UCB1 and TS, and performs similarly to ReUCB$^*$. To implement ReUCB$^*$, we set the maximum variance to $\sigma^2 = 1/4$, as suggested in Appendix G. Different from Figure 1, Figure 2 shows the regret for various $K$. As the number of arms $K$ increases, the gap between our approaches and the baselines increases.

### 6.3 Model Misspecification

Now we study what happens when TS and ReUCB$^*$ are applied to misspecified models. Note that ReUCB is also misspecified in Section 6.2, where the reward noise in Bernoulli bandits depends on the mean of the arm, meaning that it is not identical across the arms.

In the first experiment, we have a 50-armed Gaussian bandit with $\mu_k \sim N(1,0.04)$, as in Figure 1(a). We implement two variants of Thompson sampling with misspecified priors: TSm1 with prior $N(1,1)$ (misspecified $\sigma^2_0$) and TSm2 with prior $N(0,0.04)$ (misspecified $\mu_0$). In ReUCB$^*$, $\sigma_0^2 = 1$ and thus is also misspecified. Our results are reported in Figure 3(a), where TSm2 fails and has linear regret. The reason is that the misspecified prior has low variance and is downwards biased. Therefore, any initially pulled suboptimal arm is likely
Random Effect Bandits

Figure 2: $K$-armed Bernoulli bandits with $\mu_k \sim \mathcal{U}[0.2, 0.5]$. Upper row: Regret as a function of round $n$. Lower row: Distribution of the regret at the final round.

Figure 3: Model misspecification experiments. Upper row: Regret as a function of round $n$. Lower row: Regret at the final round.

In the second experiment, we have a Bernoulli bandit with $K = 20$ arms and $\mu_k \sim \text{Beta}(1, 9)$. We study two variants of Thompson sampling: TS with a correct prior and TSm with misspecified prior Beta(9, 1, 1). In ReUCB*, we set $\sigma_0^2 = 0.00818$ to match the variance of Beta(9, 1) and $\sigma^2 = 0.25$ because this is the maximum reward variance. This makes ReUCB* misspecified. Our results are reported in Figure 3(b) and are similar to Figure 3(a). We observe that TSm performs worse than ReUCB*, and that TS has much higher regret than ReUCB.

In the last experiment, we study reward-model misspecification. We have a 20-armed Gaussian bandit where the rewards are truncated to $[0, 1]$ as $r_{k,j} = \min\{\max\{r'_{k,j}, 0\}, 1\}$ for $r'_{k,j} \sim \mathcal{N}(\mu_k, 0.04)$. The mean arm rewards are generated as $\mu_k \sim \mathcal{N}(0.3, 0.01)$. We implement Bernoulli TS with prior Beta(6, 14) to match the moments of $\mathcal{N}(0.3, 0.01)$ and Gaussian TS with the correct prior $\mathcal{N}(0.3, 0.01)$. Our results are reported in Figure 3(c). ReUCB performs robustly and outperforms Thompson sampling.
7 EXPERIMENTS ON REAL DATA

In the last experiment, we evaluate ReUCB on a recommendation problem. The goal is to identify the movie that has the highest expected rating. We experiment with the MovieLens dataset (Lam and Herlocker, 2016), where we used a subset of \( K = 128 \) user groups and \( L = 128 \) movies randomly chosen from the full dataset, as described in Katariya et al. (2017). For each user group and movie, we average the ratings of all users in the group that rated the movie, and obtain the expected rating matrix \( M \) of rank 5, which is learned by a low-rank approximation on the underlying rating matrix of the user groups and movies. See details of the pre-processing in Katariya et al. (2017).

Our results are averaged over 200 runs. In each run, user \( j \) is chosen uniformly at random from \([128]\) and it represents a bandit instance in that run. The goal is to learn the most rewarding movie for user \( j \). We treat this problem as a random-effect bandit with \( K = 128 \) arms, one per movie, where the mean reward of movie \( k \) by user \( j \) is \( M_{j,k} \). The rewards are generated from \( \mathcal{N}(M_{j,k}, 0.796^2) \), where the variance 0.796 is estimated from data.

Our approach is compared to UCB1 and TS. We implement Gaussian TS with a prior \( \mathcal{N}(\mu_0, \sigma_0^2) \) that is estimated from the empirical mean rewards of all 128 arms. That is, for each user \( j \), \( \mu_0 \) and \( \sigma_0^2 \) are the empirical mean and variance of \( M_{j,1}, \ldots, M_{j,128} \). We implement UCB1 by taking the upper confidence bound with \( \sigma = 0.796 \). Our results are reported in Figure 4. We observe that ReUCB has a much lower regret than both UCB1 and TS. This indicates that ReUCB can learn the biases of different bandit instances, which represent individual users.

8 CONCLUSIONS

We propose a random-effect bandit, a novel setting where the arm means are sampled i.i.d. from an unknown distribution. Using this model, we obtain an improved estimator of arm means and design an efficient UCB-like algorithm ReUCB. ReUCB is prior-free and we show empirically that it can outperform Thompson sampling. We analyze ReUCB and prove a Bayes regret bound on its \( n \)-round regret, which improves over not using the random-effect structure.

Our initial results with random-effect models are encouraging. One limitation of our current approach is that ReUCB is not contextual. In the future work, we plan to propose random-effect contextual bandits and provide an algorithm for them. Another limitation is that our regret analysis is under the assumption that ReUCB knows \( \sigma_0^2 \) and \( \sigma^2 \). While this seems limiting, it is a weaker assumption than knowing the common mean \( \mu_0 \), which would be a standard assumption in the analysis of TS and Bayes-UCB.

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References


A Derivation of (8)

Let \( \mathbf{r}_k = (r_{k,1}, r_{k,2}, \ldots, r_{k,n_k})^\top \) be a column vector of rewards obtained by pulling arm \( k \). From modeling assumptions (1) and (2), we get

\[
\mathbf{r}_k = \mu_k \mathbf{1}_k + \mathbf{e}_k,
\]

where \( \mathbf{e}_k = (e_{k,1}, e_{k,2}, \ldots, e_{k,n_k})^\top \) and \( e_{k,j} \sim P_r(0, \sigma^2) \). The covariance matrix for vector \( \mathbf{r}_k \) is \( \mathbf{V}_k = \sigma_0^2 \mathbf{1}_k \mathbf{1}_k^\top + \sigma^2 \mathbf{I}_k \), where \( \mathbf{1}_k \) is an all-ones vector of length \( n_k \) and \( \mathbf{I}_k \) is a \( n_k \times n_k \) identity matrix. The generalized least squares estimator of \( \mu_0 \) minimizes the following loss

\[
L(\mu_0) = \sum_{k=1}^{K} (\mathbf{r}_k - \mu_0 \mathbf{1}_k)^\top \mathbf{V}_k^{-1}(\mathbf{r}_k - \mu_0 \mathbf{1}_k)
\]

with respect to \( \mu_0 \). Using the Sherman-Morrison formula,

\[
\mathbf{V}_k^{-1} = \sigma^{-2} \mathbf{I}_k - \sigma^{-4}(\sigma_0^2 + n_k \sigma^2)^{-1} \mathbf{1}_k \mathbf{1}_k^\top = \sigma^{-2} \mathbf{I}_k - \sigma^{-4} \sigma_0^2 (\sigma^2 + n_k \sigma_0^2)^{-1} \mathbf{1}_k \mathbf{1}_k^\top.
\]

Inserting the above formula into \( L(\mu_0) \) yields

\[
L(\mu_0) = \sigma^{-2} \sum_{k=1}^{K} [\| \mathbf{r}_k - \mu_0 \mathbf{1}_k \|^2 - n_k w_k (\bar{r}_k - \mu_0)^2] .
\]

The first-order derivative of \( L(\mu_0) \) with respect to \( \mu_0 \) is

\[
\frac{\partial L(\mu_0)}{\partial \mu_0} = 2 \sigma^{-2} \sum_{k=1}^{K} (1 - w_k) n_k (\bar{r}_k - \mu_0).
\]

Thus we get that \( \mu_0 \) is estimated by

\[
\bar{r}_0 = \sum_{k=1}^{K} (1 - w_k) n_k \bar{r}_k / \sum_{k=1}^{K} (1 - w_k) n_k.
\]

(17)

B Derivation of (10)

Now we derive the variance of \( \hat{\mu}_k \). We have

\[
\mathbb{E}[(\hat{\mu}_k - \mu_k)^2] = \mathbb{E}[(\tilde{\mu}_k - \mu_k + \hat{\mu}_k - \tilde{\mu}_k)^2] = \mathbb{E}[(\hat{\mu}_k - \mu_k)^2] + \mathbb{E}[(\tilde{\mu}_k - \tilde{\mu}_k)^2],
\]

where, we recall, \( \hat{\mu}_k \) in (5) and \( \tilde{\mu}_k \) in (9) differ only in that \( \mu_0 \) is estimated by \( \bar{r}_0 \), and the first step follows from \( \mathbb{E}[(\hat{\mu}_k - \mu_k)(\tilde{\mu}_k - \tilde{\mu}_k)] = 0 \) (Kachar and Harville, 1984). Now we derive the two terms of the right-hand of (18). Note that the first term is shown in (7). For the other term,

\[
\mathbb{E}[(\tilde{\mu}_k - \tilde{\mu}_k)^2] = (1 - w_k)^2 \mathbb{E}[(\bar{r}_0 - \mu_0)^2].
\]

(19)

From \( \bar{r}_0 \) in (8),

\[
\text{Var}(\bar{r}_0) = \left[ \sum_{k=1}^{K} (1 - w_k) n_k \right]^{-2} \sum_{k=1}^{K} (1 - w_k)^2 n_k^2 \text{Var}(\bar{r}_k)
\]

\[
= \left[ \sum_{k=1}^{K} (1 - w_k) n_k \right]^{-2} \sum_{k=1}^{K} (1 - w_k)^2 n_k^2 (\sigma_0^2 + \sigma^2 / n_k)
\]

\[
= \sigma^2 \left[ \sum_{k=1}^{K} (1 - w_k) n_k \right]^{-1},
\]
where the last step is from \( \sigma^2 + n_k \sigma_0^2 = (1 - w_k)^{-1} \sigma^2 \). Inserting the result above into (19), we have

\[
E[(\hat{\mu}_k - \hat{\mu}_k)^2] = \frac{(1 - w_k)^2 \sigma^2}{\sum_{k=1}^{K} n_k(1 - w_k)}.
\]

(20)

Therefore, inserting (7) & (20) into (18),

\[
E[(\hat{\mu}_k - \mu_k)^2] = w_k n_k^{-1} \sigma^2 + \frac{(1 - w_k)^2 \sigma^2}{\sum_{k=1}^{K} n_k(1 - w_k)} =: \tau_k^2,
\]

(21)

where the reason that we use the squares notation in \( \tau_k^2 \) is because \( \tau_k^2 \) is a mean squared error not smaller than 0.

C  Proofs of Proposition 1

Note that

\[
\sigma^2/n_k - \tau_k^2 = (1 - w_k) n_k^{-1} \sigma^2 - \frac{(1 - w_k)^2 \sigma^2}{\sum_{i=1}^{K} n_i(1 - w_i)} = (1 - w_k) n_k^{-1} \sigma^2 \left[ 1 - \frac{n_k(1 - w_k)}{\sum_{i=1}^{K} n_i(1 - w_i)} \right].
\]

Obviously, \( n_k(1 - w_k) < \sum_{i=1}^{K} n_i(1 - w_i) \) as long as \( n_i \geq 1 \) for all \( i \in [K] \). Thus, when \( \sigma^2 > 0 \), we get \( \tau_k^2 < \sigma^2/n_k \).

D  Estimation of \( \sigma_0^2 \) and \( \sigma^2 \)

Our BLUP estimators depend on \( \sigma_0^2 \) and \( \sigma^2 \), which may be unknown. We can estimate these quantities, and replace \( \sigma_0^2 \) and \( \sigma^2 \) in \( w_k \) with these estimates. Various methods for obtaining consistent estimators of \( \hat{\sigma}_0^2 \) and \( \hat{\sigma}^2 \) are available, including the method of moments, maximum likelihood, and restricted maximum likelihood. See Robinson (1991) for details.

We use the method of moments, which does not rely on the assumption of distributions. Unbiased quadratic estimates of \( \sigma^2 \) and \( \sigma_0^2 \) are given by

\[
\hat{\sigma}^2 = \left[ \sum_{k=1}^{K} (n_k - 1) \right]^{-1} \sum_{k=1}^{K} \sum_{j=1}^{n_k} (r_{k,j} - \bar{r}_k)^2
\]

(22)

and

\[
\hat{\sigma}_0^2 = n_\star^{-1} \sum_{k=1}^{K} n_k u_k^2
\]

(23)

where \( n_\star = \sum_{k=1}^{K} n_k - \left( \sum_{k=1}^{K} n_k \right)^{-1} \sum_{k=1}^{K} n_k^2 \) and \( u_k = \bar{r}_k - \left( \sum_{k=1}^{K} n_k \right)^{-1} \sum_{k=1}^{K} \sum_{j=1}^{n_k} r_{k,j} \).

E  Varying Reward Noise

We can generalize the standard random effect model in (2) by eliminating the assumption of identical observation noise across all arms. Instead, we allow the noise vary across arms. Specifically, the reward of arm \( k \) after the \( j \)-th pull is assumed to be generated i.i.d. as

\[
r_{k,j} \sim \mathcal{N}(\mu_k, \sigma_k^2),
\]
where, compared to model (2), variance $\sigma_k^2$ is allowed to depend on $k$. Accordingly, we have the estimate

$$\hat{\mu}_k^h = (1 - w_k^h)\tilde{r}_0 + w_k^h\tilde{r}_k$$

and $\hat{\mu}_k^h - \mu_k \mid H_t \sim \mathcal{N}(0, \tau_{k,t}^2)$, where the superscript “h” means the heteroscedasticity of reward noise among arms,

$$w_k^h = \sigma_0^2/(\sigma_0^2 + \sigma_k^2/n_k) \text{ and } \tau_{k,t}^2 = w_k^h n_k^{-1} \sigma_k^2 + (1 - w_k^h)^2 \sigma_0^2/\sum_{k=1}^K n_k(1 - w_k^h).$$

We consider an upper bound of these $\sigma_k^2$, e.g., $\max_k \sigma_k^2$, denoted by $\sigma^2$. Notice

$$\tau_{k,t}^2 = n_k^{-1} \sigma_k^2/\sigma_0^2 + (1 - w_k^h)^2 \sigma_0^2/\sum_{k=1}^K w_k.\n
Because $n_k^{-1} \sigma_k^2/\sigma_0^2 + (1 - w_k^h)^2 \sigma_0^2$ is increasing of $\sigma_k^2$ and $w_k^h$ is decreasing of $\sigma_k^2$, we have that

$$\tau_{k,t}^2 \leq \tau_{k,t}^2.$$

Therefore, we uniformly use the upper variance $\sigma^2$ across arms replacing of $\sigma_k^2$. By this way, the Bayes regret bound in Theorem 2 still holds.

**F Lemmas**

**Lemma 3.** We have that

$$\tau_{k,t}^2 \leq \frac{\sigma_0^2\sigma^2}{n_k\sigma_0^2 + \sigma^2}(1 + K^{-1}\sigma^2\sigma_0^{-2}).$$

Proof. Now we provide an upper bound on $\tau_{k,t}^2$. By using $n_k(1 - w_k)\sigma_0^2 = w_k\sigma_0^2$, we have

$$\tau_{k,t}^2 = w_k n_k^{-1}\sigma_0^2 + \frac{(1 - w_k)^2\sigma^2}{\sum_{k=1}^K w_k(1 - w_k)}
= w_k n_k^{-1}\sigma_0^2 + \frac{(1 - w_k)^2\sigma_0^2}{\sum_{k=1}^K w_k}
\leq w_k n_k^{-1}\sigma_0^2 + K^{-1}(1 - w_k)^2(\sigma_0^2 + \sigma^2)
= w_k n_k^{-1}\sigma_0^2 + K^{-1}(1 - w_k)w_k\sigma_0^2\sigma_0^{-2}(\sigma_0^2 + \sigma^2)
\leq w_k n_k^{-1}\sigma_0^2(1 + K^{-1}\sigma^2\sigma_0^{-2})
= \frac{\sigma_0^2\sigma^2}{n_k\sigma_0^2 + \sigma^2}(1 + K^{-1}\sigma^2\sigma_0^{-2})$$

where the first inequality is from $w_k \geq \sigma_0^2/(\sigma_0^2 + \sigma^2)$ and the last inequality is from $1 - w_k \leq \sigma^2/(\sigma_0^2 + \sigma^2)$ due to $n_k \geq 1$ for all $k \in [K]$.

**Lemma 4.** Let $r_{k,j} \sim \mathcal{N}(\mu_k, \sigma^2)$ and $\mu_k \sim \mathcal{N}(\mu_0, \sigma_0^2)$. Assuming that $\sigma^2$ and $\sigma_0^2$ are known, and $\mu_0$ is an improper flat prior, i.e., $p(\mu_0) \propto 1$, we have that $\mu_k \mid H_t \sim \mathcal{N}(\hat{\mu}_{k,t}, \tau_{k,t}^2)$.

Proof. Recall the following well-known identity. Let $Y \mid X = x \sim \mathcal{N}(ax + b, \sigma^2)$ and $X \sim \mathcal{N}(\mu, \sigma_0^2)$. Then

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma_0^2 + \sigma^2).$$

Obviously, if we set $X$ to $\mu_0 \mid H_t$ and $Y$ to $\mu_k \mid H_t$, we can apply this result to obtain the distribution of $\mu_k \mid H_t$ from the distributions of $\mu_k \mid \mu_0, H_t$ and $\mu_0 \mid H_t$. The distribution of $\mu_k \mid \mu_0, H_t$ is studied in Section 3.1. Under the assumptions of Lemma 4, $\mu_k \mid \mu_0, H_t$ is a Gaussian with mean in (5) and variance in (7).

Now we derive the distribution of $\mu_0 \mid H_t$. Note that $p(\mu_0) \propto 1$ is the extreme case of $\mu_0 \sim \mathcal{N}(0, \lambda)$ as $\lambda \to \infty$. Assuming $\mu_0 \sim \mathcal{N}(0, \lambda)$, $\tilde{r}_k$ can be considered to be generated from the following Bayesian model:

$$\tilde{r}_k \mid \mu_0 \sim \mathcal{N}(\mu_0, \sigma_0^2 + \sigma^2/n_k), \quad \mu_0 \sim \mathcal{N}(0, \lambda).$$
Thus, the distribution of $\mu_0 \mid H_t$ is easily obtained as
\[ \mu_0 \mid H_t \sim N(\tilde{r}_0, \sigma_0^2), \]
where
\[ \tilde{r}_0 = \sigma_0^2[\sigma^{-2} \sum_{k=1}^{K} (1 - w_k)n_k \tilde{r}_k] \quad \text{and} \quad \sigma_0^2 = [\lambda^{-1} + \sigma^{-2} \sum_{k=1}^{K} (1 - w_k)n_k]^{-1}. \]
Taking $\lambda \to \infty$, we have that
\[ \mu_0 \mid H_t \sim N(\tilde{r}_0, \sigma^2[\sum_{k=1}^{K} (1 - w_k)n_k]^{-1}). \]

Using the above results, we obtain the distribution of $\mu_k \mid H_t$ from the distributions of $\mu_k, H_t$ and $\mu_0 \mid H_t$. That distribution is a Gaussian with mean in (9) and variance in (10). This completes the proof. \hfill \□

### G Extension to sub-Gaussian

Our analysis can be extended to bounded sub-Gaussian random variables. Without loss of generality, we consider the support of $[0, 1]$ below.

**Theorem 5.** Consider ReUCB in a $K$-armed bandit with sub-Gaussian rewards $r_{k,j} - \mu_k \sim \text{subG}(\sigma^2)$ and $\mu_k - \mu_0 \sim \text{subG}(\sigma_0^2)$ with support in $[0, 1]$. Let $\sigma_0^2$ and $\sigma^2$ be known and used by ReUCB. Define $m = [1 + K^{-1}\sigma_0^2/(\sigma_0^2 + \sigma^2)]^{-1}[1 + K^{-1/2}\sigma/\sigma_0]^2$. Then (I) for any $a \geq m$, the $n$-round Bayes regret of ReUCB is
\[ R_n \leq 2 \left(1 + \frac{\sigma^2}{K\sigma_0^2}\right) \sqrt{\frac{a\sigma_0^2 \log(1 + \frac{\sigma^2}{\sigma_0^2})}{\log(1 + \frac{\sigma^2}{\sigma_0^2})}} Kn \log n + \frac{K\sigma_0^2 + \sigma^2}{\sigma_0^2} \sqrt{\frac{8n\sigma_0^2\sigma^2}{\pi(\sigma_0^2 + \sigma^2)}}, \]
(2) for any $a \geq 2m$, the $n$-round Bayes regret of ReUCB is obtained by replacing the last term above with $2K(1 + \log n)$.

**Proof.** Under the sub-Gaussian assumptions that $\mu_k - \mu_0 \sim \text{subG}(\sigma_0^2)$ and $\bar{r}_{k,t} - \mu_k \sim \text{subG}(\sigma^2/n_k)$, Lemma 6 shows that $\mu_k - \bar{\mu}_{k,t} \sim \text{subG}(\tau_{k,t})$ for $k \in [K]$, where
\[ \tau_{k,t}^2 = \sigma^2 \left[ \frac{w_k}{n_k} + \frac{(1 - w_k)^2}{\sum_{k=1}^{K} (1 - w_k)n_k} \right]^2. \]
Lemma 7 of the Appendix shows that
\[ \tau_{k,t}^2 \leq \frac{\sigma_0^2\sigma^2}{n_k\sigma_0^2 + \sigma^2} \left(1 + \sqrt{K^{-1}\sigma^2\sigma_0^{-2}}\right)^2. \]
Note that $c_{k,t} = \sqrt{2\tau_{k,t}^2 \log(1/\delta_t)}$, $E_{R,t}$, and $E_{L,t}$. We have that
\[ \mathbb{E} \left[ (\tilde{\mu}_{t,-} - \mu_t) \mathbf{1}\{E_{R,t}\} \mid H_t \right] \leq \mathbb{E} \left[ \mathbf{1}\{E_{R,t}\} \mid H_t \right], \]
where the inequality is from $(\tilde{\mu}_{t,-} - \mu_t) \in [0, 1]$ on $E_{R,t}$ due to the support $[0, 1]$. It follows that when $a \geq m$,
\[ \mathbb{E} \left[ \sum_{t=1}^{n} \mathbb{E} \left[ (\tilde{\mu}_{t,-} - \mu_t) \mathbf{1}\{E_{R,t}\} \mid H_t \right] \right] \leq \sum_{t=1}^{n} \mathbb{E} \left[ \mathbf{1}\{E_{R,t}\} \mid H_t \right] \leq K \sum_{t=1}^{n} \delta_t^{1/m} \leq K \sum_{t=1}^{n} t^{-1/2} \leq 2K\sqrt{n}, \]
where the second inequality is from $\tau_{k,t}^2/\tau_{k,t}^2 \geq 1/m$ shown in Lemma 9, and the third inequality is from the $a \geq m$. Similarly we have $\mathbb{E} \left[ \sum_{t=1}^{n} \mathbb{E} \left[ (\mu_t - \tilde{\mu}_{t,-}) \mathbf{1}\{E_{L,t}\} \mid H_t \right] \right] \leq 2K\sqrt{n}$. 

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Similarly to (14), we have that
\[
E \left[ \sum_{t=1}^{n} \sqrt{2\tau_{k,t}^2 \log(1/\delta_t)} \right] \leq \left( 1 + \frac{\sigma^2}{K\sigma_0^2} \right) \sqrt{\frac{a\sigma_0^2 \log(1 + \sigma^{-2}\sigma_0^2 n)}{\log(1 + \sigma^{-2}\sigma_0^2)}} K n \log n.
\]
Therefore, when \( a \geq m \), the regret is bounded as
\[
E \left[ R_n \right] \leq 2 \left( 1 + \frac{\sigma^2}{K\sigma_0^2} \right) \sqrt{\frac{a\sigma_0^2 \log(1 + \sigma^{-2}\sigma_0^2 n)}{\log(1 + \sigma^{-2}\sigma_0^2)}} K n \log n + 4K \sqrt{n}.
\]
Similarly, when \( a \geq 2m \), the regret is bounded as
\[
E \left[ R_n \right] \leq 2 \left( 1 + \frac{\sigma^2}{K\sigma_0^2} \right) \sqrt{\frac{a\sigma_0^2 \log(1 + \sigma^{-2}\sigma_0^2 n)}{\log(1 + \sigma^{-2}\sigma_0^2)}} K n \log n + 4K(1 + \log n).
\]

When comparing Theorems 2 and 5, the regret is of the same order. Since the assumption of \( r_{k,j} - \mu_k \sim \text{subG} (\sigma^2) \) allows for modeling arm-dependent reward noise, such as \( \sigma^2 = 1/4 \) in Bernoulli bandits, Theorem 5 holds for Bernoulli bandits. In Section 6, we experiment with Bernoulli bandits. Unlike Theorem 2, Theorem 5 requires that \( a \geq m \) or \( a \geq 2m \). We note that \( m \) in Theorem 5 is typically small, and approaches 1 as \( K \) and \( \sigma_0^2/\sigma^2 \) increase.

**Lemma 6.** We have that for \( k \in [K] \)
\[
\mu_k - \hat{\mu}_{k,t} \sim \text{subG}(\tau_{k,t}^2),
\]
where
\[
\tau_{k,t}^2 = \sigma^2 \left[ \sqrt{\frac{w_k}{n_k}} + \sqrt{\frac{(1 - w_k)^2}{\sum_{k=1}^{K}(1 - w_k)n_k}} \right]^2.
\]
Notice
\[
\hat{\mu}_{k,t} - \mu_k = \hat{\mu}_{k,t} - \hat{\mu}_{k,t} + \hat{\mu}_{k,t} - \mu_k,
\]
where \( \hat{\mu}_{k,t} - \mu_k = w_k(\hat{r}_{k,t} - \mu_k) + (1 - w_k)(\mu_k - \mu_0) \) and \( \hat{\mu}_{k,t} - \hat{\mu}_{k,t} = (1 - w_k)(\hat{r}_{0,t} - \mu_0) \). From the properties of sub-Gaussian (Fact 2) and the independence between \( \mu_0 - \mu_k \) and \( \hat{r}_{k,t} - \mu_k \), we have
\[
\hat{\mu}_{k,t} - \mu_k \sim \text{subG} \left( w_k \sigma^2/n_k \right);
\]
\[
\hat{r}_{0,t} - \mu_0 \sim \text{subG} \left( \sigma^2 \sum_{k=1}^{K}(1 - w_k)n_k \right)^{-1}.
\]
Thus, the properties of sub-Gaussian (Fact 2) tell us that
\[
\tau_{k,t}^2 = \sigma^2 \left[ \sqrt{\frac{w_k}{n_k}} + \sqrt{\frac{1 - w_k}{\sum_{k=1}^{K}(1 - w_k)n_k}} \right]^2,
\]
i.e., \( \hat{\mu}_{k,t} - \mu_k \) is sub-Gaussian with variance proxy \( \tau_{k,t}^2 \).

**Lemma 7.**
\[
\tau_{k,t}^2 \leq \frac{\sigma_0^2 \sigma^2}{n_k \sigma_0^2 + \sigma^2} \left( 1 + \sqrt{K^{-1} \sigma^2 \gamma_0^2} \right)^2.
\]
Proof. Now we provide an upper bound on $\tau_k^2$. By making using of $n_k(1 - w_k)\sigma_0^2 = w_k\sigma^2$, we have that

\[
\tau_{k,t}^2 = \sigma^2 \left[ \sqrt{w_k n_k^{-1}} + \sqrt{\frac{(1 - w_k)^2}{\sum_{k=1}^{K} n_k(1 - w_k)}} \right]^2 \\
= \left[ \sqrt{\sigma^2 w_k n_k^{-1}} + \frac{(1 - w_k)^2 \sigma_0^2}{\sum_{k=1}^{K} w_k} \right]^2 \\
\leq \left[ \sqrt{\sigma^2 w_k n_k^{-1}} + \sqrt{K^{-1}(1 - w_k)^2(\sigma_0^2 + \sigma^2)} \right]^2 \\
= \left[ \sqrt{\sigma^2 w_k n_k^{-1}} + \sqrt{K^{-1}n_k^{-1}(1 - w_k)w_k\sigma^2\sigma_0^{-2}(\sigma_0^2 + \sigma^2)} \right]^2 \\
\leq \left[ \frac{\sigma_0^2\sigma^2}{n_k\sigma_0^2 + \sigma^2} \right]^2 \left[ 1 + \sqrt{K^{-1}(\sigma_0^2 + \sigma^2)} \right]^2,
\]

where the first inequality is from $w_k \geq \sigma_0^2/(\sigma_0^2 + \sigma^2)$, and the last inequality is from $1 - w_k \leq \sigma^2/(\sigma_0^2 + \sigma^2)$.

Lemma 8.

\[
\tau_{k,t}^2 \geq \frac{\sigma_0^2\sigma^2}{n_k\sigma_0^2 + \sigma^2} (1 + K^{-1}\sigma_0^2/(\sigma_0^2 + \sigma^2)).
\]

Proof. Now we provide a lower bound on $\tau_k^2$. Similarly, we have that

\[
\tau_{k,t}^2 = w_k n_k^{-1}\sigma^2 + \frac{(1 - w_k)^2 \sigma_0^2}{\sum_{k=1}^{K} w_k} \\
\geq w_k n_k^{-1}\sigma^2 + K^{-1}(1 - w_k)^2 \sigma_0^2 \\
= w_k n_k^{-1}\sigma^2 + K^{-1}n_k^{-1}(1 - w_k)w_k\sigma^2 \\
\geq w_k n_k^{-1}\sigma^2(1 + K^{-1}\sigma_0^2/(\sigma_0^2 + \sigma^2)) \\
= \frac{\sigma_0^2\sigma^2}{n_k\sigma_0^2 + \sigma^2} (1 + K^{-1}\sigma_0^2/(\sigma_0^2 + \sigma^2)),
\]

where the first inequality is from $w_k \leq 1$, and the last inequality is from $1 - w_k \geq \sigma_0^2/(\sigma_0^2 + \sigma^2)$.

Lemma 9.

\[
\frac{\tau_{k,t}^2}{\tau_k^2} \geq \frac{1 + \sigma_0^2/(K(\sigma_0^2 + \sigma^2))}{(1 + K^{-1/2}\sigma/\sigma_0)^2}.
\]

Proof. From Lemmas 8 & 7, we use the upper bound of $\tau_{k,t}^2$ and the lower bound of $\tau_{k,t}^2$. Then the result is proved.

H Some Facts

Let $X$ and $Y$ be sub-Gaussian with variance proxies $\sigma^2$ and $\tau^2$, respectively. Then (1) a$X$ is sub-Gaussian with variance proxy $a^2\sigma^2$; (2) $X + Y$ is sub-Gaussian with variance proxy $(\sigma + \tau)^2$; and (3) if $X$ and $Y$ are independent, $X + Y$ is sub-Gaussian with variance proxy $\sigma^2 + \tau^2$.

I Additional Experiments
Figure 5: Performance of UCB algorithms on the 50-armed Gaussian bandit with \( \mu_k \sim \mathcal{N}(1, 0.04) \), as in Figure 1(a). Left column: Regret performance as a function of round \( n \). Right column: Distribution of the regret at the final round. “UCB(0)” denotes the extreme case of ReUCB, i.e., ReUCB\( ^\infty \) by taking \( w_k = 1 \) that behaves as UCB1 with \( a = 1 \). “B-UCB” in the right figure denotes BayesUCB. The results are summarized over 1000 runs.