A PROOFS OF LEMMAS

Lemma 1. If $X$ is a real-valued random variable with finite mean then

$$\lim_{x \to -\infty} xF(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} x(1 - F(x)) = 0$$

Lemma 2. Let $X, Y$ be independent random variables a.c. with respect to the Lebesgue measure. Then

$$E[\max(X, Y)] = \int_{-\infty}^{\infty} z \left( f_Y(z) F_X(z) + f_x(z) F_Y(z) \right) dz.$$  

Proof. We start by noting, for $x < 0$,

$$0 \geq xF(x) = x \int_{-\infty}^{x} f(z) dz \geq \int_{-\infty}^{x} zf(z) dz$$ (1)

Since $E[X]$ is finite, we can calculate

$$\lim_{x \to -\infty} \int_{-\infty}^{x} zf(z) dz = \lim_{x \to -\infty} \left( E[X] - \int_{-\infty}^{x} f(z) dz \right) = 0$$ (2)

Applying the squeeze theorem to (1) yields

$$\lim_{x \to -\infty} xF(x) = 0.$$ (3)

The other limit can be obtained by applying this to $-X$. □

Proof. Let $Z = \max(X, Y)$, then

$$F_Z(z) = F_{\max(X,Y)} = F_X(z) F_Y(z), \quad \text{for } z \in \mathbb{R}.$$ Thus,

$$E[\max(X, Y)] = \int_{-\infty}^{\infty} z \frac{d}{dz} F_X(z) F_Y(z) dz$$ (4)

$$= \int_{-\infty}^{\infty} z(f_X(z) F_Y(z) + f_Y(z) F_X(z)) dz.$$ (5)

B EXPECTED VOLUME OF GUMBEL BOX

Note that, with Lemma 2 in hand, we almost instantly can calculate the expected volume of a Gumbel box. If $X \sim \text{Gumbel}_{\max}(\mu_x, \beta)$ and $Y \sim \text{Gumbel}_{\min}(\mu_y, \beta)$, Lemma 2 implies

$$E[\max(0, Y - X)] = \int_{-\infty}^{\infty} (1 - F_{\min}(z; \mu_y)) F_{\max}(z; \mu_x) dz$$ (6)

$$= \beta \int_{-\infty}^{\infty} \exp\left(-e^{-u} - e^{-\frac{u - \mu_y}{\beta}}\right) du.$$ (7)

The remaining steps (which we include here for convenience) are to make the substitution $u = \frac{z - (\mu_x + \mu_y)/2}{\beta}$:

$$= \beta \int_{-\infty}^{\infty} \exp\left(-e^{-u} - e^{-\frac{u - \mu_y}{\beta}}\right) du.$$ (8)

$$= 2\beta \int_{0}^{\infty} \exp(-2e^{-u - \frac{u - \mu_y}{\beta}} - \cosh u) du.$$ (9)

By setting $z = 2e^{-u - \frac{\mu_y}{\beta}}$ this is a known integral representation of the modified Bessel function of the second kind of order zero, $K_0(z)$ (DLMF, eq 10.32.9).

C EXPLICIT CALCULATION OF INTERSECTION OF GUMBEL BOX

We can compute this explicitly for Gumbel boxes, in which case we have

$$Z^- \sim \text{Gumbel}_{\max}(\mu_Z, \beta)$$ (10)
$$Z^+ \sim \text{Gumbel}_{\min}(\mu_Z, \beta),$$ (11)
where
\[
\mu_Z^- = \beta \ln(e^{\frac{\mu_X^-}{\beta}} + e^{\frac{\mu_Y^-}{\beta}}), \quad \text{and} \quad
\mu_Z^+ = -\beta \ln(e^{-\frac{\mu_X^+}{\beta}} + e^{-\frac{\mu_Y^+}{\beta}}).
\]

Note that
\[
\ln F_{Z^-}(z) = -\exp\left[\frac{z - \mu_Z^-}{\beta}\right]
= -\exp\left[\frac{z - \beta \ln(e^{\frac{\mu_X^-}{\beta}} + e^{\frac{\mu_Y^-}{\beta}})}{\beta}\right]
= -e^{\frac{z - \mu_X^-}{\beta}} - e^{\frac{z - \mu_Y^-}{\beta}},
\]
and
\[
\ln F_{Z^+}(z) = -\exp\left[\frac{z + \mu_Z^+}{\beta}\right]
= -\exp\left[\frac{z + \beta \ln(e^{-\frac{\mu_X^+}{\beta}} + e^{-\frac{\mu_Y^+}{\beta}})}{\beta}\right]
= -e^{\frac{z - \mu_X^+}{\beta}} - e^{\frac{z - \mu_Y^+}{\beta}}.
\]

Thus, for \( z \in \mathbb{R} \), we have
\[
\ln[(1 - F_{Z^+}(z))F_{Z^-}(z)] =
= -e^{\frac{z - \mu_X^-}{\beta}} - e^{\frac{z - \mu_Y^+}{\beta}} - e^{\frac{z - \mu_X^+}{\beta}} - e^{\frac{z - \mu_Y^-}{\beta}}
= \ln[(1 - F_{X^+}(z))F_{X^-}(z)(1 - F_{Y^+}(z))F_{Y^-}(z)].
\]

Therefore,
\[
\mathbb{E}[\max(0, Z^+ - Z^-)] = \int_{\mathbb{R}} (1 - F_{Z^+}(z))F_{Z^-}(z) \, dz
= \int_{\mathbb{R}} (1 - F_{X^+}(z))F_{X^-}(z)(1 - F_{Y^+}(z))F_{Y^-}(z) \, dz.
\]