Simple Combinatorial Algorithms for Combinatorial Bandits: Corruptions and Approximations (Supplementary material)

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1 MISSING PROOFS FOR SUBSECTION 4.1

1.1 PROOF OF LEMMA 4.1

Proof. Because $\Delta_i^m \geq 2^{-\frac{m-1}{4}}$, we get the desired upper bound for n_i^m by applying it to Line 4 of Algorithm 1. For N^m, the lower bound comes because $N^m \geq n_*^m = \lambda d^2 K 2^{\frac{m-1}{2}}$. For the upper bound, we have $N^m = n_*^m + (\sum_i n_i^m) \leq 2\lambda d^2 K 2^{\frac{m-1}{2}}$. Then the upper bound for M follows trivially. □

1.2 PROOF OF LEMMA 4.2

Proof. The proof of this lemma is similar to the Lemma 4 in Gupta et al. [2019]. We provide it here for the sake of completeness.

For each arm i, we define a random variable $I_{t,i} = \mathbb{I}[Z_t = Z_i^m]$ to be the indicator of whether Z_i^m is chosen at time t. We define $c_{t,i}$ to be the corruption put on arm i on round t, so we have $\tilde{R}_{t,i} = R_{t,i} + c_{t,i}$ and our observed value is $I_{t,i}(R_{t,i} + c_{t,i})$. We define $E_m = [T_{m-1}+1, T_m]$ as the set of time step within epoch m as an abbreviation.

In the following concentration event, we choose probability $\beta = \frac{\delta}{8K \log_2 T}$

Concentration of $A_i^m = \sum\limits_{t \in E_m} I_{t,i} R_{t,i}$ By definition, we have $\mathbb{E}\left[I_{t,i} \cdot R_{t,i}\right] = q_i^m \cdot \mu_i$, so the expectation of the sum is $n_i^m \mu_i$. Then, by standard Chernoff-Hoeffding inequality:

$$Pr\left\{ \left| \frac{A_i^m}{n_i^m} - \mu_i \right| \geq \sqrt{\frac{3\mu_i \ln \frac{2}{\beta}}{n_i^m}} \right\} \leq \beta$$

Concentration of $B_i^m = \sum\limits_{t \in E_m} I_{t,i} c_{t,i}$ Note that $\mathbb{E}[I_{t,i}] = q_i^m$, so $\{(I_{t,i} - q_i^m)c_{t,i}\}_{t \in E_m}$ is a martingale difference sequence, with filtration be all the random variables generated before time t. By calculation, we have the following bound for its sum of variance,

$$V = \mathbb{E}[\sum_{t \in E_m} ((I_{t,i} - q_i^m)c_{t,i})^2] \leq q_i^m \sum_{t \in E_m} |c_{t,i}| \leq q_i^m C_i^m$$

Then, we apply Freedman-type concentration inequality for martingales: With probability at least $1 - \frac{\beta}{2}$:

$$\left| \frac{B_i^m}{n_i^m} \right| \leq \frac{q_i^m C_i^m}{n_i^m} + \frac{V + \ln \frac{4}{\beta}}{n_i^m} \leq \frac{2q_i^m C_i^m}{n_i^m} + \frac{\ln \frac{4}{\beta}}{n_i^m}$$

Because $n_i^m \geq \lambda \geq \ln \frac{4}{\beta}$, we can further enlarge the second term by taking its square root, and the resulting inequality is that

$$\left|\frac{B_i^m}{n_i^m}\right| \leq \frac{2C_i^m}{N^m} + \sqrt{\frac{\ln \frac{4}{\beta}}{n_i^m}}$$

Merging these two concentration event, we can get

$$|\hat{\mu}_i - \mu_i| = \left|\frac{A_i^m + B_i^m}{n_i^m} - \mu_i\right| \leq \frac{2C_i^m}{N^m} + \frac{\Delta_i^m}{16d}$$

Next, for the concentration bound on $\tilde{n}_i^m$, note that $\mathbb{E}[\tilde{n}_i^m] = n_i^m$, and we again use standard Chernoff inequality for the random variable $\tilde{n}_i^m = \sum\limits_{t \in E_m} I_{t,i}$:

$$Pr\left\{\left|\sum_{t \in E_m} I_{t,i} - n_i^m\right| \geq \sqrt{3n_i^m \ln \frac{2}{\beta}}\right\} \leq \beta$$

Because $n_i^m \geq \lambda \geq 12 \ln \frac{2}{\beta}$, this deviation is smaller than $\frac{n_i^m}{2}$. □

1.3 PROOF OF LEMMA 4.5

Proof. We can prove it by induction. Recall that $\Delta_i^{m+1} = \max\left(2^{-\frac{m}{4}}, \overline{r}_*^m - \underline{r}_i^m, \frac{\Delta_i^m}{2}\right)$. We only need to verify the second term and the third term. We check the second term first. Here Z denotes $\underset{Z \in \mathcal{M}}{\operatorname{argmax}} \sum_{j \in Z} \left(\hat{\mu}_j^m + \frac{1}{16d}\Delta_j^m\right)$, which is different from Z_*^m and Z_i^* denotes $\underset{Z \in \mathcal{M} \wedge i \in Z}{\operatorname{argmax}} \mu(Z)$. Event $\mathcal{E}$ is repeatedly used in the proof to give upper and lower bound for $\hat{\mu}$.

We apply the definition of $\overline{r}_*^m$ and $\underline{r}_i^m$ in Line 10 and Line 11 in Algorithm 1 and then expand them by using the induction argument.

$$\begin{aligned}
&\overline{r}_*^m - \underline{r}_i^m \\
&= \sum_{j \in Z}\left(\hat{\mu}_j^m + \frac{1}{16d}\Delta_j^m\right) - \sum_{j \in Z_i^{m+1}}\left(\hat{\mu}_j^m - \frac{1}{16d}\Delta_j^m\right) \\
&\leq \left(\mu(Z) + \frac{1}{8d}\sum_{j \in Z}\Delta_j^m + \frac{2C^m}{N^m}\right) \\
&\quad - \left(\mu(Z_i^*) - \frac{1}{8d}\sum_{j \in Z_i^*}\Delta_j^m - \frac{2C^m}{N^m}\right) \\
&\leq \mu(Z) + \frac{1}{8d}\sum_{j \in Z} 2(\Delta_j + 2^{-\frac{m-1}{4}} + \rho_{m-1}) \\
&\quad - \mu(Z_i^*) + \frac{1}{8d}\sum_{j \in Z_i^*} 2(\Delta_j + 2^{-\frac{m-1}{4}} + \rho_{m-1}) + \frac{4C^m}{N^m} \qquad (1)
\end{aligned}$$

Then we arrange all the terms and do some straightforward calculations.

$$\begin{aligned}
&\overline{r}_*^m - \underline{r}_i^m \\
&\le \mu(Z) + \frac{\Delta(Z)}{4} + \frac{2^{-\frac{m-1}{4}}}{4} + \frac{\rho_{m-1}}{4} \\
&\quad - \mu(Z_i^*) + \frac{\Delta(Z_i^*)}{4} + \frac{2^{-\frac{m-1}{4}}}{4} + \frac{\rho_{m-1}}{4} + \frac{4C^m}{N^m} && (2) \\
&\le \mu(Z) + \frac{\Delta(Z)}{4} - \mu(Z_i^*) + \frac{\Delta_i}{4} + \frac{2^{-\frac{m-1}{4}}}{2} + \frac{\rho_{m-1}}{2} + \frac{4C^m}{N^m} && (3) \\
&\le \frac{5}{4}\Delta_i + \frac{2^{-\frac{m-1}{4}}}{2} + 2\rho_m && (4) \\
&\le 2\Delta_i + 2 \times 2^{-\frac{m}{4}} + 2\rho_m
\end{aligned}$$

In (2), we use the property that $\forall j \in Z, \Delta_j \le \Delta(Z)$, and in (3), we notice that $\Delta(Z_i^*) = \Delta_i$ by definition. In (4), we need the fact that $\mu(Z^*) = \mu(Z) + \Delta(Z)$.

Next, the third term also meets the upper bound because $\frac{\rho_{m-1}}{2} \le \rho_m$ □

1.4 PROOF OF LEMMA 4.6

Proof. We use induction to prove the second term. Again, here Z denotes $\underset{Z}{\operatorname{argmax}} \sum_{j\in Z} \left(\hat{\mu}_j^m + \frac{1}{16d}\Delta_j^m\right)$.

$$\begin{aligned}
\overline{r}_*^m - \underline{r}_i^m &= \sum_{j\in Z}\left(\hat{\mu}_j^m + \frac{1}{16d}\Delta_j^m\right) - \sum_{j\in Z_i^{m+1}}\left(\hat{\mu}_j^m - \frac{1}{16d}\Delta_j^m\right) \\
&\ge \sum_{j\in Z^*}\left(\hat{\mu}_j^m + \frac{1}{16d}\Delta_j^m\right) - \left(\mu(Z_i^{m+1}) + \frac{2C^m}{N^m}\right) \\
&\ge \left(\mu(Z^*) - \frac{2C^m}{N^m}\right) - \left(\mu(Z_i^{m+1}) + \frac{2C^m}{N^m}\right) \\
&= \mu(Z^*) - \mu(Z_i^{m+1}) - \frac{4C^m}{N^m} \\
&= \Delta(Z_i^{m+1}) - \frac{4C^m}{N^m} \\
&\ge \Delta_i - \frac{4C^m}{N^m} && (5)
\end{aligned}$$

Note that in Line (5) $\Delta_i \le \Delta(Z_i^{m+1})$, because $i \in Z_i^{m+1}$. □

1.5 PROOF OF LEMMA 4.7

Proof.

$$\sum_{j\in Z_i^*}\left(\hat{\mu}_j^{m-1} - \frac{1}{16d}\Delta_j^{m-1}\right) \le \sum_{j\in Z_i^m}\left(\hat{\mu}_j^{m-1} - \frac{1}{16d}\Delta_j^{m-1}\right)$$

$$\mu(Z_i^*) - \frac{1}{8d}\sum_{j\in Z_i^*}\Delta_j^{m-1} - \frac{2C^{m-1}}{N^{m-1}} \le \mu(Z_i^m) + \frac{2C^{m-1}}{N^{m-1}}$$

We rearrange this inequality and use the equality $\Delta(Z_i^m) = \Delta_i + \mu(Z_i^*) - \mu(Z_i^m)$.

$$\begin{aligned}\Delta(Z_i^m) &\leq \frac{4C^{m-1}}{N^{m-1}} + \Delta_i + \frac{2}{8d}\sum_{j\in Z_i^*}\left(\Delta_j + 2^{-\frac{m-2}{4}} + \rho_{m-2}\right) \\ &\leq \frac{4C^{m-1}}{N^{m-1}} + \Delta_i + \frac{\Delta_i}{4} + \frac{2^{-\frac{m-2}{4}}}{4} + \frac{\rho_{m-2}}{4} \qquad (6) \\ &\leq \frac{5}{4}\Delta_i + 2\rho_{m-1} + \frac{2^{-\frac{m-2}{4}}}{4}\end{aligned}$$

In Line (6), because $j \in Z_i^*$, by Proposition 4.4, $\Delta_j \leq \Delta(Z_i^*) = \Delta_i$. □

1.6 PROOF OF LEMMA 4.8

Proof.

$$\begin{aligned}\sum_{j\in Z_*}\left(\hat{\mu}_j^{m-1} - \frac{1}{16d}\Delta_j^{m-1}\right) &\leq \sum_{j\in Z_*^m}\left(\hat{\mu}_j^{m-1} - \frac{1}{16d}\Delta_j^{m-1}\right) \\ \mu(Z^*) - \frac{1}{8d}\sum_{j\in Z^*}\Delta_j^{m-1} - \frac{2C^{m-1}}{N^{m-1}} &\leq \mu(Z_*^m) + \frac{2C^{m-1}}{N^{m-1}} \\ \mu(Z^*) - \mu(Z_*^m) &\leq \frac{4C^{m-1}}{N^{m-1}} + \frac{2^{-\frac{m-2}{4}}}{4} + \frac{\rho_{m-2}}{4} \qquad (7) \\ \Delta(Z_*^m) &\leq 2\rho_{m-1} + \frac{2^{-\frac{m-2}{4}}}{4}\end{aligned}$$

In (7), no Δ_j term exists because for $j \in Z^*$, $\Delta_j = 0$. □

1.7 PROOF OF PROPOSITION 4.9

Proof.

$$\begin{aligned}\sum_{m=1}^{M}\lambda d^2 K 2^{\frac{m-1}{2}}\rho_m &= \sum_{m=1}^{M}\lambda d^2 K 2^{\frac{m-1}{2}}\sum_{s=1}^{m}\frac{2C^s}{2^{m-s}N^s} \\ &= \sum_{s=1}^{M}\frac{C^s}{N^s}\sum_{m=s}^{M}\frac{\lambda d^2 K 2^{\frac{m+1}{2}}}{2^{m-s}} \\ &\leq \sum_{s=1}^{M}\frac{C^s}{\lambda d^2 K 2^{\frac{s-1}{2}}}\sum_{m=s}^{M}\frac{\lambda d^2 K 2^{\frac{m+1}{2}}}{2^{m-s}} \\ &\leq \sum_{s=1}^{M}2C^s\sum_{m=s}^{M}\frac{1}{2^{\frac{m-s}{2}}} \\ &\leq \sum_{s=1}^{M}C^s\frac{2}{1-\frac{1}{\sqrt{2}}} \\ &\leq O(C)\end{aligned}$$

□

1.8 PROOF OF LEMMA 4.10

Proof. The proof is almost the same as Lemma 4.1. Note $N^m \leq 4N^{m-1}$ because $N^m = n_*^m + \sum_{i\in[K]} n_i^m$ where $n_*^m = \sqrt{2}n_*^{m-1}$ and $n_i^m \leq 4n_i^{m-1}$ (by applying the third term in Line 14 of algorithm 1 to Line 4). □

1.9 PROOF OF LEMMA 4.13

Proof. For epoch $m \leq m_1$, $Z^* \in \{0,1\}^{K_m}$ is admissible for the oracle A, so

$$\mu(Z_*^m) \geq \hat{\mu}^m(Z_*^m) - \frac{2^{-m}}{16} \geq \alpha\hat{\mu}^m(Z^*) - \frac{2^{-m}}{16} \geq \alpha\text{OPT} - \frac{2^{-m}}{8}$$

□

1.10 PROOF OF LEMMA 4.16

Proof.

$$\begin{aligned}
\mu(Z_i^{m+1}) &\geq \hat{\mu}^m(Z_i^{m+1}) - \frac{2^{-m}}{16} \\
&\geq \hat{\mu}^m(Z_*^{m+1}) - \frac{2^{-m}}{16} - \frac{2^{-m}}{4} && (8) \\
&\geq \mu(Z_*^{m+1}) - \frac{2^{-m}}{16} - \frac{2^{-m}}{16} - \frac{2^{-m}}{4} \\
&\geq \alpha\text{OPT} - \frac{2^{-m}}{2} && \text{(Apply Corollary 4.15)}
\end{aligned}$$

□

Line (8) holds because Line 8 in Algorithm 2 provides a lower bound for any arm i still in K_{m+1}.