Abstract

In this paper, we revisit the problem of Differentially Private Stochastic Convex Optimization (DP-SCO) and provide excess population risks for some special classes of functions that are faster than the previous results of general convex and strongly convex functions. In the first part of the paper, we study the case where the population risk function satisfies the Tsybakov Noise Condition (TNC) with some parameter \( \theta > 1 \). Specifically, we first show that under some mild assumptions on the loss functions, there is an algorithm whose output could achieve an upper bound of \( O\left(\frac{1}{\sqrt{n}} + \frac{d}{n\epsilon}\right)\tau^{\theta_1} \) and \( O\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d\log(1/\delta)}}{n\epsilon}\right)\tau^{\theta_2} \) for \( \epsilon \)-DP and \((\epsilon, \delta)\)-DP, respectively when \( \theta \geq 2 \), here \( n \) is the sample size and \( d \) is the dimension of the space. Then we address the inefficiency issue, improve the upper bounds by Poly(\log n) factors and extend to the case where \( \theta \geq \bar{\theta} > 1 \) for some known \( \bar{\theta} \).

Next we show that the excess population risk of population functions satisfying TNC with parameter \( \theta \geq 2 \) is always lower bounded by \( \Omega\left(\frac{d}{n\epsilon}\right)\tau^{\theta_1} \) and \( \Omega\left(\frac{\sqrt{d\log(1/\delta)}}{n\epsilon}\right)\tau^{\theta_2} \) for \( \epsilon \)-DP and \((\epsilon, \delta)\)-DP, respectively, which matches our upper bounds. In the second part, we focus on a special case where the population risk function is strongly convex. Unlike the previous studies, here we assume the loss function is non-negative and the optimal value of population risk is sufficiently small. With these additional assumptions, we propose a new method whose output could achieve an upper bound of \( O\left(\frac{d\log(1/\delta)}{n^2\epsilon^2} + \frac{1}{\sqrt{n}}\right) \) and \( O\left(\frac{d^2}{n^2\epsilon^2} + \frac{1}{\sqrt{n}}\right) \) for any \( \tau > 1 \) in \((\epsilon, \delta)\)-DP and \( \epsilon \)-DP model respectively if the sample size \( n \) is sufficiently large. These results circumvent their corresponding lower bounds in (Feldman et al., 2020) for general strongly convex functions. Finally, we conduct experiments of our new methods on real world data. Experimental results also provide new insights into established theories.

Keywords: Differential Privacy, Stochastic Convex Optimization

* Part of the work was done when Jinyan Su was a research intern at KAUST.
† Extended abstract. Full version appears as (Su et al., 2022).
1. Introduction

Preserving the privacy of training data has become an important consideration and now is a challenging task for machine learning algorithms. To address the privacy issue, Differential Privacy (DP) (Dwork et al., 2006), which roots in cryptography, is a strong mathematical scheme for privacy preserving. It allows for rich statistical and machine learning analysis, and is now becoming a de facto notation for private data analysis. Methods to guarantee differential privacy have been widely studied, and recently adopted in industry (Tang et al., 2017; Ding et al., 2017).

As one of the most important problems in Machine Learning and Differential Privacy community, the Empirical Risk Minimization problem in the DP model, i.e., DP-ERM, has been studied quite well in the last decade, starting from (Chaudhuri et al., 2011), such as (Bassily et al., 2014; Wang et al., 2017, 2019a; Wu et al., 2017; Kasiviswanathan and Jin, 2016; Kifer et al., 2012; Smith et al., 2017; Wang et al., 2018, 2019b; Asi et al., 2021a). Besides DP-ERM, its population (or expected) version, namely Differentially Private Stochastic Convex Optimization (DP-SCO), has received much attention in recent years, starting from (Bassily et al., 2014). DP-SCO is defined as the follows.

**Definition 1 (DP-SCO (Bassily et al., 2014))** Given a dataset $S = \{x_1, \cdots, x_n\}$ from a data universe $X$ where $x_i$ are i.i.d. samples from some unknown distribution $\mathcal{D}$, a convex loss function $f(\cdot, \cdot)$, and a convex constraint set $W \subseteq \mathbb{R}^d$, Differentially Private Stochastic Convex Optimization (DP-SCO) is to find $w^{\text{priv}}$ so as to minimize the population risk, i.e., $F(w) = \mathbb{E}_{x \sim \mathcal{D}}[f(w, x)]$ with the guarantee of being differentially private. The utility of the algorithm is measured by the (expected) excess population risk, that is $\mathbb{E}_{A}[F(w^{\text{priv}})] - \min_{w \in W} F(w)$, where the expectation of $A$ is taken over all the randomness of the algorithm. Besides the population risk, we can also measure the empirical risk of dataset $S$: $\bar{F}(w, S) = \frac{1}{n} \sum_{i=1}^n f(w, x_i)$.

Specifically, (Bassily et al., 2019) first provides the optimal rate of DP-SCO with general convex loss functions in $(\epsilon, \delta)$-DP, which is quite different from the optimal rate in DP-ERM. Later, (Feldman et al., 2020) extends this problem to strongly convex and (or) non-smooth cases by providing a general localization technique. Moreover, their methods have linear time complexity if the loss functions are smooth. For non-smooth loss functions, (Kulkarni et al., 2021) recently proposes a new method which only need subquadratic gradient complexity. While there are already a large number of studies on DP-SCO, the problem is still far from well understood. A key observation is that, all of the previous works only focus on the the case where the loss functions are either general convex or strongly convex. However, there are also many problems that are even stronger than strongly convex functions, or fall between convex and strongly convex functions. In the non-private counterpart, various studies have attempted to get faster rates by imposing additional assumptions on the loss functions. And it has been shown that it is indeed possible to achieve rates that are faster than the rates of general convex loss functions (Yang et al., 2018; Koren and Levy, 2015; van Erven et al., 2015), or it could even achieve the same rate as in the strongly convex case even if the function is not strongly convex (Karimi et al., 2016; Liu et al., 2018; Xu et al., 2017). Motivated by this, our question is,

For the problem of DP-SCO with special classes of population risk functions, is it possible to achieve faster rates of the excess population risk than the optimal ones of general convex and (or) strongly convex cases?
In this paper, we will mainly study the case where the population risk satisfies the Tsybakov Noise Condition (TNC) (Ramdas and Singh, 2012; Liu et al., 2018)\(^1\), which includes strongly convex functions, SVM and linear regression as special cases. Moreover, it has been studied quite well and has been shown that it could achieve faster rates than the optimal one of general convex loss functions in the non-private case. Specifically,

**Definition 2** For a convex function \(F(\cdot)\), let \(W_\ast = \arg\min_{w \in W} F(w)\) denote the optimal set and for any \(w \in W\), let \(w^\ast = \arg\min_{u \in W_\ast} \|u - w\|_2\) denote the projection of \(w\) onto the optimal set \(W_\ast\). Function \(F\) satisfies \((\theta, \lambda)\)-TNC for some \(\theta > 1\) and \(\lambda > 0\) if for any \(w \in W\) the following inequality holds

\[
F(w) - F(w^\ast) \geq \lambda \|w - w^\ast\|_2^\theta.
\]

(1)

Our contributions can be summarized as follows.

- In the first part of the paper, we study the problem where the population risk satisfying \((\theta, \lambda)\)-TNC and propose three methods.

1. When \(\theta \geq 2\) and is known, we first propose a method that could achieve an excess population risk of \(\tilde{O}\left(\left(\frac{1}{\sqrt{n}} + \frac{d}{\sqrt{n}e}\right)^{\frac{\theta}{2}}\right)\) and \(\tilde{O}\left(\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{\sqrt{n}e}\right)^{\frac{\theta}{2}}\right)\) in \(\epsilon\)-DP and \((\epsilon, \delta)\)-DP model respectively under the assumption that the loss function is smooth and Lipschitz, where \(n\) is the sample size of the data and \(d\) is the dimension of the space. Our idea is based on the localization technique proposed by (Feldman et al., 2020), which provides an algorithm, namely Phased-SGD for DP-SCO with general convex loss functions. We propose an adaptive stochastic approximation algorithm. The updates are divided into \(m\) stages. At each stage, the Phased-SGD algorithm is applied with a number of samples. Each employment of the Phased-SGD algorithm is warm-started by the initial point that is returned from the last stage.

2. However, in practice, the main difficulty on implementing the previous Algorithm is we need to implement the projection onto the intersection of two balls in each iteration of the Phased-SGD in each phase. To overcome the challenge, instead of considering the original stochastic function, we focus on the problem with an additional and adaptive strongly convex regularization in each stage. Specifically, we first divide the whole algorithm into \(m\) stages. In each stage we hope to find a private estimator \(w^k\) such that \(w^k \approx \arg\min_{w \in W} F(w) + \frac{1}{2\gamma_k}\|w - w^{k-1}\|_2^2\) with \(\gamma_k\) changing with \(k\). Note that due to the additional \(\ell_2\) regularization, now the function is strongly convex. Thus, instead of using the original Phased-SGD for general convex loss, here we use a strongly convex version of Phased-SGD, which is adopted from (Feldman et al., 2020). Moreover, since now we have an additional \(\ell_2\)-norm regularization, here we do not need the projection onto the intersection of two balls during updates compared with the previous algorithm.

3. Finally, we propose another method to resolve the inefficiency issue under the assumption that \(\theta\) is known. The idea of our algorithm is as the following: assuming that the value of \(\theta\) is unknown, but \(\theta\) is lower bounded by some known constant \(\tilde{\theta} > 1\), namely \(\theta \geq \tilde{\theta} > 1\). We first divide the whole dataset into \(k = \lfloor (\log_2 2) \cdot \log \log n \rfloor\) disjoint subsets, where

\(^1\) In some related work it is also called the Error Bound Condition or the Growth Condition (Liu et al., 2018; Xu et al., 2017).
the $i$-th subset has $n_i = 2^{i-1}n/(\log n)^{\log_2 2}$ samples; then we repeat the Phased-SGD for $k$ times where each phase runs on the $i$-th subset and is initialized at the output of the previous phase. We also show that our algorithm improves the previous upper bounds of error by $\text{Poly}(\log n)$ factors and it outperforms the previous methods practically.

- Next, we focus on the lower bounds of the excess population risk. Specifically, for any $\theta \geq 2$, we show that there is a population risk function satisfying TNC with parameter $\theta$ such that for any $\epsilon$-DP ($\epsilon, \delta$)-DP algorithm, its output achieves an excess risk of $\Omega((\frac{d}{n\epsilon})^{\frac{1}{\theta} - 1})$ with high probability.

In the second part of the paper, we will focus on the problem where the population risk function is strongly convex, which is a special case of TNC functions with $\theta = 2$. Unlike the previous studies, here we assume the loss function is non-negative and the optimal value of the population is sufficiently small. With these additional assumptions, we propose a new method whose output could achieve an upper bound of $O(\frac{d\log(1/\delta)}{n\epsilon^2} + \frac{1}{n\tau})$ and $O(\frac{d^2}{n^2\epsilon^2} + \frac{1}{n\tau})$ for any $\tau > 1$ in $(\epsilon, \delta)$-DP and $\epsilon$-DP model respectively if the sample size $n$ is sufficiently large. These rates circumvent their corresponding lower bounds for general strong convex functions in (Feldman et al., 2020), i.e., $\Theta(\frac{d^2}{n^2\epsilon^2} + \frac{1}{n})$ for $\epsilon$-DP and $\Theta(\frac{d\log(1/\delta)}{n\epsilon^2} + \frac{1}{n})$ for $(\epsilon, \delta)$-DP.

There are two parts in the algorithm. In the first part, we perform the original Iterated Phased-SGD on the first half of the data to get a good solution to the optimal parameter $w^*$. After that we perform a new method, namely Epoch-DP-SGD on the second half of the data, which may also be used in other problems.

2. Related Work

Starting from (Chaudhuri et al., 2011), a long list of works have attacked the problems of DP-ERM from different perspectives: (Bassily et al., 2014; Iyengar et al., 2019; Zhou et al., 2020; Song et al., 2020; Wang et al., 2017; Zhang et al., 2017) studied the problems in the low dimensional case and the central model, (Kasiviswanathan and Jin, 2016; Kifer et al., 2012; Talwar et al., 2015; Wang and Gu, 2020; Cai et al., 2020) considered the problems in the high dimensional sparse case and the central model, (Smith et al., 2017; Duchi et al., 2013; Wang et al., 2020a; Duchi et al., 2018) focused on the problems in the local model. However, almost all of these works only focus the case where the empirical risk function is either general convex or strongly convex. For special class of functions, (Wang et al., 2017) studies the empirical risk functions satisfying Polyak-Lojasiewicz (PL) condition, which is weaker than strongly convexity and show that it is possible to achieve an excess empirical risk of $O(\frac{d\log(1/\delta)}{n\epsilon^2})$, which is the same as the strongly convex loss. And the PL condition is equivalent to TNC with parameter $\theta = 2$. Thus, in this paper we extend the result from the empirical risk to the population risk function.

For DP-SCO, besides the related work we mentioned in the previous section, there is another direction which studies some special cases of DP-SCO. For example, (Bassily et al., 2021) and (Asi et al., 2021a) consider the case where the underlying constraint set $\mathcal{W}$ has specific geometric structures, such as polyhedron. (Guzmán et al., 2021) studies the (non)smooth and (non)convex generalized linear loss. (Wang et al., 2020b) and (Kamath et al., 2021) focus on the case where the distribution of the data or the gradient of the loss function is heavy-tailed. However, none of these works study the case where the population risk satisfies TNC. (Liu et al., 2021) recently studies...
the theoretical guarantees of the PATE model (Papernot et al., 2016) under the assumption that the population risk function satisfies TNC and shows that it is possible to achieve faster rates than in the convex case (Bassily et al., 2018). However, since here we focus on a different problem, their results cannot be used to DP-SCO.

Concurrent Work: We notice that (Asi et al., 2021b) also studies DP-SCO with TNC population risk functions concurrently. However, compared with its results there are several critical differences.

1) The idea of Algorithm 2 in (Asi et al., 2021b) is similar to one algorithm in our paper. However, the idea of proof and the choice of parameters are quite different.

2) The same as our first algorithm, Algorithm 2 in (Asi et al., 2021b) is also inefficient and has poor performance in practice. To resolve the issue, we also develop two other algorithms.

3) For \((\epsilon, \delta)\)-DP model, (Asi et al., 2021b) only shows the worst-case lower bound under the assumption \(\theta \geq 1 + c\) for some constant \(c > 0\) while in this paper we also extend the result to \(\theta \geq 2\). Although the hard instance in (Asi et al., 2021b) is similar to ours, the proofs of lower bounds are different.

4) In this paper, we also provide experimental results on the problem which has not been studied in (Asi et al., 2021b).

5) Besides TNC population risk functions, in this paper we also provide faster rates of DP-SCO with strongly convex loss function with additional assumptions which also has not been studied in (Asi et al., 2021b).

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