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Data-Driven Controller Synthesis of Unknown Nonlinear Polynomial Systems via Control Barrier Certificates

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Abstract

In this work, we propose a data-driven approach to synthesize safety controllers for continuous-time nonlinear polynomial-type systems with unknown dynamics. The proposed framework is based on notions of so-called *control barrier certificates*, constructed from data while providing a guaranteed confidence of 1 on the safety of unknown systems. Under a certain rank condition, we synthesize polynomial state-feedback controllers to ensure the safety of the unknown system only via a *single trajectory* collected from it. We demonstrate the effectiveness of our proposed results by applying them to a nonlinear polynomial-type system with unknown dynamics.

Keywords: Data-driven controller synthesis, Nonlinear polynomial-type systems, Control barrier certificates, Safety property

1. Introduction

In the past two decades, formal methods have become a promising approach to synthesize controllers for complex dynamical systems enforcing high-level logic properties, *e.g.,* those expressed as linear temporal logic (LTL) formulae (Pnueli, 1977), in a reliable and cost-effective way. Nevertheless, the closed-form characterization of these controllers over continuous-time continuous-space systems is not available in general. Hence, formal controller synthesis for those complex systems against complex properties is inherently very challenging due to their continuous state and input sets.

In order to mitigate the encountered difficulty, notions of *barrier certificates* were introduced in (Prajna and Jadbabaie, 2004; Prajna et al., 2007) as a discretization-free approach for formal analysis of complex dynamical systems. Concretely, barrier certificates are Lyapunov-like functions defined over the state set of the system enforcing a set of inequalities on both the function itself and its Lie derivative alongside the flow of the system. As a key insight, an appropriate level set of a barrier certificate separates an unsafe region from all system trajectories starting from a given set of initial states. Consequently, such a function provides a formal (probabilistic) certificate for the safety of the system. Barrier certificates have been so far widely employed for the formal verification and controller synthesis of non-stochastic (Borrmann et al., 2015; Wang et al., 2017; Ames et al., 2019) and stochastic dynamical systems (Zhang et al., 2010; Ahmadi et al., 2018; Clark, 2019; Jagtap et al., 2020b; Nejati et al., 2020), to name a few.

Note that all the above-mentioned results require some models of the systems to provide the corresponding analyses. Nevertheless, closed-form mathematical models for many physical systems are either not available or equally complex to be of any practical use. Accordingly, one cannot employ these model-based techniques to analyze and design controllers for complex systems with unknown models. Although there have been some techniques to provide analysis frameworks by approximating underlying dynamics, acquiring an accurate model for complex dynamical systems is complicated, time-consuming, and expensive, in general (see e.g. (Hou and Wang, 2013, and references herein)). Due to these difficulties, data-driven techniques have received significant attentions in the past decade to bypass the modeling phase and directly employ system measurements for the verification or controller synthesis.

There have been several results on formal analysis and controller synthesis for unknown systems via *indirect data-driven approaches*, i.e., those which leverage system identification techniques followed by model-based controller synthesis approaches. In this regard, a data-driven approach based on Gaussian processes to learn models of quadrotors operating in partially unknown environments is proposed in (Wang et al., 2018). A safe reinforcement learning framework for safety-critical control tasks is presented in (Cheng et al., 2019), in which Gaussian processes are employed to model the system dynamics and its uncertainties. A data-driven approach to synthesize controllers enforcing signal temporal logic specifications is studied in (Sadraddini and Belta, 2018), where a set-valued piece-wise affine model is learned to contain all possible behaviors of an unknown system. A learning-based approach for the construction of symbolic models for nonlinear control systems to enforce safety specifications is proposed in (Hashimoto et al., 2020). A data-driven approach utilizing Gaussian processes to learn unknown control affine nonlinear systems together with a probabilistic bound on the accuracy of the learned model is presented in (Jagtap et al., 2020a). An optimization-based framework for learning control laws from data to enforce safety properties is studied in (Lindemann et al., 2020).

There have also been some results in recent years on the formal analysis of unknown systems via *direct data-driven approaches*, i.e., those that bypass the system identification phase and directly employ system measurements for the verification and control analysis. A data-driven approach for stability analysis of black-box linear switched systems is proposed in (Kenanian et al., 2019), in which a stability-like guarantee is provided based on both the number of observations and the required level of confidence. As an extension of (Kenanian et al., 2019), a data-driven computation of invariant sets for discrete time-invariant black-box systems is proposed in (Wang and Jungers, 2019). A data-enabled predictive control algorithm for unknown stochastic linear systems is presented in (Coulson et al., 2019b). A data-driven verification approach for partially-known dynamics with non-deterministic inputs and noisy observations is proposed in (Haesaert et al., 2015). Reinforcement learning schemes to synthesize correct policies for continuous-space Markov decision processes with unknown models are studied in (Lavaei et al., 2020; Kazemi and Soudjani, 2020).

Recently, other *direct* data-driven approaches which are developed on top of *behavioral approaches* (Willems and Polderman, 1997) have been proposed to solve linear quadratic regulation (LQR) problems (De Persis and Tesi, 2019), to design model-reference controllers for linear systems (Breschi et al., 2021), and to stabilize polynomial-type systems (Guo et al., 2021), switched linear systems (Rotulo et al., 2021), and linear time-varying systems (Nortmann and Mylvaganam, 2020). Recently, data-driven approaches for solving LQR problems and synthesizing robust controllers are proposed in (De Persis and Tesi, 2021; Berberich et al., 2020a,b), in which underlying unknown dynamics are affected by exogenous disturbances. A data-driven technique to learn control laws for nonlinear polynomial-type systems directly from data is proposed in (Guo et al., 2020), in which input-output measurements are collected in an experiment over a finite-time period. Nevertheless, none of these approaches consider state and input constraints. Given both input and output constraints, a data-enabled predictive control algorithm is proposed in (Coulson et al., 2019a) for synthesizing controllers for linear systems tracking desired trajectories. However, there is no formal safety guarantee when disturbances are involved in the system dynamics. Recently, data-driven approaches to synthesize state-feedback controllers making a compact polyhedral set containing the origin robustly invariant are proposed in (Bisoffi et al., 2020b,a). These results are conservative in the sense that when there is no controller for the given compact polyhedral set, one might be able to find controllers making subsets of this polytope robustly invariant. In addition, these techniques require an individual constraint for each vertex of the polytope (cf. (Bisoffi et al., 2020b, Section 4) and (Bisoffi et al., 2020a, Theorems 1, 2)). Unfortunately, given any arbitrary polytope, the number of vertices grows exponentially with respect to its dimension and the number of hyperplanes in the worst case scenario (Dyer, 1983). There are also some recent results (e.g., (Noroozi et al., 2022; Lavaei et al., 2021; Salamati et al., 2021)), in which data-driven approaches are proposed for synthesizing barrier certificates by leveraging performance bounds for scenario programs (Mohajerin Esfahani et al., 2014). However, they only focus on safety verification. Additionally, the safety guarantees in these results require a large number of independent and identically distributed data sampled from the state sets (instead of a single trajectory collected from the system, which is the case in our work).

The main contribution of our work is to propose a data-driven approach to synthesize safety controllers for continuous-time nonlinear polynomial-type systems with unknown models. In our proposed framework, we leverage notions of control barrier certificates constructed from data and provide guaranteed confidence of 1 on the safety of unknown systems. Under a certain rank condition, which is closely related to the condition of *persistency of excitation* (Willems et al., 2005), we synthesize polynomial-type state-feedback controllers to ensure the safety of unknown systems only by using a *single trajectory* collected from systems. To illustrate the effectiveness of our proposed approaches, we apply them to a nonlinear polynomial system with unknown dynamics.

The rest of the paper is structured as follows. Section 2 is dedicated to describe nonlinear polynomial systems, including mathematical notations, problem description, and the formal definition of control barrier certificates. In Section 3, we propose our data-driven approach to synthesize safety controllers for unknown nonlinear polynomial systems. We verify our proposed results via a nonlinear polynomial system with an unknown model in Section 4. Finally, we conclude the paper in Section 5.

2. Continuous-Time Nonlinear Polynomial Systems

2.1. Notations

Sets of non-negative and positive integers are denoted by $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{N}_{\geq 1} := \{1, 2, 3, \dots\}$, respectively. Moreover, symbols $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$ denote, respectively, sets of real, positive, and nonnegative real numbers. We use $\mathbb{R}^n$ to denote an n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ to denote the space of real matrices with n rows and m columns. Given N vectors $x_i \in \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}_{\geq 1}$, and $i \in \{1,\dots,N\}$, we use $x = [x_1;\dots;x_N]$ to denote the corresponding column vector of the dimension $\sum_i n_i$. A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is said to be positive definite, denoted by $P \succ 0$, if its all eigenvalues are positive. We denote by $\mathbb{I}_n$ an identity matrix in $\mathbb{R}^{n\times n}$.

2.2. Continuous-Time Nonlinear Polynomial Systems

In this work, we consider continuous-time nonlinear polynomial systems (ct-NPS) as formalized in the following definition.

Definition 1 *A continuous-time nonlinear polynomial system (ct-NPS) is described by*

$$\Sigma\colon \dot{x} = A\mathcal{M}(x) + Bu, \tag{1}$$

where $A \in \mathbb{R}^{n\times N}$, $B \in \mathbb{R}^{n\times m}$, $\mathcal{M}(x) \in \mathbb{R}^N$ is a vector of monomials in state $x \in X$, and $u \in U$ is the control input, with $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ being the state and input sets, respectively.

We assume that matrices A, B are both unknown and we employ the term *unknown model* to refer to this type of systems in (1). With this definition in hand, we now state the main problem that we aim to solve in this paper.

Problem 2 *Consider a ct-NPS in (1) with unknown matrices A, B, and an initial and unsafe sets $X_0, X_1 \subset X$, respectively. Synthesize a matrix polynomial $F(x)$ such that controller $u = F(x)\mathcal{M}(x)$ makes the unknown ct-NPS (1) safe in the sense that its trajectories starting from X_0 never reach X_1.*

In order to address Problem 2, we present a formal definition of control barrier certificates for ct-NPS, which is adapted from (Prajna and Jadbabaie, 2004).

2.3. Control Barrier Certificates (CBC)

Definition 3 *Consider a ct-NPS Σ, and $X_0, X_1 \subseteq X$ as its initial and unsafe sets, respectively. A function $\mathcal{B} : X \to \mathbb{R}$ is called a control barrier certificate (CBC) for Σ if there exist $\gamma_1, \gamma_2 \in \mathbb{R}_{>0}$, with $\gamma_2 > \gamma_1$, such that*

$$\mathcal{B}(x) \leq \gamma_1, \qquad \forall x \in X_0, \tag{2}$$
$$\mathcal{B}(x) \geq \gamma_2, \qquad \forall x \in X_1, \tag{3}$$

and $\forall x \in X, \exists u \in U$, such that

$$\mathsf{L}\mathcal{B}(x) \leq 0, \tag{4}$$

where $\mathsf{L}\mathcal{B}$ is the Lie derivative of $\mathcal{B} : X \to \mathbb{R}$ with respect to dynamics as in (1), which is defined as

$$\mathsf{L}\mathcal{B}(x) = \partial_x\mathcal{B}(x)(A\mathcal{M}(x) + Bu), \tag{5}$$

with $\partial_x\mathcal{B}(x) = \left[\frac{\partial\mathcal{B}(x)}{\partial x_i}\right]_i$.

We denote by $x_{x_0u}(t)$ the state of Σ reached at time $t \in \mathbb{R}_{\geq 0}$ under an input u and from an initial condition $x_0 = x(0)$. Inspired by (Prajna and Jadbabaie, 2004), we present the next theorem showing how to use CBC to ensure that the state evolution of Σ starting from any initial state in X_0 will never reach the unsafe region X_1 for an infinite time horizon.

Theorem 4 *Consider a ct-NPS Σ. Suppose $\mathcal{B}$ is a CBC for Σ as in Definition 3. Then, one gets $x_{x_0u}(t) \notin X_1$ for any $x_0 \in X_0$ and any $t \in \mathbb{R}_{\geq 0}$, where the control input u is chosen in a way that (4) holds.*

3. Data-Driven Synthesis of Safety Controller

In this section, we propose our data-driven approach to synthesize safety controllers for unknown ct-NPS in (1). To do so, we first fix the structure of our CBC to be quadratic in the form of $\mathcal{B}(x) = \mathcal{M}(x)^\top P \mathcal{M}(x)$, with $P \succ 0$. We then collect input-output data from unknown ct-NPS over the time interval $[t_0, t_0 + (T-1)\tau]$, where $T \in \mathbb{N}_{>0}$ is the number of collected samples, and $\tau \in \mathbb{R}_{>0}$ is the sampling time:

$$\mathcal{U}_{0,T} = [u(t_0) \;\; u(t_0+\tau) \;\; \dots \;\; u(t_0+(T-1)\tau)], \tag{6}$$
$$\mathcal{X}_{0,T} = [x(t_0) \;\; x(t_0+\tau) \;\; \dots \;\; x(t_0+(T-1)\tau)], \tag{7}$$
$$\mathcal{X}_{1,T} = [\dot{x}(t_0) \;\; \dot{x}(t_0+\tau) \;\; \dots \;\; \dot{x}(t_0+(T-1)\tau)]. \tag{8}$$

Remark 5 *Note that $\mathcal{X}_{1,T}$ contains derivatives of the state at sampling times, which are in general not available as measurements. In order to tackle this issue, one can use appropriate filters for the approximation of derivatives via the available approaches proposed in the relevant literature (e.g., Larsson et al. (2008); Padoan and Astolfi (2015)). Although there exists an error involved in approximating the derivatives of the state at sampling times, we do not provide any analysis for this error for the sake of better readability of the paper.*

Inspired by (Guo et al., 2020), we present the following lemma to obtain data-based representation of closed-loop ct-NPS (1) with polynomial controllers $u = F(x)\mathcal{M}(x)$, where $F(x)$ is a matrix polynomial, which will be synthesized.

Lemma 6 *Let matrix $Q(x)$ be a $(T \times N)$ matrix polynomial such that*

$$\mathbb{I}_N = \mathcal{N}_{0,T} Q(x),$$

with

$$\mathcal{N}_{0,T} = [\mathcal{M}(x(t_0)) \;\; \mathcal{M}(x(t_0+\tau)) \;\; \dots \;\; \mathcal{M}(x(t_0+(T-1)\tau))]$$

being an $(N \times T)$ full row-rank matrix, constructed from the vector $\mathcal{M}(x)$ and samples $\mathcal{X}_{0,T}$. If one sets $u = F(x)\mathcal{M}(x) = \mathcal{U}_{0,T} Q(x)\mathcal{M}(x)$, then the closed-loop system $\dot{x} = A\mathcal{M}(x) + Bu$ has the following data-based representation:

$$\dot{x} = \mathcal{X}_{1,T} Q(x)\mathcal{M}(x), \; \textit{equivalently}, \; A + BF(x) = \mathcal{X}_{1,T} Q(x).$$

Proof: Since $F(x) = \mathcal{U}_{0,T}Q(x)$, the closed-loop ct-NPS can be written as

$$(A + BF(x))\mathcal{M}(x) = [B \quad A]\begin{bmatrix} F(x) \\ \mathbb{I}_N \end{bmatrix}\mathcal{M}(x) = [B \quad A]\begin{bmatrix} \mathcal{U}_{0,T} \\ \mathcal{N}_{0,T} \end{bmatrix}Q(x)\mathcal{M}(x) = \mathcal{X}_{1,T}Q(x)\mathcal{M}(x),$$

with $\mathcal{X}_{1,T} = [B \ A]\begin{bmatrix} \mathcal{U}_{0,T} \\ \mathcal{N}_{0,T} \end{bmatrix}$ and $\mathcal{U}_{0,T}$ as in (6). Hence, $\dot{x} = \mathcal{X}_{1,T}Q(x)\mathcal{M}(x)$, equivalently, $A + BF(x) = \mathcal{X}_{1,T}Q(x)$ is the data-based representation of the closed-loop ct-NPS, which completes the proof. ■

Remark 7 *Note that in order to enforce $\mathcal{N}_{0,T}$ to be full row rank, the number of samples T should be at least N. Since the matrix $\mathcal{N}_{0,T}$ is constructed from sampled data, this assumption is readily verifiable.*

By employing the data-based representation in Lemma 6, we propose the following theorem, as the main result of the work, to construct a CBC from data and synthesize the control gain $F(x)$ making the unknown ct-NPS in (1) safe.

Theorem 8 *Consider an unknown ct-NPS Σ as in (1), i.e., $\dot{x} = A\mathcal{M}(x) + Bu$, with its data-based representation $\dot{x} = \mathcal{X}_{1,T}Q(x)\mathcal{M}(x)$. Suppose there exists a matrix polynomial $H(x) \in \mathbb{R}^{T\times N}$ such that*

$$\mathcal{N}_{0,T}H(x) = P^{-1}, \quad \textit{with } P \succ 0. \tag{9}$$

If the following conditions are satisfied

$$\forall x \in X_0,\ \mathcal{M}(x)^\top\big[\mathcal{N}_{0,T}H(x)\big]^{-1}\mathcal{M}(x) \leq \gamma_1, \tag{10}$$

$$\forall x \in X_1, \mathcal{M}(x)^\top\big[\mathcal{N}_{0,T}H(x)\big]^{-1}\mathcal{M}(x) \geq \gamma_2, \tag{11}$$

$$\forall x \in X, J(x) := -\Big[\frac{\partial\mathcal{M}}{\partial x}\mathcal{X}_{1,T}H(x) + H(x)^\top X_{1,T}^\top(\frac{\partial\mathcal{M}}{\partial x})^\top\Big] \succeq 0, \tag{12}$$

then $\mathcal{B}(x) = \mathcal{M}(x)^\top[\mathcal{N}_{0,T}H(x)]^{-1}\mathcal{M}(x)$ is a CBC and $u = \mathcal{U}_{0,T}H(x)(\mathcal{N}_{0,T}H(x))^{-1}\mathcal{M}(x)$ is its corresponding safety controller for the unknown ct-NPS.

Proof Since $\mathcal{B}(x) = \mathcal{M}(x)^\top P\mathcal{M}(x)$ and $P^{-1} = \mathcal{N}_{0,T}H(x)$, it is straightforward that conditions (10)-(11) imply (2)-(3). We now proceed with showing condition (4), as well. Considering (4) and (5), one has

$$\begin{aligned}\mathsf{L}\mathcal{B}(x) &= \mathcal{M}(x)^\top P\frac{\partial\mathcal{M}}{\partial x}(A + BF(x))\mathcal{M}(x) + \mathcal{M}(x)^\top(A + BF(x))^\top(\frac{\partial\mathcal{M}}{\partial x})^\top P\mathcal{M}(x) \\ &= \mathcal{M}(x)^\top P\Big[\frac{\partial\mathcal{M}}{\partial x}(A + BF(x))P^{-1} + P^{-1}(A + BF(x))^\top(\frac{\partial\mathcal{M}}{\partial x})^\top\Big]P\mathcal{M}(x).\end{aligned}$$

Since $P^{-1} = \mathcal{N}_{0,T}H(x)$, then $P^{-1}P = \mathbb{I}_N = \mathcal{N}_{0,T}H(x)P$. Since $\mathbb{I}_N = \mathcal{N}_{0,T}Q(x)$, then $Q(x) = H(x)P$ and, accordingly, $Q(x)P^{-1} = H(x)$. Since $A + BF(x) = \mathcal{X}_{1,T}Q(x)$, then

$$(A + BF(x))P^{-1} = \mathcal{X}_{1,T}Q(x)P^{-1} = \mathcal{X}_{1,T}H(x).$$

Therefore,

$$\mathsf{L}\mathcal{B}(x) = \mathcal{M}(x)^\top P\Big[\frac{\partial \mathcal{M}}{\partial x}\mathcal{X}_{1,T}H(x) + H(x)^\top X_{1,T}^\top(\frac{\partial \mathcal{M}}{\partial x})^\top\Big]P\mathcal{M}(x) = -\mathcal{M}(x)^\top P\big[J(x)\big]P\mathcal{M}(x).$$

If $J(x) \succeq 0$, then $\mathsf{L}\mathcal{B}(x) \preceq 0$ and condition (4) is satisfied. Consequently,

$$\mathcal{B}(x) = \mathcal{M}(x)^\top[\mathcal{N}_{0,T}H(x)]^{-1}\mathcal{M}(x)$$

is a CBC and

$$u = \mathcal{U}_{0,T}Q(x)\mathcal{M}(x) = \mathcal{U}_{0,T}H(x)(\mathcal{N}_{0,T}H(x))^{-1}\mathcal{M}(x)$$

is its corresponding safety controller for the unknown ct-NPS, which completes the proof. ■

Remark 9 *It is worth mentioning that conditions* (10)-(12) *correspond to conditions* (2)-(4)*, which are standard for synthesizing control barrier certificates.*

In the remainder of this section, we discuss the implementation of Theorem 8. Here, we consider the state set X, initial set X_0, and unsafe set X_1 as

$$X = \bigcup_{i=1}^{m_x} X_i, \text{ with } X_i := \{x \in \mathbb{R}^n \,\big|\, g_{ik}(x) \geq 0, k = 1, \dots, \mathsf{k}\}, \tag{13}$$

$$X_0 = \bigcup_{i=1}^{m_0} X_{0i}, \text{ with } X_{0i} := \{x \in \mathbb{R}^n \,\big|\, f_{ik}(x) \geq 0, k = 1, \dots, \mathsf{k}_0\}, \tag{14}$$

$$X_1 = \bigcup_{i=1}^{m_1} X_{1i}, \text{ with } X_{1i} := \{x \in \mathbb{R}^n \,\big|\, h_{ik}(x) \geq 0, k = 1, \dots, \mathsf{k}_1\}, \tag{15}$$

with $g_{ik}(x)$, $f_{ik}(x)$, and $h_{ik}(x)$ being polynomial. The input set U is defined as

$$U := \{u \in \mathbb{R}^m \,\big|\, b_j^\top u \leq 1, \text{ with } j = 1, \dots, \mathcal{J}\}, \tag{16}$$

with $b_j \in \mathbb{R}^m$ being some constant vectors. Additionally, we raise the following corollary which is required for our implementation results.

Corollary 10 *Consider a CBC $\mathcal{B}(x) = \mathcal{M}(x)^\top P\mathcal{M}(x)$ with $P \succ 0$ as in Definition 3 for a ct-NPS Σ in* (1)*, and $\tilde\gamma \in \mathbb{R}_{>0}$. If $\mathcal{M}(x(0))^\top P\mathcal{M}(x(0)) \leq \tilde\gamma$, then $\mathcal{M}(x(t))^\top P\mathcal{M}(x(t)) \leq \tilde\gamma$ for all $t \in \mathbb{R}_{>0}$.*

Corollary 10 can readily be verified with the help of non-positiveness of $\mathsf{L}\mathcal{B}(x)$ (4). By employing Corollary 10, we are ready to show the next result for computing a CBC and its associated safety controller.

Corollary 11 *Consider a ct-NPS Σ as in* (1)*, sets X, X_0, and X_1 as in* (13)-(15)*, respectively, an input set U as in* (16)*, and data $\mathcal{U}_{0,T}$, $\mathcal{X}_{1,T}$, and $\mathcal{N}_{0,T}$ as in* (6)*,* (8)*, and Lemma 6, respectively.*

If there exist a positive definite matrix $P \in \mathbb{R}^{N\times N}$, a matrix polynomial $H(x) \in \mathbb{R}^{T\times N}$, and $\gamma_1, \gamma_2 \in \mathbb{R}_{>0}$, with $\gamma_2 > \gamma_1$ such that

$$-\mathcal{M}(x)^\top P\mathcal{M}(x) - \sum_{k=1}^{\mathsf{k}_0}\lambda'_{ik}(x)f_{ik}(x) + \gamma_1, \forall i \in [1, m_0], \forall k \in [1, \mathsf{k}_0], \tag{17}$$

$$\mathcal{M}(x)^\top P\mathcal{M}(x) - \sum_{k=1}^{\mathsf{k}_1}\lambda''_{ik}(x)h_{ik}(x) - \gamma_2, \forall i \in [1, m_1], \forall k \in [1, \mathsf{k}_1], \tag{18}$$

$$-[\frac{\partial Z}{\partial x}\mathcal{X}_{1,T}H(x) + H(x)^\top\mathcal{X}_{1,T}^\top(\frac{\partial Z}{\partial x})^\top] - \sum_{k=1}^{\mathsf{k}}\lambda_{ik}(x)g_{ik}(x)\mathbb{I}_N, \forall i \in [1, m_x], \forall k \in [1, \mathsf{k}], \tag{19}$$

$$1 - b_j^\top\mathcal{U}_{0,T}H(x)PM(x) - \lambda_u(x)\Big(\gamma_1 - \mathcal{M}(x)^\top P\mathcal{M}(x)\Big), \forall j = 1, \ldots, \mathcal{J}, \tag{20}$$

are sum-of-square (SOS), with $\lambda_{ik}(x)$, $\lambda'_{ik}(x)$, $\lambda''_{ik}(x)$, and $\lambda_u(x)$ being SOS polynomials, and $\mathbb{I}_N = P\mathcal{N}_{0,T}H(x)$, then $\mathcal{B}(x) = \mathcal{M}(x)^\top P\mathcal{M}(x)$ is a CBC for Σ with the corresponding safety controller $u = \mathcal{U}_{0,T}H(x)P\mathcal{M}(x)$.

Proof It is straightforward that if (17) holds, then one has $\mathcal{M}(x)^\top P\mathcal{M}(x) + \sum_{k=1}^{\mathsf{k}_0}\lambda'_{ik}(x)f_{ik}(x) \leq \gamma_1$, $\forall i \in [1, m_0], \forall k \in [1, \mathsf{k}_0]$. Since $\lambda'_{ik}(x)$ are SOS polynomials, then $\sum_{k=1}^{\mathsf{k}}\lambda_{ik}(x)g_{ik}(x)$ are non-negative given the definition of X_0 in (14). Hence, $\mathcal{M}(x)^\top P\mathcal{M}(x) \leq \gamma_1$ holds $\forall x \in X_0$, indicating that (10) holds with $P = \big[\mathcal{N}_{0,T}H(x)\big]^{-1}$. Similarly, (18) implies that $\mathcal{M}(x)^\top P\mathcal{M}(x) - \sum_{k=1}^{\mathsf{k}_1}\lambda''_{ik}(x)h_{ik}(x) \geq \gamma_2$, $\forall i \in [1, m_1], \forall k \in [1, \mathsf{k}_1]$. Since $\lambda''_{ik}(x)$ are SOS polynomials, one has $\sum_{k=1}^{\mathsf{k}_1}\lambda''_{ik}(x)h_{ik}(x) \geq 0$, and accordingly $\mathcal{M}(x)^\top P\mathcal{M}(x) \geq \gamma_2$ for all $x \in X_1$, indicating that (11) holds with $P = \big[\mathcal{N}_{0,T}H(x)\big]^{-1}$. Next, we show that (19) implies that

$$J_i(x) := -[\frac{\partial Z}{\partial x}\mathcal{X}_{1,T}H(x)+H(x)^\top\mathcal{X}_{1,T}^\top(\frac{\partial Z}{\partial x})^\top] \succeq 0 \tag{21}$$

hold for all $x \in X_i, i \in [1, m_x]$. First, (19) is SOS implying that $J_i(x) - \sum_{k=1}^{\mathsf{k}}\lambda_{ik}(x)g_{ik}(x)\mathbb{I}_N \succeq 0$. Since $\lambda_{ik}(x)$ are SOS polynomials for all $k \in [1, \mathsf{k}]$, $\sum_{k=1}^{\mathsf{k}}\lambda_{ik}(x)g_{ik}(x)$ are non-negative over X_i. Then, one can readily verify that $J_i(x) \succeq 0$, $\forall x \in X_i$, and (12) holds accordingly. Finally, we show that (20) ensures that $u = \mathcal{U}_{0,T}H(x)P\mathcal{M}(x) \in U$ for all $x \in \mathcal{B}_1(x)$ with $\mathcal{B}_1(x) := \{x \in \mathbb{R}^n | \mathcal{M}(x)^\top P\mathcal{M}(x) \leq \gamma_1\}$, and $\mathbb{I}_N = P\mathcal{N}_{0,T}H(x)$. Note that we only need to consider the set $\mathcal{B}_1(x)$ instead of the whole state set X since Corollary 10 shows that state trajectories of the system stay inside the set $\mathcal{B}_1(x)$. Considering the definition of U as in (16), $u \in U$ requires that

$$b_j^\top\mathcal{U}_{0,T}H(x)P\mathcal{M}(x) \leq 1, \tag{22}$$

holds $\forall j = 1, \ldots, \mathcal{J}$, and $\forall x \in \mathcal{B}_1(x)$. Note that (20) implies that $b_j^\top\mathcal{U}_{0,T}H(x)PM(x) + \lambda_u(x)(\gamma_1 - \mathcal{M}(x)^\top P\mathcal{M}(x)) \leq 1$. Hence, (22) holds since λ_u is an SOS polynomial. ■

Remark 12 *Observe that one can employ existing software tools in the relevant literature such as* SOSTOOLS *(Papachristodoulou et al., 2013) together with a semidefinite programming (SDP) solver such as* SeDuMi *(Sturm, 1999) to readily enforce conditions (17)-(20) over the sets X_0, X_1, and X, while searching for the matrix polynomial $H(x)$ and matrix P.*

Remark 13 *Remark that condition (20) is a bilinear matrix inequality (BMI) due to having a bilinearity between decision matrices H and P. In order to resolve this problem, one can first obtain a candidate for P based on (17) and (18), and then try to search for appropriate $H(x)$ such that (19) and (20) hold. As an alternative approach, one can also use the technique proposed in (Hassibi et al., 1999) to linearize the BMI using a first-order perturbation approximation and then solve the linearized version. In this case, similar to all local methods for solving BMIs, the choice of the initial value is important for the convergence.*

4. Case Study

Here, we focus on the following nonlinear polynomial system borrowed from (Guo et al., 2020):

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_1^2 + u, \end{aligned} \tag{23}$$

which is of the form of (1), with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathcal{M}(x) = \begin{bmatrix} x_2 \\ x_1^2 \end{bmatrix},$$

and $n = N = 2$. Here, we consider the state set $X = [-20, 20] \times [-20, 20]$, the initial set $X_0 = [-2.5, 2.5] \times [-2.5, 2.5]$, the unsafe set $X_1 = [-20, 20] \times [10, 20] \cup [-20, 0] \times [-20, -10] \cup [3.5, 7] \times [-4, 0]$, and the input set $U = [-30, 30]$. We assume that both matrices A and B are unknown and treat this system as a black-box one. To collect data, we initialize the system at

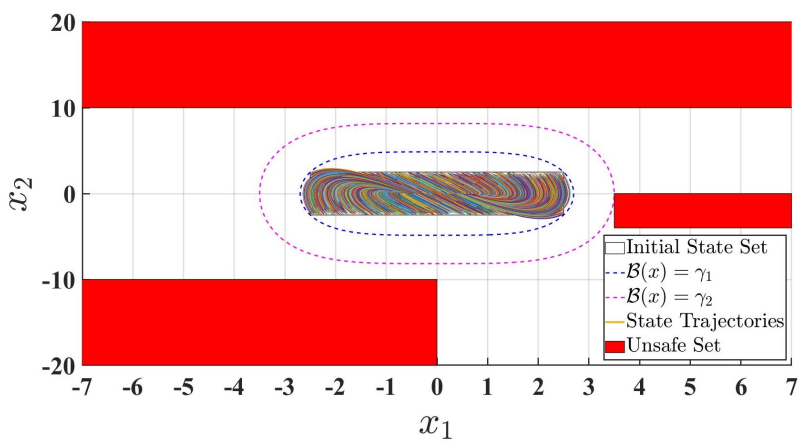

Figure 1: Several state trajectories, initial set X_0, unsafe set X_1, and level sets $\mathcal{M}(x)^\top P \mathcal{M}(x) = \gamma_1$ and $\mathcal{M}(x)^\top P \mathcal{M}(x) = \gamma_2$.

$x(0) = [2; 3]$ and simulate the system with inputs that are randomly selected from the input set following a uniform distribution. The data are collected with a sampling time $\tau = 0.02s$ and they are as follows

$$\mathcal{U}_{0,5} = \begin{bmatrix} 0.8134 & 3.6710 & -0.4437 & -1.9421 & -0.7241 \end{bmatrix},$$

$$\mathcal{X}_{0,5} = \begin{bmatrix} 2 & 2.0610 & 2.1246 & 2.1906 & 2.2581 \\ 3 & 3.0987 & 3.2597 & 3.3439 & 3.4040 \end{bmatrix},$$

$$\mathcal{X}_{1,5} = \begin{bmatrix} 3 & 3.0987 & 3.2597 & 3.3439 & 3.4040 \\ 4.8134 & 7.9186 & 4.0701 & 2.8565 & 4.3747 \end{bmatrix}.$$

Accordingly, one has

$$\mathcal{N}_{0,5} = \begin{bmatrix} 3 & 3.0987 & 3.2597 & 3.3439 & 3.4040 \\ 4 & 4.2476 & 4.5137 & 4.7986 & 5.0988 \end{bmatrix},$$

with $\mathcal{N}_{0,5}$ being defined as in Lemma 6 with $T = 5$. With the help of Theorem 8 and Corollary 11, we obtain

$$H(x) = \begin{bmatrix} 0.4266 & 0.07214x_1 - 0.3641 \\ -0.2245 & -0.1001x_1 - 0.0333 \\ 0.2831 & 0.0047x_1 - 0.3398 \\ 0.1311 & 0.0097x_1 - 0.3049 \\ -0.5217 & 0.01334x_1 + 0.9762 \end{bmatrix}, \quad P = \begin{bmatrix} 5.8938 & 0 \\ 0 & 2.6160 \end{bmatrix},$$

with $\gamma_1 = 139.03$, and $\gamma_2 = 392.56$. The associated safety controller is

$$u = -0.8877x_1^3 - x_1^2 - 2.8264x_2. \tag{24}$$

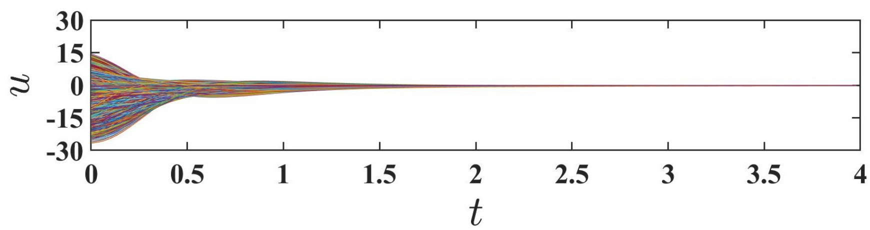

Figure 2: Several input trajectories of the system.

For the simulation results, we randomly select 10^5 initial states from the initial state set and simulate the system for 4 seconds, while the controller in (24) is applied in the closed-loop. We depicted some state and input trajectories in Figures 1 and 2, respectively. Moreover, we also depicted in Figure 1 the initial set X_0, the unsafe set X_1, and the corresponding level sets specified by γ_1 and γ_2 as in Corollary 11. One can readily see that the system in (23) is safe and the input constraint is also satisfied.

5. Conclusion

In this work, we proposed a data-driven approach for safety controller synthesis of continuous-time nonlinear polynomial systems with unknown models. Our proposed framework utilized notions of control barrier certificates constructed from data. Under a certain rank condition, we synthesized polynomial state-feedback controllers to ensure the safety of unknown systems only via a *single trajectory* collected from systems. We demonstrated the effectiveness of our proposed approaches by applying them to a nonlinear polynomial system with unknown dynamics.

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