Hitting Times for Continuous-Time Imprecise-Markov Chains (Supplementary Material)

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A PROOFS AND LEMMAS FOR SECTION 4

For certain operators, we note that subspace restriction distributes over operator composition:

Lemma 2. *Let M and N be operators on $\mathbb{R}^{\mathcal{X}}$ such that $N|_A = I$. Then $\bigl(MN\bigr)|_{A^c} = M|_{A^c}N|_{A^c}$.*

Proof. Fix any $f \in \mathbb{R}^{A^c}$. Then

$$M|_{A^c}N|_{A^c}f = M\Bigl(\bigl((Nf\!\uparrow_{\mathcal{X}})|_{A^c}\bigr)\!\uparrow_{\mathcal{X}}\Bigr)|_{A^c}\,.$$

Note that since $N|_A = I$ and $f\!\uparrow_{\mathcal{X}}(x) = 0$ for all $x \in A$, we also have $Nf\!\uparrow_{\mathcal{X}}(x) = 0$ for all $x \in A$. Hence in particular, it holds that $\bigl((Nf\!\uparrow_{\mathcal{X}})|_{A^c}\bigr)\!\uparrow_{\mathcal{X}} = Nf\!\uparrow_{\mathcal{X}}$. We therefore find that

$$M|_{A^c}N|_{A^c}f = \bigl(MNf\!\uparrow_{\mathcal{X}}\bigr)|_{A^c} = (MN)|_{A^c}f\,,$$

which concludes the proof. □

This can be used in particular for certain operators associated with $Q \in \mathcal{Q}$ and the associated lower- and upper rate operators:

Lemma 3. *Fix any $\Delta \geq 0$ and any $Q \in \mathcal{Q}$. Then*

$$(I + \Delta Q)|_A = (I + \Delta \underline{Q})|_A = (I + \Delta\overline{Q})|_A = I\,.$$

Proof. Fix any $Q \in \mathcal{Q}$, and first choose any $f \in \mathbb{R}^{\mathcal{X}}$ and $x \in A$. By Assumption 1 and the definition of rate matrices, we have $Q(x,y) = 0$ for all $y \in \mathcal{X}$, whence $Qf(x) = \sum_{y\in\mathcal{X}} Q(x,y)f(y) = 0$. Since $Q \in \mathcal{Q}$ is arbitrary, we also have $\underline{Q}f(x) = 0$ and $\overline{Q}f(x) = 0$. It follows that

$$f(x) = (I{+}\Delta Q)f(x) = (I{+}\Delta\underline{Q})f(x) = (I{+}\Delta\overline{Q})f(x)\,.$$

Since this is true for all $f \in \mathbb{R}^{\mathcal{X}}$ and all $x \in A$, the result is now immediate. □

Corollary 3. *For all $Q \in \mathcal{Q}$ and $t \in \mathbb{R}_{\geq 0}$ it holds that*

$$e^{Qt}|_A = e^{\underline{Q}t}|_A = e^{\overline{Q}t}|_A = I\,.$$

Proof. Use Lemma 3 and the definitions of e^{Qt}, $e^{\underline{Q}t}$, $e^{\overline{Q}t}$. □

Lemma 4. *Let M and N be operators on $\mathbb{R}^{\mathcal{X}}$ such that $M|_A = I = N|_A$. Then $\|M|_{A^c} - N|_{A^c}\| \leq \|M - N\|$.*

Proof. Fix any $f \in \mathbb{R}^{A^c}$ with $\|f\| = 1$. Then $\|f\!\uparrow_{\mathcal{X}}\| = 1$. Moreover, since $f\!\uparrow_{\mathcal{X}}(x) = 0$ for all $x \in A$ and since $M|_A = I = N|_A$, we have that $(Mf\!\uparrow_{\mathcal{X}})(x) = 0 = (Nf\!\uparrow_{\mathcal{X}})(x)$ for all $x \in A$. Hence we find

$$\begin{aligned}
&\|(M|_{A^c} - N|_{A^c})f\| \\
&\quad= \bigl\|\bigl((M-N)f\!\uparrow_{\mathcal{X}}\bigr)|_{A^c}\bigr\| \\
&\quad= \|(M-N)f\!\uparrow_{\mathcal{X}}\| \\
&\quad\leq \sup\bigl\{\|(M-N)g\| \,:\, g \in \mathbb{R}^{\mathcal{X}}, \|g\| = 1\bigr\} \\
&\quad= \|M-N\|\,.
\end{aligned}$$

The result follows since $f \in \mathbb{R}^{A^c}$ is arbitrary. □

Proof of Proposition 5. Fix $Q \in \mathcal{Q}$ and let G be its subgenerator. First fix any $t \in \mathbb{R}_{\geq 0}$ and any $\epsilon > 0$. Then by definition of e^{Qt}, for all $n \in \mathbb{N}$ large enough it holds that

$$\bigl\|e^{Qt} - \bigl(I + {}^t\!/\!_nQ\bigr)^n\bigr\| < \epsilon\,.$$

Moreover, by Lemmas 2 and 3 we have

$$\Bigl(\bigl(I + {}^t\!/\!_nQ\bigr)^n\Bigr)|_{A^c} = \bigl(I + {}^t\!/\!_nG\bigr)^n\,,$$

and, by Corollary 3, that $e^{Qt}|_A = I$. Hence by Lemma 4 we find

$$\bigl\|e^{Gt} - \bigl(I + {}^t\!/\!_nG\bigr)^n\bigr\| \leq \bigl\|e^{Qt} - \bigl(I + {}^t\!/\!_nQ\bigr)^n\bigr\| < \epsilon\,.$$

Since $\epsilon > 0$ is arbitrary, we have

$$e^{Gt} = \lim_{n\to+\infty}\bigl(I + {}^t\!/\!_nG\bigr)^n\,.$$

Accepted for the 38th Conference on Uncertainty in Artificial Intelligence (UAI 2022).

This concludes the proof of the first claim.

To see that $(e^{Gt})_{t\in\mathbb{R}_{\geq 0}}$ is a semigroup, note that $(e^{Qt})_{t\in\mathbb{R}_{\geq 0}}$ is a semigroup, then apply Lemma 2 and Corollary 3. □

Proof of Proposition 6. The proof is completely analogous to the proof of Proposition 5; simply replace Q with either $\underline{Q}$ or $\overline{Q}$ as appropriate. □

Proof of Proposition 7. Let $\epsilon := \min_{x\in A^c} e^{\underline{Q}t}\mathbb{I}_A(x)$; then $\epsilon > 0$ due to Assumption 2. Fix any $f \in \mathbb{R}^{A^c}$ with $\|f\| = 1$. By definition, we have $e^{\overline{G}t}f = e^{\overline{Q}t}|_{A^c}f = \left(e^{\overline{Q}t}f{\uparrow}_{\mathcal{X}}\right)|_{A^c}$.

Let $\mathcal{T}_t$ denote the set of transition matrices that dominates $e^{\underline{Q}t}$. Due to Proposition 3, there is some $T \in \mathcal{T}_t$ such that $Tf{\uparrow}_{\mathcal{X}} = e^{\overline{Q}t}f{\uparrow}_{\mathcal{X}}$. Fix any $x \in A^c$. Then, using that $f{\uparrow}_{\mathcal{X}}(y) = 0$ for all $y \in A$, together with the fact that T is a transition matrix, we have

$$|Tf{\uparrow}_{\mathcal{X}}(x)| = \left|\sum_{y\in\mathcal{X}} T(x,y)f{\uparrow}_{\mathcal{X}}(y)\right| \leq \sum_{y\in A^c} T(x,y)\,,$$

and hence $|Tf{\uparrow}_{\mathcal{X}}(x)| \leq T\mathbb{I}_{A^c}(x)$. We have $\mathbb{I}_A + \mathbb{I}_{A^c} = \mathbf{1}$ and $T\mathbf{1}(x) = 1$ since T is a transition matrix. Using the linear character of T, we find that

$$T\mathbb{I}_{A^c}(x) = T(\mathbf{1} - \mathbb{I}_A)(x) = 1 - T\mathbb{I}_A(x)\,.$$

Since $T \in \mathcal{T}_t$ and $x \in A^c$ we have

$$0 < \epsilon = \min_{y\in A^c} e^{\underline{Q}t}\mathbb{I}_A(y) \leq e^{\underline{Q}t}\mathbb{I}_A(x) \leq T\mathbb{I}_A(x)\,.$$

Combining the above we find that

$$|Tf{\uparrow}_{\mathcal{X}}(x)| \leq T\mathbb{I}_{A^c}(x) = 1 - T\mathbb{I}_A(x) \leq 1 - \epsilon\,.$$

Since this is true for all $x \in A^c$, we find that $\|(Tf{\uparrow}_{\mathcal{X}})|_{A^c}\| \leq 1 - \epsilon$. Moreover, since $Tf{\uparrow}_{\mathcal{X}} = e^{\overline{Q}t}f{\uparrow}_{\mathcal{X}}$, it follows that $\left\|(e^{\overline{Q}t}f{\uparrow}_{\mathcal{X}})|_{A^c}\right\| \leq 1 - \epsilon$, or in other words, that

$$\left\|e^{\overline{G}t}f\right\| \leq 1 - \epsilon\,, \qquad \text{with } \epsilon > 0.$$

The result follows since $f \in \mathbb{R}^{A^c}$ with $\|f\| = 1$ is arbitrary. □

Proof of Lemma 1. Let $\rho(e^G) := \max_{\lambda\in\sigma(e^G)} |\lambda|$ denote the spectral radius of e^G. We know from Section 4 that $\left\|e^G\right\| < 1$, and hence we have $\rho(e^G) \leq \left\|e^G\right\| < 1$ [Taylor and Lay, 1958, Thm V.3.5]. This implies that $|\lambda| < 1$ for all $\lambda \in \sigma(e^G)$.

By the spectral mapping theorem [Engel and Nagel, 2000, Lemma I.3.13] we then have $e^{\operatorname{Re}\lambda} < 1$ for all $\lambda \in \sigma(G)$, or in other words, that $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(G)$. □

B PROOFS AND LEMMAS FOR SECTION 4.1

Proof of Proposition 11. This proof is a straightforward generalization of an argument in [Engel and Nagel, 2000, Prop I.3.12].

Let first $q := \left\|e^{\overline{G}}\right\|$; then $0 < q < 1$ due to Proposition 7. Define

$$m := \sup_{s\in[0,1]} \left\|e^{\overline{G}s}\right\|\,.$$

Then $m \geq 1$ since $m \geq \left\|e^{\overline{G}0}\right\| = \|I\| = 1$. Moreover, $m \leq 1$ due to Proposition 7, and hence $m = 1$. Now set $M := {}^1\!/\!_q$ and $\xi := -\log q$; then $\xi > 0$ since $q < 1$.

Fix any $t \in \mathbb{R}_{\geq 0}$. If $t = 0$ then the result is trivial, so let us suppose that $t > 0$. Then there are $k \in \mathbb{N}_0$ and $s \in [0,1)$ such that $t = k + s$. Using the semigroup property, we have

$$\left\|e^{\overline{G}t}\right\| = \left\|e^{\overline{G}(s+k)}\right\| \leq \left\|e^{\overline{G}s}\right\| \left\|e^{\overline{G}}\right\|^k \leq mq^k = e^{k\log q}\,.$$

We have $k = t - s$ and $s \in [0,1)$, and so

$$\begin{aligned}\left\|e^{\overline{G}t}\right\| &\leq e^{k\log q}\\ &= e^{(t-s)\log q}\\ &= e^{t\log q}e^{-s\log q}\\ &= e^{-\xi t}e^{-s\log q}\\ &\leq e^{-\xi t}e^{-\log q} = \frac{1}{q}e^{-\xi t} = Me^{-\xi t}\,,\end{aligned}$$

which concludes the proof. □

Proof of Proposition 12. It follows from the definition that for any upper transition operator $\overline{T}$ and any non-negative $f \in \mathbb{R}^{\mathcal{X}}$, also $\overline{T}f$ is non-negative. In the sequel, we will therefore say that upper transition operators *preserve non-negativity*. Since $e^{\overline{Q}t}$ is an upper transition operator, this property clearly extends also to $e^{\overline{G}t}$.

Now fix $f, g \in \mathbb{R}^{A^c}$ and $t \in \mathbb{R}_{\geq 0}$. By preservation of non-negativity we have for any $x \in A^c$ that

$$\left|e^{\overline{G}t}\,|f+g|\right|(x) = e^{\overline{G}t}\,|f+g|\,(x)\,.$$

Moreover, we clearly have $|f+g| \leq |f| + |g|$, and so by the monotonicity of upper transition operators, we have

$$e^{\overline{G}t}\,|f+g|\,(x) \leq e^{\overline{G}t}(|f| + |g|)(x)\,.$$

Finally, by the subadditivity of upper transition operators, we find that

$$e^{\overline{G}t}(|f| + |g|)(x) \leq e^{\overline{G}t}\,|f|\,(x) + e^{\overline{G}t}\,|g|\,(x)\,.$$

Again by preservation of non-negativity we have

$$e^{\overline{G}t}\,|f|\,(x) + e^{\overline{G}t}\,|g|\,(x) = \left| e^{\overline{G}t}\,|f|\,(x) + e^{\overline{G}t}\,|g| \right|(x)\,.$$

Because this is true for all $x \in A^c$, we find that

$$\begin{aligned}\left\| e^{\overline{G}t}\,|f+g| \right\| &\leq \left\| e^{\overline{G}t}\,|f| + e^{\overline{G}t}\,|g| \right\| \\ &\leq \left\| e^{\overline{G}t}\,|f| \right\| + \left\| e^{\overline{G}t}\,|g| \right\|\,.\end{aligned}$$

Multiplying both sides with $e^{\xi t}$ and noting that $t \in \mathbb{R}_{\geq 0}$ is arbitrary, we find that

$$\begin{aligned}\|f+g\|_* &= \sup_{t\in\mathbb{R}_{\geq 0}} \left\| e^{\xi t} e^{\overline{G}t}\,|f+g| \right\| \\ &\leq \sup_{t\in\mathbb{R}_{\geq 0}} \left\| e^{\xi t} e^{\overline{G}t}\,|f| \right\| + \left\| e^{\xi t} e^{\overline{G}t}\,|g| \right\| \\ &\leq \sup_{t\in\mathbb{R}_{\geq 0}} \left\| e^{\xi t} e^{\overline{G}t}\,|f| \right\| + \sup_{t\in\mathbb{R}_{\geq 0}} \left\| e^{\xi t} e^{\overline{G}t}\,|g| \right\| \\ &= \|f\|_* + \|g\|_*\,.\end{aligned}$$

Hence we have established that $\|\cdot\|_*$ satisfies the triangle inequality.

Next, fix any $f \in \mathbb{R}^{A^c}$ and $c \in \mathbb{R}$. Then

$$\begin{aligned}\|cf\|_* &= \sup_{t\in\mathbb{R}_{\geq 0}} \left\| e^{\xi t} e^{\overline{G}t}\,|cf| \right\| \\ &= \sup_{t\in\mathbb{R}_{\geq 0}} \left\| e^{\xi t} e^{\overline{G}t}\,|c|\,|f| \right\| \\ &= |c| \sup_{t\in\mathbb{R}_{\geq 0}} \left\| e^{\xi t} e^{\overline{G}t}\,|f| \right\| = |c|\,\|f\|_*\,.\end{aligned}$$

So $\|\cdot\|_*$ is absolutely homogeneous.

Finally, fix $f \in \mathbb{R}^{A^c}$ and suppose that $\|f\|_* = 0$. It holds that

$$0 = \|f\|_* \geq \left\| e^{\xi 0} e^{\overline{G}0}\,|f| \right\| \geq 0\,,$$

whence it holds that $\left\| e^{\xi 0} e^{\overline{G}0}\,|f| \right\| = 0$. This implies that also $\left\| e^{\overline{G}0}\,|f| \right\| = 0$. Since $e^{\overline{G}0} = I$, we have

$$0 = \left\| e^{\overline{G}0}\,|f| \right\| = \||f|\| = \|f\|\,,$$

whence $f = 0$. Hence $\|\cdot\|_*$ separates $\mathbb{R}^{A^c}$. ☐

Lemma 5. *For any $Q \in \mathcal{Q}$ with subgenerator G, any $f \in \mathbb{R}^{A^c}$, and any $t \geq 0$, it holds that $\left\| e^{Gt} f \right\|_* \leq \left\| e^{\overline{G}t} f \right\|_*$.*

Proof. Choose $f \in \mathbb{R}^{A^c}$. Let T be any matrix with non-negative entries. Then $|Tf(x)| \leq |T\,|f|\,(x)|$ for all $x \in A^c$. In particular, we have

$$\begin{aligned}|Tf(x)| &= \left| \sum_{y\in A^c} T(x,y) f(y) \right| \\ &\leq \sum_{y\in A^c} |T(x,y)|\,|f(y)| \\ &= T\,|f|\,(x) = |T\,|f|\,(x)|\,,\end{aligned}$$

where the final two equalities follow from the fact that T only has non-negative entries. Since this is true for any matrix T with non-negative entries, we have in particular that $\left| e^{Gt} f \right|(x) \leq e^{Gt}\,|f|\,(x)$. Similarly, it holds that

$$\begin{aligned}\left| e^{\overline{G}t} f \right|(x) &= \left| \sup_{T\in\mathcal{T}_t} Tf(x) \right| \\ &\leq \sup_{T\in\mathcal{T}_t} |Tf(x)| \\ &\leq \sup_{T\in\mathcal{T}_t} T\,|f|\,(x) = e^{\overline{G}t}\,|f|\,(x)\,.\end{aligned}$$

It follows that, for any $s \in \mathbb{R}_{\geq 0}$, we have

$$e^{\overline{G}s} \left| e^{Gt} f \right|(x) \leq e^{\overline{G}s} e^{Gt}\,|f|\,(x)\,.$$

Due to preservation of non-negativity, and since this is true for any $x \in A^c$, we have

$$\left\| e^{\overline{G}s} \left| e^{Gt} f \right| \right\| \leq \left\| e^{\overline{G}s} e^{Gt}\,|f| \right\|\,.$$

Now let $f \in \mathbb{R}^{A^c}$ be such that $\|f\|_* = 1$ and $\left\| e^{Gt} \right\|_* = \left\| e^{Gt} f \right\|_*$; this f clearly exists since $\mathbb{R}^{A^c}$ is finite-dimensional. Then we have

$$\begin{aligned}\left\| e^{Gt} \right\|_* &= \left\| e^{Gt} f \right\|_* \\ &= \sup_{s\in\mathbb{R}_{\geq 0}} \left\| e^{\xi s} e^{\overline{G}s} \left| e^{Gt} f \right| \right\| \\ &\leq \sup_{s\in\mathbb{R}_{\geq 0}} \left\| e^{\xi s} e^{\overline{G}s} e^{Gt}\,|f| \right\| = \left\| e^{Gt}\,|f| \right\|_* \leq \left\| e^{Gt} \right\|_*\,,\end{aligned}$$

where the final inequality used that $\||f|\|_* = \|f\|_* = 1$. Hence we have found that $\left\| e^{Gt} \right\|_* = \left\| e^{Gt}\,|f| \right\|_*$.

Since $e^{Qt} \in \mathcal{T}_t$ by Equation (7), we also have

$$e^{Gt}\,|f| \leq e^{\overline{G}t}\,|f|\,.$$

By monotonicity of upper transition operators, this implies that

$$e^{\overline{G}s} e^{Gt}\,|f| \leq e^{\overline{G}s} e^{\overline{G}t}\,|f|$$

and, due to the preservation of non-negativity, we have

$$e^{\overline{G}s} e^{Gt}\,|f| = \left| e^{\overline{G}s} e^{Gt}\,|f| \right|\,,$$

and

$$e^{\overline{G}s} e^{\overline{G}t}\,|f| = \left| e^{\overline{G}s} e^{\overline{G}t}\,|f| \right|\,.$$

Hence for all $x \in A^c$ we have

$$\left| e^{\overline{G}s} e^{Gt} |f| \right| (x) \leq \left| e^{\overline{G}s} e^{\overline{G}t} |f| \right| (x) ,$$

or in other words, that

$$\left\| e^{\overline{G}s} e^{Gt} |f| \right\| \leq \left\| e^{\overline{G}s} e^{\overline{G}t} |f| \right\| .$$

Since this holds for all $s \in \mathbb{R}_{\geq 0}$, we have

$$\begin{aligned} \left\| e^{Gt} \right\|_* = \left\| e^{Gt} |f| \right\|_* &= \sup_{s \in \mathbb{R}_{\geq 0}} \left\| e^{\xi s} e^{\overline{G}s} e^{Gt} |f| \right\| \\ &\leq \sup_{s \in \mathbb{R}_{\geq 0}} \left\| e^{\xi s} e^{\overline{G}s} e^{\overline{G}t} |f| \right\| \\ &= \left\| e^{\overline{G}t} |f| \right\|_* \leq \left\| e^{\overline{G}t} \right\|_* , \end{aligned}$$

which concludes the proof. ☐

Proof of Proposition 13. The argument is analogous to the well-known case for linear quasicontractive semigroups; for a similar result, see e.g. [Renardy and Rogers, 2006, Thm 12.21]. So, fix any $t \in \mathbb{R}_{\geq 0}$ and $f \in \mathbb{R}^{A^c}$.

Using a similar argument as used in the proof of Lemma 5, we use the preservation of non-negativity and the monotonicity of upper transition operators, to find for any $s \in \mathbb{R}_{\geq 0}$ that

$$\left\| e^{\xi s} e^{\overline{G}s} \left| e^{\overline{G}t} f \right| \right\| \leq \left\| e^{\xi s} e^{\overline{G}s} e^{\overline{G}t} |f| \right\| .$$

Hence we have

$$\begin{aligned} \left\| e^{\overline{G}t} f \right\|_* &= \sup_{s \in \mathbb{R}_{\geq 0}} \left\| e^{\xi s} e^{\overline{G}s} \left| e^{\overline{G}t} f \right| \right\| \\ &\leq \sup_{s \in \mathbb{R}_{\geq 0}} \left\| e^{\xi s} e^{\overline{G}s} e^{\overline{G}t} |f| \right\| \\ &= e^{-\xi t} e^{\xi t} \sup_{s \in \mathbb{R}_{\geq 0}} \left\| e^{\xi s} e^{\overline{G}(s+t)} |f| \right\| \\ &= e^{-\xi t} \sup_{s \in \mathbb{R}_{\geq 0}} \left\| e^{\xi (s+t)} e^{\overline{G}(s+t)} |f| \right\| \\ &= e^{-\xi t} \sup_{s \in \mathbb{R}_{\geq t}} \left\| e^{\xi (s)} e^{\overline{G}s} |f| \right\| \leq e^{-\xi t} \left\| f \right\|_* , \end{aligned}$$

where for the second equality we used the semigroup property. Since $f \in \mathbb{R}^{A^c}$ is arbitrary, this implies that

$$\left\| e^{\overline{G}t} \right\|_* = \sup \left\{ \left\| e^{\overline{G}t} f \right\|_* \, : \, f \in \mathbb{R}^{A^c}, \|f\|_* = 1 \right\} \leq e^{-\xi t} ,$$

which completes the proof. ☐

Proof of Proposition 14. This is immediate from Lemma 5 and Proposition 13. ☐

C PROOFS AND LEMMAS FOR SECTION 5

The following result is well-known, but we state it here for convenience:

Lemma 6. *Let T be a linear bounded operator on a Banach space with norm $\|\cdot\|_*$. Suppose that $\|T\|_* < 1$ and that $(I - T)^{-1}$ exists. Then*

$$\left\| (I - T)^{-1} \right\|_* \leq \frac{1}{1 - \|T\|_*} .$$

Proof. Since $\|T\|_* < 1$ we have $(I - T)^{-1} = \sum_{k=0}^{+\infty} T^k$. Taking norms,

$$\left\| (I - T)^{-1} \right\|_* = \left\| \sum_{k=0}^{+\infty} T^k \right\|_* \leq \sum_{k=0}^{+\infty} \|T\|_*^k = \frac{1}{1 - \|T\|_*} ,$$

where the final step used the value of the geometric series and that $\|T\|_* < 1$. ☐

Lemma 7. *There is some $C > 0$ such that for any $\Delta > 0$ with $\Delta\xi < 1$, and any $Q \in \mathcal{Q}$ with subgenerator G, it holds that $\left\| (I - e^{G\Delta})^{-1} \right\| < C/\Delta$.*

Proof. Let $\xi > 0$ be as in Proposition 11, and let $\|\cdot\|_*$ be the norm from Equation (8). Since $\mathbb{R}^{A^c}$ is finite-dimensional the norms $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent, and hence there is some $c > 0$ such that $\|f\| \leq c \|f\|_*$ for all $f \in \mathbb{R}^{A^c}$. Set $C := 2c/\xi$; then $C > 0$ since $\xi > 0$.

Fix any $\Delta > 0$ such that $\Delta\xi < 1$, and any $Q \in \mathcal{Q}$ with subgenerator G. It follows from Proposition 14 that $\left\| e^{G\Delta} \right\|_* \leq e^{-\xi\Delta}$. Using a standard quadratic bound on the negative scalar exponential, we have

$$\left\| e^{G\Delta} \right\|_* \leq e^{-\xi\Delta} \leq 1 - \xi\Delta + \frac{1}{2}\Delta^2\xi^2 < 1 - \frac{\Delta\xi}{2} < 1 , \tag{14}$$

where the third inequality used that $\Delta\xi < 1$. Notice that $\left\| e^{G\Delta} \right\|_* \leq e^{-\xi\Delta} < 1$. Moreover, $(I - e^{G\Delta})^{-1}$ exists by Proposition 8. By the norm equivalence, we have

$$\left\| (I - e^{G\Delta})^{-1} \right\| \leq c \left\| (I - e^{G\Delta})^{-1} \right\|_* , \tag{15}$$

and, by Lemma 6, that

$$\left\| (I - e^{G\Delta})^{-1} \right\|_* \leq \frac{1}{1 - \left\| e^{G\Delta} \right\|_*} .$$

Using Equation (14) we obtain

$$\left\| (I - e^{G\Delta})^{-1} \right\|_* \leq \frac{1}{1 - \left\| e^{G\Delta} \right\|_*} < \frac{1}{1 - 1 + \frac{\Delta\xi}{2}} = \frac{1}{\Delta} \frac{2}{\xi} .$$

Combining with Equation (15) yields

$$\left\| (I - e^{G\Delta})^{-1} \right\| < c \frac{1}{\Delta} \frac{2}{\xi} = \frac{C}{\Delta} ,$$

which concludes the proof. ☐

Proof of Proposition 15. Let $\xi, C > 0$ be as in Lemma 7, and let $\delta := 1/\xi$ and $L := C\left\|\mathcal{Q}\right\|^2$ with $\left\|\mathcal{Q}\right\| := \sup_{Q\in\mathcal{Q}}\left\|Q\right\|$; note that $\left\|\mathcal{Q}\right\| \in \mathbb{R}_{\geq 0}$ since $\mathcal{Q}$ is bounded by assumption. Observe that we must have $\left\|\mathcal{Q}\right\| > 0$ due to Assumption 2, whence $L > 0$.

Choose any $\Delta \in (0, \delta)$ and $Q \in \mathcal{Q}$. It is immediate from the definitions that $h^Q(x) = 0 = h^Q_\Delta(x)$ for all $x \in A$ and all $Q \in \mathcal{Q}$, so it remains to bound the norm on A^c.

Let G be the subgenerator of Q on A^c. By Proposition 9 we have that $h^Q_\Delta|_{A^c} = (I - e^{G\Delta})^{-1}\Delta\mathbf{1}$. Using the definition of h^Q this implies that

$$h^Q_\Delta|_{A^c} - e^{G\Delta}h^Q_\Delta|_{A^c} = \Delta\mathbf{1} = -\Delta G h^Q|_{A^c}\,.$$

Re-ordering terms we have

$$h^Q_\Delta|_{A^c} = e^{G\Delta}h^Q_\Delta|_{A^c} - \Delta G h^Q|_{A^c}\,.$$

Let $B = e^{G\Delta} - (I + \Delta G)$. We find that

$$\begin{aligned}
&h^Q_\Delta|_{A^c} - h^Q|_{A^c} \\
&\quad= e^{G\Delta}h^Q_\Delta|_{A^c} - \Delta G h^Q|_{A^c} - h^Q|_{A^c} \\
&\quad= e^{G\Delta}h^Q_\Delta|_{A^c} - (I + \Delta G)h^Q|_{A^c} \\
&\quad= e^{G\Delta}(h^Q_\Delta|_{A^c} - h^Q|_{A^c}) + \big(e^{G\Delta} - (I + \Delta G)\big)h^Q|_{A^c} \\
&\quad= e^{G\Delta}(h^Q_\Delta|_{A^c} - h^Q|_{A^c}) + B h^Q|_{A^c}\,.
\end{aligned}$$

We see that the difference on the left-hand side occurs again on the right-hand side. Hence we can substitute the same expansion $n \in \mathbb{N}$ times to get

$$\begin{aligned}
&h^Q_\Delta|_{A^c} - h^Q|_{A^c} \\
&\quad= e^{G\Delta(n+1)}(h^Q_\Delta|_{A^c} - h^Q|_{A^c}) + \sum_{k=0}^{n} e^{G\Delta k} B h^Q|_{A^c}\,.
\end{aligned}$$

Since we know from Section 4 that $\lim_{t\to+\infty} e^{Gt} = 0$, we see that the left summand vanishes as we take $n \to +\infty$ and, using Proposition 8, we have $(I - e^{Q\Delta})^{-1} = \sum_{k=0}^{+\infty} e^{G\Delta k}$. So, passing to this limit and taking norms, we find

$$\begin{aligned}
\left\|h^Q_\Delta|_{A^c} - h^Q|_{A^c}\right\| &= \left\|(I - e^{G\Delta})^{-1} B h^Q|_{A^c}\right\| \\
&\leq \left\|(I - e^{G\Delta})^{-1}\right\| \left\|B\right\| \left\|h^Q|_{A^c}\right\|\,.
\end{aligned}$$

Using Lemmas 3 and 4 and Corollary 3, we have

$$\left\|B\right\| = \left\|e^{G\Delta} - (I + \Delta G)\right\| \leq \left\|e^{Q\Delta} - (I + \Delta Q)\right\|\,,$$

and so, due to [Krak, 2021, Lemma B.8], we have $\left\|B\right\| \leq \Delta^2\left\|Q\right\|^2$. Since $Q \in \mathcal{Q}$ it follows that $\left\|Q\right\| \leq \left\|\mathcal{Q}\right\|$, and so $\left\|B\right\| \leq \Delta^2\left\|\mathcal{Q}\right\|^2$. Since $\Delta < \delta$ we have $\Delta\xi < 1$, whence $\left\|(I - e^{G\Delta})^{-1}\right\| < C/\Delta$ due to Lemma 7. In summary we find

$$\left\|h^Q_\Delta|_{A^c} - h^Q|_{A^c}\right\| < \frac{C}{\Delta}\Delta^2\left\|\mathcal{Q}\right\|^2\left\|h^Q|_{A^c}\right\| = \Delta L\left\|h^Q\right\|\,,$$

which concludes the proof. □

Proposition 17. *[Krak, 2020, Prop 7] Fix any $\Delta > 0$, and let $\mathcal{T}_\Delta$ denote the set of transition matrices that dominate $e^{\underline{Q}\Delta}$. Choose any $T_0 \in \mathcal{T}_\Delta$. For all $n \in \mathbb{N}_0$, let h_n be the (unique) non-negative solution to $h_n = \Delta\mathbb{I}_{A^c} + \mathbb{I}_{A^c}T_n h_n$, and let $T_{n+1} \in \mathcal{T}_\Delta$ be such that $T_{n+1}h_n = e^{\underline{Q}\Delta}h_n$.*

Then $\lim_{n\to+\infty} h_n = \underline{h}_\Delta$.

Proof. The preconditions of the reference actually require every T_n to be an extreme point of $\mathcal{T}_\Delta$, but inspection of the proof of [Krak, 2020, Prop 7] shows that this is not required; the superfluous condition is only used to streamline the statement of an algorithmic result further on in that work. □

We next need some results that involve transition matrices ${}^{P}T_t^s$ corresponding to (not-necessarily homogeneous) Markov chains $P \in \mathbb{P}^{\mathrm{M}}_{\mathcal{Q}}$. We recall from Section 2.2 that these are defined for any $t, s \in \mathbb{R}_{\geq 0}$ with $t \leq s$ as

$${}^{P}T_t^s(x, y) := P(X_s = y \,|\, X_t = x) \quad \text{for all } x, y \in \mathcal{X}.$$

Lemma 8. *Consider the sequence $(h_n)_{n\in\mathbb{N}_0}$ constructed as in Proposition 17. For any $n \in \mathbb{N}_0$, there is a Markov chain $P_{n+1} \in \mathcal{P}^{\mathrm{M}}_{\mathcal{Q}}$ with corresponding transition matrix ${}^{(n+1)}T_0^\Delta$ such that ${}^{(n+1)}T_0^\Delta h_n = e^{\underline{Q}\Delta}h_n$.*

Hence in particular, we can choose the co-sequence $(T_n)_{n\in\mathbb{N}}$ in Proposition 17 to be $({}^{(n)}T_0^\Delta)_{n\in\mathbb{N}}$.

Proof. This follows from [Krak, 2021, Cor 6.24] and the fact that $\mathcal{Q}$ is non-empty, compact, convex, and has separately specified rows. □

Proposition 18. *For all $\Delta > 0$ there is a Markov chain $P \in \mathcal{P}^{\mathrm{M}}_{\mathcal{Q}}$ with corresponding transition matrix $T = {}^{P}T_0^\Delta$, such that the unique solution h to $h = \Delta\mathbb{I}_{A^c} + \mathbb{I}_{A^c}Th$ satisfies $h = \underline{h}_\Delta$.*

Proof. Let $\mathcal{T}^{\mathrm{M}}_\Delta := \{{}^{P}T_0^\Delta \,:\, P \in \mathcal{P}^{\mathrm{M}}_{\mathcal{Q}}\}$, and let $(h_n)_{\in\mathbb{N}}$ be as in Proposition 17, with the co-sequence $(T_n)_{n\in\mathbb{N}}$ chosen as in Lemma 8 to consist of transition matrices corresponding to Markov chains in $\mathcal{P}^{\mathrm{M}}_{\mathcal{Q}}$. Then $(T_n)_{n\in\mathbb{N}}$ lives in $\mathcal{T}^{\mathrm{M}}_\Delta$.

The set $\mathcal{T}^{\mathrm{M}}_\Delta$ is compact by [Krak, 2021, Cor 5.18] and the fact that $\mathcal{Q}$ is non-empty, compact, and convex. Hence we can find a subsequence $(T_{n_j})_{j\in\mathbb{N}}$ with $\lim_{j\to+\infty} T_{n_j} =: T \in \mathcal{T}^{\mathrm{M}}_\Delta$. Since $T \in \mathcal{T}^{\mathrm{M}}_\Delta$, there is a Markov chain $P \in \mathcal{P}^{\mathrm{M}}_{\mathcal{Q}}$ with corresponding transition matrix $T = {}^{P}T_0^\Delta$.

Moreover, since $T, T_{n_j} \in \mathcal{T}^{\mathrm{M}}_\Delta$, it follows from [Krak, 2021, Cor 6.24] that the transition matrices T and all T_{n_j} dominate the lower transition operator $e^{\underline{Q}\Delta}$. Together with Assumption 2, this allows us to invoke [Krak, 2020, Prop 6], by which we can let h be the unique solution to $h = \Delta\mathbb{I}_{A^c} + \mathbb{I}_{A^c}Th$, and it holds for any $j \in \mathbb{N}$ that $h_{n_j}|_A = 0$, and

$$h_{n_j}|_{A^c} = (I - T_{n_j}|_{A^c})^{-1}\mathbf{1}\Delta\,.$$

Similarly, it holds that $h|_A = 0$, and

$$h|_{A^c} = (I - T|_{A^c})^{-1}\mathbf{1}\Delta\,.$$

Since $\lim_{h\to+\infty} T_{n_j} = T$ and by continuity of the map $M \mapsto (I-M)^{-1}$—which holds since all these inverses exist—it follows that $h|_{A^c} = \lim_{j\to+\infty} h_{n_j}|_{A^c}$. Since also $h|_A = h_{n_j}|_A$, it follows that $\lim_{j\to+\infty} h_{n_j} = h$.

By Proposition 17 we have $\lim_{n\to+\infty} h_n = \underline{h}_\Delta$, and hence we conclude that $\underline{h}_\Delta = \lim_{j\to+\infty} h_{n_j} = h$. □

Proposition 19. *Fix any $t \geq 0$ and consider any Markov chain $P \in \mathcal{P}^{\mathrm{M}}_{\mathcal{Q}}$ with transition matrix ${}^PT_0^t$. Choose any $\epsilon > 0$. Then there is some $m \in \mathbb{N}$ such that for all $n \geq m$ there are $Q_1, \ldots, Q_n \in \mathcal{Q}$, such that*

$$\left\| {}^PT_0^t - \prod_{i=1}^n (I + {}^t\!/\!_n Q_i) \right\| < \epsilon\,.$$

Proof. The result is trivial if $t = 0$, so let us consider the case where $t > 0$. Let $\epsilon' := {}^\epsilon\!/\!_{2t}$. By [Krak, 2021, Lemma 5.12] there is some $m \in \mathbb{N}$ such that for all $n \geq m$ and with $\Delta := {}^t\!/\!_n$, for all $i = 1, \ldots, n$ there is some $Q_i \in \mathcal{Q}$ such that

$$\left\| {}^PT_{(i-1)\Delta}^{i\Delta} - (I + \Delta Q_i) \right\| \leq \Delta\epsilon'\,.$$

Since P is a Markov chain, we can factor its transition matrices [Krak, 2021, Prop 5.1] as

$${}^PT_0^t = {}^PT_0^\Delta\, {}^PT_\Delta^{2\Delta} \cdots {}^PT_{t-\Delta}^t = \prod_{i=1}^n {}^PT_{(i-1)\Delta}^{i\Delta}\,.$$

Using [Krak, 2021, Lemma B.5] for the first inequality, we have

$$\begin{aligned}
&\left\| {}^PT_0^t - \prod_{i=1}^n (I + \Delta Q_i) \right\| \\
&= \left\| \prod_{i=1}^n {}^PT_{(i-1)\Delta}^{i\Delta} - \prod_{i=1}^n (I + \Delta Q_i) \right\| \\
&\leq \sum_{i=1}^n \left\| {}^PT_{(i-1)\Delta}^{i\Delta} - (I + \Delta Q_i) \right\| \\
&\leq \sum_{i=1}^n \Delta\epsilon' = n\frac{t}{n}\frac{\epsilon}{2t} = \frac{\epsilon}{2}\,,
\end{aligned}$$

which concludes the proof. □

Lemma 9. *Consider a sequence $(Q_n)_{n\in\mathbb{N}}$ in $\mathcal{Q}$ with limit $Q_* := \lim_{n\to+\infty} Q_n$. For all $n \in \mathbb{N}$, let h_n denote the minimal non-negative solution to $\mathbb{I}_A h_n = \mathbb{I}_{A^c} + \mathbb{I}_{A^c} Q_n h_n$, and let h_* denote the minimal non-negative solution to $\mathbb{I}_A h_* = \mathbb{I}_{A^c} + \mathbb{I}_{A^c} Q_* h_*$. Then $h_* = \lim_{n\to+\infty} h_n$.*

Proof. Since $\mathcal{Q}$ is closed, we have $Q_* \in \mathcal{Q}$. Let $(G_n)_{n\in\mathbb{N}}$ and G_* denote the subgenerators of $(Q_n)_{n\in\mathbb{N}}$ and Q_*, respectively. Then G_*^{-1} and G_n^{-1}, $n \in \mathbb{N}$ exist by Corollary 1, and hence we also have $\lim_{n\to+\infty} G_n^{-1} = G_*^{-1}$. Right-multiplying with $-\mathbf{1}$ and applying Proposition 10 gives

$$\lim_{n\to+\infty} h_n|_{A^c} = \lim_{n\to+\infty} -G_n^{-1}\mathbf{1} = -G_*^{-1}\mathbf{1} = h_*|_{A^c}\,.$$

Finally, by definition we trivially have $h_n(x) = 0 = h_*(x)$ for all $x \in A$. Hence also $\lim_{n\to+\infty} h_n|_A = h_*|_A$. □

Lemma 10. *[Krak et al., 2019, Cor 13] Fix any $\Delta > 0$ and let $\underline{h}_\Delta$ be the minimal non-negative solution to the non-linear system* (12). *Let $\mathcal{T}_\Delta$ denote the set of transition matrices that dominate $e^{\underline{Q}\Delta}$ and, for all $T \in \mathcal{T}_\Delta$, let h_T denote the minimal non-negative solution to the linear system $h_T = \Delta\mathbb{I}_{A^c} + \mathbb{I}_{A^c} T h_T$. Then it holds that*

$$\underline{h}_\Delta = \inf_{T\in\mathcal{T}_\Delta} h_T\,.$$

Proof of Proposition 16. We only give the proof for the lower hitting times, i.e. that $\lim_{\Delta\to 0^+} \|\underline{h}_\Delta - \underline{h}\| = 0$. The argument for the upper hitting times is completely analogous.

Choose any two sequences $(\Delta_n)_{n\in\mathbb{N}}$ and $(\epsilon_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ such that $\lim_{n\to+\infty} \Delta_n = 0$ and $\lim_{n\to+\infty} \epsilon_n = 0$. We will assume without loss of generality that $\Delta_n \|\mathcal{Q}\| \leq 1$ for all $n \in \mathbb{N}$, where $\|\mathcal{Q}\| = \sup_{Q\in\mathcal{Q}} \|Q\|$.

Now first fix any $n \in \mathbb{N}$, and consider $\underline{h}_{\Delta_n}$. By Proposition 18 there is a Markov chain $P_n \in \mathcal{P}^{\mathrm{M}}_{\mathcal{Q}}$ with transition matrix $T_n := {}^{P_n}T_0^{\Delta_n}$ such that the unique solution h_n to $h_n = \Delta_n\mathbb{I}_{A^c} + \mathbb{I}_{A^c} T_n h_n$ satisfies $h_n = \underline{h}_{\Delta_n}$.

By Proposition 19, there are $m_n \in \mathbb{N}$ with $m_n \geq n$ and $Q_1^{(n)}, \ldots, Q_{m_n}^{(n)}$ in $\mathcal{Q}$ such that, with

$$\Phi_n := \prod_{i=1}^{m_n} \left(I + \frac{\Delta_n}{m_n} Q_i^{(n)} \right)\,,$$

it holds that $\|T_n - \Phi_n\| < \epsilon_n$. Now define

$$Q_n := \sum_{i=1}^{m_n} \frac{1}{m_n} Q_i^{(n)}\,.$$

Then $Q_n \in \mathcal{Q}$ since $\mathcal{Q}$ is convex. Let h_{Q_n} denote the minimal non-negative solution to $\mathbb{I}_A h_{Q_n} = \mathbb{I}_{A^c} + \mathbb{I}_{A^c} Q_n h_{Q_n}$.

By repeating this construction for all $n \in \mathbb{N}$, we obtain a sequence $(Q_n)_{n\in\mathbb{N}}$ in $\mathcal{Q}$. Since $\mathcal{Q}$ is (sequentially) compact, we can consider a subsequence $(Q_{n_j})_{j\in\mathbb{N}}$ such that $\lim_{j\to+\infty} Q_{n_j} =: Q_* \in \mathcal{Q}$.

Let h_* be the minimal non-negative solution to $\mathbb{I}_A h_* = \mathbb{I}_{A^c} + \mathbb{I}_{A^c} Q_* h_*$. We now need to estimate some norm bounds that hold by choosing j large enough. Let $K = 5$ and fix any $\delta > 0$.

Since $(Q_{n_j})_{j\in\mathbb{N}}$ converges to Q_*, it follows from Lemma 9 that for j large enough, we have

$$\left\| h_{Q_{n_j}} - h_* \right\| < \frac{\delta}{K} \tag{16}$$

Since h_* is bounded, this also implies that the sequence $(h_{Q_{n_j}})_{j\in\mathbb{N}}$ is eventually uniformly bounded above in norm by some constant $M \geq 0$, say.

For all $j \in \mathbb{N}$, let $\hat{h}_{n_j}$ be such that $\hat{h}_{n_j}|_{A^c} := (I - e^{G_{n_j}\Delta_{n_j}})^{-1}\Delta_{n_j}\mathbf{1}$ and $\hat{h}_{n_j}|_A := 0$. Then

$$\hat{h}_{n_j} = \Delta_{n_j}\mathbb{I}_{A^c} + \mathbb{I}_{A^c}e^{Q_{n_j}\Delta_{n_j}}\hat{h}_{n_j} .$$

For j large enough we eventually have $\Delta_{n_j}\xi < 1$, and so by Proposition 15, we then have

$$\begin{aligned}\left\|\hat{h}_{n_j} - h_{Q_{n_j}}\right\| &< \Delta_{n_j}L\left\|h_{Q_{n_j}}\right\| \\ &\leq \Delta_{n_j}LM ,\end{aligned}$$

with L, M independent of j. Hence for j large enough we have

$$\left\|\hat{h}_{n_j} - h_{Q_{n_j}}\right\| < \frac{\delta}{K} . \tag{17}$$

Let next $\tilde{h}_{n_j}$ be the minimal non-negative solution to $\tilde{h}_{n_j} = \Delta_{n_j}\mathbb{I}_{A^c} + \mathbb{I}_{A^c}\Phi_{n_j}\tilde{h}_{n_j}$. Since $m_{n_j} \geq n_j$, for j large enough we have $\left\|\Phi_{n_j}|_{A^c}\right\| < 1$ due to Assumption 2.

By [Krak, 2021, Lemmas B.8 and B.12] we have

$$\left\|\Phi_{n_j} - e^{Q_{n_j}\Delta_{n_j}}\right\| \leq 2\Delta_{n_j}^2 \left\|\mathcal{Q}\right\|^2 ,$$

and so, for any $\epsilon > 0$, we can choose j large enough so that eventually $\left\|\Phi_{n_j} - e^{Q_{n_j}\Delta_{n_j}}\right\| < \epsilon$. Using the continuity of the map $T \mapsto (I-T)^{-1}$ on operators T for which this inverse exists, for large enough j we therefore find that

$$\begin{aligned}&\left\|\tilde{h}_{n_j}|_{A^c} - \hat{h}_{n_j}|_{A^c}\right\| \\ &= \left\|\left((I - \Phi_{n_j}|_{A^c})^{-1} - (I - e^{Q_{n_j}\Delta_{n_j}}|_{A^c})^{-1}\right)\Delta_{n_j}\mathbf{1}\right\| \\ &< \Delta_{n_j}\frac{\delta}{K} \leq \frac{\delta}{K} .\end{aligned}$$

Since $\tilde{h}_{n_j}|_A = 0 = \hat{h}_{n_j}|_A$, this implies that then also

$$\left\|\tilde{h}_{n_j} - \hat{h}_{n_j}\right\| < \frac{\delta}{K} . \tag{18}$$

Next, we recall that $\underline{h}_{\Delta_{n_j}} = h_{n_j}$, and

$$\left\|T_{n_j} - \Phi_{n_j}\right\| < \epsilon_{n_j} .$$

Hence by continuity of the map $T \mapsto (I-T)^{-1}$ on operators T for which this inverse exists, for large enough j we find that

$$\begin{aligned}&\left\|h_{n_j}|_{A^c} - \tilde{h}_{n_j}|_{A^c}\right\| \\ &= \left\|\left((I - T_{n_j}|_{A^c})^{-1} - (I - \Phi_{n_j}|_{A^c})^{-1}\right)\Delta_{n_j}\mathbf{1}\right\| \\ &< \Delta_{n_j}\frac{\delta}{K} \leq \frac{\delta}{K} .\end{aligned}$$

Since $h_{n_j}|_A = 0 = \tilde{h}_{n_j}|_A$, this implies that also

$$\left\|h_{n_j} - \tilde{h}_{n_j}\right\| < \frac{\delta}{K} . \tag{19}$$

Putting Equations (16)–(19) together, we find that for any large enough j it holds that

$$\begin{aligned}\left\|\underline{h}_{\Delta_{n_j}} - h_*\right\| &= \left\|h_{n_j} - h_*\right\| \\ &\leq \left\|h_{n_j} - \tilde{h}_{n_j}\right\| \\ &\quad + \left\|\tilde{h}_{n_j} - \hat{h}_{n_j}\right\| \\ &\quad + \left\|\hat{h}_{n_j} - h_{Q_{n_j}}\right\| \\ &\quad + \left\|h_{Q_{n_j}} - h_*\right\| \\ &< 4\frac{\delta}{K} .\end{aligned} \tag{20}$$

Since $\delta > 0$ is arbitrary this clearly implies that

$$\lim_{j\to+\infty}\underline{h}_{\Delta_{n_j}} = h_* . \tag{21}$$

Next, let us show that $h_* = \underline{h}$. To this end, assume *ex absurdo* that there is some $Q \in \mathcal{Q}$ such that $h^Q(x) < h_*(x)$ for some $x \in A^c$. Let $\delta := h_*(x) - h_Q(x) > 0$. Due to Corollary 2, for any $\Delta > 0$ small enough it holds that

$$\left\|h_\Delta^Q - h^Q\right\| < \frac{\delta}{K} .$$

This implies in particular that for large enough j it holds that $h^Q_{\Delta_{n_j}}(x) < h^Q(x) + \delta/K$. Moreover, it follows from Equation (20) that for large enough j we have $\underline{h}_{\Delta_{n_j}}(x) > h_*(x) - 4\delta/K$. It holds that $h_Q(x) = h_*(x) - \delta$, and hence, since $K = 5$, we find that that for large enough j,

$$\begin{aligned}h^Q_{\Delta_{n_j}}(x) &< h^Q(x) + \delta/K \\ &= h_*(x) - \delta + \delta/K \\ &= h_*(x) - K\frac{\delta}{K} + \delta/K \\ &= h_*(x) - (K-1)\frac{\delta}{K} \\ &= h_*(x) - 4\frac{\delta}{K} < \underline{h}_{\Delta_{n_j}}(x) .\end{aligned}$$

In other words, and using Lemma 10, we then have

$$h^Q_{\Delta_{n_j}}(x) < \underline{h}_{\Delta_{n_j}}(x) = \inf_{T\in\mathcal{T}_{\Delta_{n_j}}} h_T(x) \leq h^Q_{\Delta_{n_j}}(x) ,$$

where the last step used that $e^{Q\Delta_{n_j}} \in \mathcal{T}_{\Delta_{n_j}}$. From this contradiction we conclude that our earlier assumption must be wrong, and so it holds that $h_*(x) \leq h^Q(x)$ for all $x \in \mathcal{X}$ and $Q \in \mathcal{Q}$. This implies that $h_* \leq \underline{h}$. Since it clearly also holds that $\underline{h} \leq h_*$ because $Q_* \in \mathcal{Q}$, this implies that, indeed as claimed, $h_* = \underline{h}$.

In summary, at this point we have shown that for any sequence $(\Delta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ with $\lim_{n\to+\infty}\Delta_n = 0$, there is a subsequence such that $\lim_{j\to+\infty}\underline{h}_{\Delta_{n_j}} = \underline{h}$.

So, finally, suppose *ex absurdo* that $\lim_{\Delta\to 0^+}\underline{h}_\Delta \neq \underline{h}$. Then there is some sequence $(\Delta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ such that $\lim_{n\to+\infty}\Delta_n = 0$, and some $\epsilon > 0$, such that $\left\|\underline{h}_{\Delta_n} - \underline{h}\right\| \geq \epsilon$ for all $n \in \mathbb{N}$. By the above result, there is a subsequence such that $\lim_{j\to+\infty}\underline{h}_{\Delta_{n_j}} = \underline{h}$, which is a contradiction. □

Proof of Theorem 1. The crucial approach of this proof is to emulate Erreygers [2021, Sec 6.3] and consider *discretized* and *truncated* hitting times. By taking appropriate limits of such approximations, we then recover the "real" hitting times. We however need to be a bit careful with these constructions, since lower (and upper) expectation operators for continuous-time imprecise-Markov chains are not necessarily continuous with respect to arbitrary limits of such approximations [Erreygers, 2021, Chap 5]. This—fairly long—proof is therefore roughly divided into two parts; first, we construct a specific sequence of approximations, and establish the relevant continuity properties with respect to this sequence. Then, in the second part of this proof, we use this continuity to establish the main claim of this theorem.

To this end, for any $t \in \mathbb{R}_{\geq 0}$ and $\Delta \in \mathbb{R}_{>0}$, we first consider a fixed-step grid ν_Δ^t over $[0, t]$ with step-size Δ, as

$$\nu_\Delta^t := \left\{ i\Delta \,:\, i \in \mathbb{N}_0, i\Delta \leq t \right\}. \tag{22}$$

We define the associated *approximate* hitting time functions $\tau_\Delta^t : \Omega_{\mathbb{R}_{\geq 0}} \to \mathbb{R}$ for all $\omega \in \Omega_{\mathbb{R}_{\geq 0}}$ as

$$\tau_\Delta^t(\omega) := \min\Big(\left\{ s \in \nu_\Delta^t \,:\, \omega(s) \in A \right\} \cup \{t\}\Big). \tag{23}$$

Then by [Erreygers, 2021, Lemma 6.19], as we take the time-horizon t to infinity and the step-size Δ to zero, we have the point-wise limit to the actual hitting time function $\tau_{\mathbb{R}_{\geq 0}}$, in that

$$\tau_{\mathbb{R}_{\geq 0}}(\omega) = \lim_{t\to+\infty, \Delta\to 0^+} \tau_\Delta^t(\omega) \quad \text{for all } \omega \in \Omega_{\mathbb{R}_{\geq 0}}. \tag{24}$$

Let us now construct a specific sequence of approximate hitting time functions that will converge to this limit. To this end, first fix an arbitrary sequence $(\epsilon_n)_{n\in\mathbb{N}_0}$ in $\mathbb{R}_{>0}$ such that $\lim_{n\to+\infty}\epsilon_n = 0$. Moreover, for any $n \in \mathbb{N}_0$, we introduce the (discrete-time) *truncated* hitting time $\tau_{0:n} : \Omega_{\mathbb{N}_0} \to \mathbb{R}$, defined for all $\omega \in \Omega_{\mathbb{N}_0}$ as

$$\tau_{0:n}(\omega) := \min\Big(\left\{ t \in \{0, \ldots, n\} \,:\, \omega(t) \in A \right\} \cup \{n\}\Big).$$

Now fix any $k \in \mathbb{N}_0$, let $\Delta_k := 2^{-k}$, and let $\mathcal{T}_k$ denote the set of transition matrices that dominate $e^{\underline{Q}\Delta_k}$. We now consider discrete-time imprecise-Markov chains parameterized by $\mathcal{T}_k$. As discussed in [Krak et al., 2019], for all $n \in \mathbb{N}_0$ there are functions[1] $\underline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{V}}[\tau_{0:n} \mid X_0]$ and $\overline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{V}}[\tau_{0:n} \mid X_0]$ in $\mathbb{R}^{\mathcal{X}}$ such that

$$\begin{aligned}\underline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{V}}[\tau_{0:n} \mid X_0] &\leq \underline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{I}}[\tau_{0:n} \mid X_0] \\ &\leq \overline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{I}}[\tau_{0:n} \mid X_0] \leq \overline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{V}}[\tau_{0:n} \mid X_0]\end{aligned}$$

that, moreover, satisfy

$$\lim_{n\to+\infty} \underline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{V}}[\tau_{0:n} \mid X_0] = \underline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{V}}[\tau_{\mathbb{N}_0} \mid X_0] = \underline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{HM}}[\tau_{\mathbb{N}_0} \mid X_0]$$

and

$$\lim_{n\to+\infty} \overline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{V}}[\tau_{0:n} \mid X_0] = \overline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{V}}[\tau_{\mathbb{N}_0} \mid X_0] = \overline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{HM}}[\tau_{\mathbb{N}_0} \mid X_0].$$

We already noted in Section 5 that the functions $\underline{h}_{\Delta_k}$ and $\overline{h}_{\Delta_k}$ from Equations (12) and (13) satisfy

$$\underline{h}_{\Delta_k} = \Delta_k \overline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{HM}}[\tau_{\mathbb{N}_0} \mid X_0] \text{ and } \overline{h}_{\Delta_k} = \Delta_k \overline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{HM}}[\tau_{\mathbb{N}_0} \mid X_0].$$

Combining the above, we find that

$$\lim_{n\to+\infty} \Delta_k \underline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{V}}[\tau_{0:n} \mid X_0] = \underline{h}_{\Delta_k}$$

and

$$\lim_{n\to+\infty} \Delta_k \overline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{V}}[\tau_{0:n} \mid X_0] = \overline{h}_{\Delta_k}.$$

Hence for all $k \in \mathbb{N}_0$, we can now choose $t_k \in \mathbb{N}_0$ large enough such that $t_k \geq k$, and so that with $n_k = 2^k t_k$ we have both

$$\left\| \Delta_k \underline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{V}}[\tau_{0:n_k} \mid X_0] - \underline{h}_{\Delta_k} \right\| < \epsilon_k, \tag{25}$$

and

$$\left\| \Delta_k \overline{\mathbb{E}}_{\mathcal{T}_k}^{\mathrm{V}}[\tau_{0:n_k} \mid X_0] - \overline{h}_{\Delta_k} \right\| < \epsilon_k. \tag{26}$$

With these selections, we now define the sequence $(\tau_k)_{k\in\mathbb{N}_0}$ of approximate hitting times as $\tau_k := \tau_{\Delta_k}^{t_k}$ for all $k \in \mathbb{N}_0$. Clearly we have $\lim_{k\to+\infty}\Delta_k = 0$, and since $t_k \geq k$ we also find that $\lim_{k\to+\infty} t_k = +\infty$. Hence by Equation (24) we have the pointwise limit

$$\tau_{\mathbb{R}_{\geq 0}}(\omega) = \lim_{k\to+\infty} \tau_k(\omega) \quad \text{for all } \omega \in \Omega_{\mathbb{R}_{\geq 0}}. \tag{27}$$

Having constructed this specific sequence that converges to the "true" hitting time function, we will now demonstrate the relevant continuity properties of the lower- and upper expectations of interest, with respect to this sequence.

To this end, we define $\hat{\tau} : \Omega_{\mathbb{R}_{\geq 0}} \to \mathbb{R} \cup \{+\infty\}$ as

$$\hat{\tau}(\omega) := \sup_{t\in\mathbb{N}_0} \sup_{n\in\mathbb{N}_0} \tau_{\Delta_n}^t(\omega) \quad \text{for all } \omega \in \Omega_{\mathbb{R}_{\geq 0}}. \tag{28}$$

[1] These represent lower and an upper expectations with respect to a *game-theoretic imprecise-Markov chain*, but the details don't concern us here.

Then for all $k \in \mathbb{N}_0$ we have $\tau_k(\omega) = \tau^{t_k}_{\Delta_k}(\omega) \leq \hat{\tau}(\omega)$ for all $\omega \in \Omega_{\mathbb{R}_{\geq 0}}$. Moreover, since every τ_k is non-negative, it holds in fact that $|\tau_k(\omega)| \leq \hat{\tau}(\omega)$ for all $k \in \mathbb{N}_0$ and $\omega \in \Omega_{\mathbb{R}_{\geq 0}}$. This means that if we can show that the upper expectation $\overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\hat{\tau} \,|\, X_0 = x]$ is bounded for all $x \in \mathcal{X}$, then we can use the imprecise version of the dominated convergence theorem [Erreygers, 2021, Thm 5.32] to take lower- and upper expectations of the limit in Equation (27). So, we will now show that this boundedness indeed holds.

We note that, for fixed $t \in \mathbb{N}_0$, $\tau^t_{\Delta_n}$ is monotonically *decreasing* as we increase $n \in \mathbb{N}_0$. To see this, first consider the grids $\nu^t_{\Delta_n}$ and $\nu^t_{\Delta_{n+1}}$ over $[0, t]$. For any $s \in \nu^t_{\Delta_n}$ there is some $i \in \mathbb{N}_0$ such that $s = i\Delta_n$, and since $\Delta_n = 2^{-n} = 2\Delta_{n+1}$, we find that also $s = 2i\Delta_{n+1} \in \nu^t_{\Delta_{n+1}}$. Hence we conclude that $\nu^t_{\Delta_n} \subseteq \nu^t_{\Delta_{n+1}}$. From this set inclusion, we also clearly have for any $\omega \in \Omega_{\mathbb{R}_{\geq 0}}$ that

$$\{s \in \nu^t_{\Delta_n} \,:\, \omega(s) \in A\} \subseteq \{s \in \nu^t_{\Delta_{n+1}} \,:\, \omega(s) \in A\}\,,$$

and so together with the fact that $s \leq t$ for all $s \in \nu^t_{\Delta_{n+1}}$, it then follows from Equation (23) that $\tau^t_{\Delta_n}(\omega) \geq \tau^t_{\Delta_{n+1}}(\omega)$.

Using this observation, we immediately find that for any $t \in \mathbb{N}_0$ and $\omega \in \Omega_{\mathbb{R}_{\geq 0}}$ it holds that

$$\sup_{n \in \mathbb{N}_0} \tau^t_{\Delta_n}(\omega) = \tau^t_{\Delta_0}(\omega) = \tau^t_1(\omega)\,,$$

and so from Equation (28), we have

$$\hat{\tau}(\omega) = \sup_{t \in \mathbb{N}_0} \tau^t_1(\omega)\,.$$

Next, we observe that τ^t_1 is monotonically increasing as we increase $t \in \mathbb{N}_0$. Indeed, for any $t \in \mathbb{N}_0$ the grid ν^t_1 over $[0, t]$ simply constitutes the set $\nu^t_1 = \{0, \ldots, t\}$. Hence that τ^t_1 is monotonically increasing as we increase $t \in \mathbb{N}_0$, follows immediately from Equation (23). In particular, this implies that the sequence $(\tau^t_1)_{t \in \mathbb{N}_0}$ converges *monotonically* to $\hat{\tau}$. Moreover, we have $\tau^0_1(\omega) = 0$ for all $\omega \in \Omega_{\mathbb{R}_{\geq 0}}$, and so we find that identically

$$\underline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau^0_1 \,|\, X_0] = \inf_{P \in \mathcal{P}^{\mathrm{I}}_{\mathcal{Q}}} \mathbb{E}_P[\tau^0_1 \,|\, X_0] = 0\,.$$

Hence by the continuity of upper expectations with respect to monotonically increasing convergent sequences of functions that are bounded below [Erreygers, 2021, Thm 5.31], we have

$$\overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\hat{\tau} \,|\, X_0] = \lim_{t \to +\infty, t \in \mathbb{N}_0} \overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau^t_1 \,|\, X_0]\,. \tag{29}$$

Now, for every $t \in \mathbb{N}_0$, τ^t_1 only depends on finitely many time-points; indeed, $\tau^t_1(\omega)$ only depends on the value of $\omega(s)$ for $s \in \{0, \ldots, t\}$. Using [Krak, 2021, Thm 7.2], this means that the lower- and upper expectations of these functions with respect to the imprecise-Markov chain $\mathcal{P}^{\mathrm{I}}_{\mathcal{Q}}$, can also be expressed as lower (resp. upper) expectations of this function with respect to an induced *discrete*-time imprecise-Markov chain. Indeed, since the step-size used in these approximating functions is uniformly equal to one, and using the obvious correspondence between τ^t_1 and $\tau_{0:t}$, it is not difficult to see that

$$\overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau^t_1 \,|\, X_0] = \overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{T}}[\tau_{0:t} \,|\, X_0] \quad \text{for all } t \in \mathbb{N}_0, \tag{30}$$

where $\mathcal{T}$ is the set of transition matrices that dominates $e^{\underline{Q}}$.

We now again invoke the previously mentioned results from [Krak et al., 2019]; for any $t \in \mathbb{N}_0$, there is a function $\overline{\mathbb{E}}^{\mathrm{V}}_{\mathcal{T}}[\tau_{0:t} \,|\, X_0] \in \mathbb{R}^{\mathcal{X}}$ that satisfies

$$\overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{T}}[\tau_{0:t} \,|\, X_0] \leq \overline{\mathbb{E}}^{\mathrm{V}}_{\mathcal{T}}[\tau_{0:t} \,|\, X_0]\,. \tag{31}$$

Combining Equations (29), (30), and (31), and using [Krak et al., 2019, Prop 7] to establish the limit on the final right-hand side, we have

$$\begin{aligned}\overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\hat{\tau} \,|\, X_0] &= \lim_{t \to +\infty, t \in \mathbb{N}_0} \overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau^t_1 \,|\, X_0] \\ &= \lim_{t \to +\infty, t \in \mathbb{N}_0} \overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{T}}[\tau_{0:t} \,|\, X_0] \\ &\leq \lim_{t \to +\infty, t \in \mathbb{N}_0} \overline{\mathbb{E}}^{\mathrm{V}}_{\mathcal{T}}[\tau_{0:t} \,|\, X_0] = \overline{\mathbb{E}}^{\mathrm{V}}_{\mathcal{T}}[\tau_{\mathbb{N}_0} \,|\, X_0]\,.\end{aligned}$$

By [Krak et al., 2019, Thm 12] it holds that

$$\overline{\mathbb{E}}^{\mathrm{V}}_{\mathcal{T}}[\tau_{\mathbb{N}_0} \,|\, X_0] = \overline{\mathbb{E}}^{\mathrm{HM}}_{\mathcal{T}}[\tau_{\mathbb{N}_0} \,|\, X_0]\,,$$

and, moreover, that there is some homogeneous discrete-time Markov chain $P \in \mathcal{P}^{\mathrm{HM}}_{\mathcal{T}}$ with associated transition matrix $T = {}^PT \in \mathcal{T}$ and hitting times $h = \mathbb{E}_P[\tau_{\mathbb{N}_0} \,|\, X_0]$ such that $h = \overline{\mathbb{E}}^{\mathrm{HM}}_{\mathcal{T}}[\tau_{\mathbb{N}_0} \,|\, X_0]$. Putting this together, we find that

$$\overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\hat{\tau} \,|\, X_0 = x] \leq h(x) \quad \text{for all } x \in \mathcal{X}. \tag{32}$$

By Proposition 1, h is also the minimal non-negative solution to the system

$$h = \mathbb{I}_{A^c} + \mathbb{I}_{A^c} T h\,. \tag{33}$$

It is immediate from the definition that $h|_A = 0$, and since $\hat{\tau}$ is clearly non-negative, we obtain from Equation (32) that for all $x \in A$ we have

$$0 \leq \overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\hat{\tau} \,|\, X_0 = x] \leq h(x) = 0\,,$$

or in other words, that $\overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\hat{\tau} \,|\, X_0 = x] = 0$ for all $x \in A$. So, it remains to bound this upper expectation on A^c.

By our Assumption 2, it holds for all $x \in A^c$ that $e^{\underline{Q}}\mathbb{I}_A(x) > 0$. Since $e^{\underline{Q}}$ is the lower transition operator corresponding to $\mathcal{T}$ due to Proposition 3, it follows that $e^{\underline{Q}}$ satisfies conditions C1–C3 and R1 from Reference [Krak, 2020]. We now recall that $T = {}^PT \in \mathcal{T}$. Since the preconditions C1–C3 and R1 of this reference are all satisfied, we can now invoke [Krak, 2020, Lemma 10], which states that the inverse operator $(I - T|_{A^c})^{-1}$ exists.

We note that $h|_A = 0$, and so $h = (h|_{A^c})\uparrow_{\mathcal{X}}$. Hence in particular, we have $T|_{A^c} h|_{A^c} = (Th)|_{A^c}$. From Equation (33), we now find that

$$h|_{A^c} = \mathbf{1} + (Th)|_{A^c} = \mathbf{1} + T|_{A^c} h|_{A^c}\,,$$

and so re-ordering terms, we have $(I - T|_{A^c})h|_{A^c} = \mathbf{1}$. Using the existence of the inverse operator established above, we obtain

$$h|_{A^c} = (I - T|_{A^c})^{-1}\mathbf{1}\,.$$

Since $(I - T|_{A^c})$ is an invertible bounded linear operator, also clearly $(I - T|_{A^c})^{-1}$ is bounded. Hence we have

$$\|h|_{A^c}\| = \left\|(I - T|_{A^c})^{-1}\mathbf{1}\right\| \leq \left\|(I - T|_{A^c})^{-1}\right\| < +\infty\,.$$

From Equation (32) we find that $\overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\hat{\tau} \mid X_0 = x] < +\infty$ for all $x \in A^c$. In summary, at this point we have shown that $\overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\hat{\tau} \mid X_0 = x]$ is bounded for all $x \in \mathcal{X}$. Since we already established that $\hat{\tau}$ absolutely dominates the sequence $(\tau_k)_{k\in\mathbb{N}_0}$, we can now finally use the limit (27) and the dominated convergence theorem [Erreygers, 2021, Thm 5.32] to establish that

$$\limsup_{k\to+\infty} \underline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau_k \mid X_0] \leq \underline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau_{\mathbb{R}_{\geq 0}} \mid X_0]\,, \tag{34}$$

and

$$\overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau_{\mathbb{R}_{\geq 0}} \mid X_0] \leq \liminf_{k\to+\infty} \overline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau_k \mid X_0]\,. \tag{35}$$

This concludes the first part of this proof. Our next step will be to identify the limits superior and inferior in the above inequalities as corresponding to, respectively, $\underline{h}$ and $\overline{h}$.

Let us start by obtaining the required result for the lower expectation. From the definition of the limit superior, there is a convergent subsequence such that

$$\underline{s} := \lim_{j\to+\infty} \underline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau_{k_j} \mid X_0] = \limsup_{k\to+\infty} \underline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau_k \mid X_0]\,. \tag{36}$$

Now fix any $j \in \mathbb{N}_0$, and consider the approximate function $\tau_{k_j} = \tau_{\Delta_{k_j}}^{t_{k_j}}$. As before, this function really only depends on the system at finitely many time points; specifically, those on the grid $\nu_{\Delta_{k_j}}^{t_{k_j}}$ over $[0, t_{k_j}]$. We can therefore again use [Krak, 2021, Thm 7.2], to express the lower- and upper expectations of this function with respect to the imprecise-Markov chain $\mathcal{P}^{\mathrm{I}}_{\mathcal{Q}}$, as lower- and upper expectations of a function with respect to an induced *discrete*-time imprecise-Markov chain. Since the step size of this grid is now equal to Δ_{k_j} rather than one, this requires a bit more effort than before. In particular, we now need to compensate for the step-size Δ_{k_j} of the grid. Indeed, the corresponding discrete-time imprecise-Markov chain should consider steps that are implicitly of this "length", so we consider the model induced by the set $\mathcal{T}_{k_j}$ of transition matrices that dominate $e^{\underline{Q}\Delta_{k_j}}$. It then remains to find an appropriate translation $\tilde{\tau}_{k_j}$ of τ_{k_j} to the domain $\Omega_{\mathbb{N}_0}$.

As a first observation, we note that this "translation" $\tilde{\tau}_{k_j} : \Omega_{\mathbb{N}_0} \to \mathbb{R}$ should depend on the same number of time points as τ_{k_j}. We note that since $t_k \in \mathbb{N}_0$ it holds that $t_k \in \nu_{\Delta_{k_j}}^{t_{k_j}}$. Hence it follows from Equation (22) that $\nu_{\Delta_{k_j}}^{t_{k_j}}$ contains exactly $\Delta_{k_j}^{-1} t_{k_j} = 2^{k_j} t_{k_j} = n_{k_j}$ time points, in addition to the origin 0, and that τ_{k_j} depends exactly on these time points. Indeed, inspection of Equation (23) reveals that, by re-scaling to compensate for the step size Δ_{k_j}, the quantity $\Delta_{k_j}^{-1}\tau_{k_j}(\omega)$ simply represents the natural index of the discrete grid element of $\nu_{\Delta_{k_j}}^{t_{k_j}}$ on which $\omega \in \Omega_{\mathbb{R}_{\geq 0}}$ did (or did not) initially hit A. Adapting Equation (23), we therefore define for any $\omega \in \Omega_{\mathbb{N}_0}$ that

$$\tilde{\tau}_{k_j}(\omega) := \min\Big(\big\{s \in \nu_{\Delta_{k_j}}^{t_{k_j}} \,:\, \omega\big(\Delta_{k_j}^{-1}s\big) \in A\big\} \cup \{t_{k_j}\}\Big)\,.$$

We see that, as required, $\Delta_{k_j}^{-1}\tilde{\tau}_{k_j}(\omega)$ is again simply the identity of the step on which $\omega \in \Omega_{\mathbb{N}_0}$ did (or did not) initially hit A. This implies the relation to the discrete-time truncated hitting time $\tau_{0:n_{k_j}}$; for any $\omega \in \Omega_{\mathbb{N}_0}$ we have

$$\begin{aligned}
&\tilde{\tau}_{k_j}(\omega) \\
&= \Delta_{k_j}\Delta_{k_j}^{-1}\tilde{\tau}_{k_j}(\omega) \\
&= \Delta_{k_j} \min\Big(\big\{s \in \Delta_{k_j}^{-1}\nu_{\Delta_{k_j}}^{t_{k_j}} \,:\, \omega(s) \in A\big\} \cup \{\Delta_{k_j}^{-1}t_{k_j}\}\Big) \\
&= \Delta_{k_j} \min\Big(\big\{s \in \{0, \ldots, n_{k_j}\} \,:\, \omega(s) \in A\big\} \cup \{n_{k_j}\}\Big) \\
&= \Delta_{k_j}\tau_{0:n_{k_j}}(\omega)\,,
\end{aligned}$$

and so we simply have that $\tilde{\tau}_{k_j} = \Delta_{k_j}\tau_{0:n_{k_j}}$. Following the discussion in [Krak, 2021, Chap 7], and [Krak, 2021, Thm 7.2] in particular, we therefore find the identity

$$\underline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau_{k_j} \mid X_0] = \Delta_{k_j}\underline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{T}_{k_j}}[\tau_{0:n_{k_j}} \mid X_0] \quad \text{for all } j \in \mathbb{N}_0. \tag{37}$$

We again recall from Krak et al. [2019] the objects $\underline{\mathbb{E}}^{\mathrm{V}}_{\mathcal{T}_{k_j}}[\tau_{0:n_{k_j}} \mid X_0]$ in $\mathbb{R}^{\mathcal{X}}$ satisfying

$$\underline{\mathbb{E}}^{\mathrm{V}}_{\mathcal{T}_{k_j}}[\tau_{0:n_{k_j}} \mid X_0] \leq \underline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{T}_{k_j}}[\tau_{0:n_{k_j}} \mid X_0] \quad \text{for all } j \in \mathbb{N}_0.$$

Hence from Equations (36) and (37) we now find that

$$\begin{aligned}
\underline{s} &= \lim_{j\to+\infty} \underline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau_{k_j} \mid X_0] \\
&= \lim_{j\to+\infty} \Delta_{k_j}\underline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{T}_{k_j}}[\tau_{0:n_{k_j}} \mid X_0] \\
&\geq \lim_{j\to+\infty} \Delta_{k_j}\underline{\mathbb{E}}^{\mathrm{V}}_{\mathcal{T}_{k_j}}[\tau_{0:n_{k_j}} \mid X_0]\,.
\end{aligned} \tag{38}$$

We now note that, for all $j \in \mathbb{N}_0$, we have

$$\begin{aligned}
&\left\|\Delta_{k_j}\underline{\mathbb{E}}^{\mathrm{V}}_{\mathcal{T}_{k_j}}[\tau_{0:n_{k_j}} \mid X_0] - \underline{h}\right\| \\
&\quad\leq \left\|\Delta_{k_j}\underline{\mathbb{E}}^{\mathrm{V}}_{\mathcal{T}_{k_j}}[\tau_{0:n_{k_j}} \mid X_0] - \underline{h}_{\Delta_{k_j}}\right\| + \left\|\underline{h}_{\Delta_{k_j}} - \underline{h}\right\| \\
&\quad< \epsilon_{k_j} + \left\|\underline{h}_{\Delta_{k_j}} - \underline{h}\right\|\,,
\end{aligned}$$

where we used Equation (25) for the final inequality. Using that $\lim_{j\to+\infty}\epsilon_{k_j}=0$ and $\lim_{j\to+\infty}\Delta_{k_j}=0$, together with Proposition 16, we see that both summands vanish as we increase $j\in\mathbb{N}_0$, and so we have

$$\lim_{j\to+\infty}\Delta_{k_j}\underline{\mathbb{E}}^{\mathrm{V}}_{\mathcal{T}_{k_j}}[\tau_{0:n_{k_j}}\,|\,X_0]=\underline{h}\,. \tag{39}$$

We already established in Section 5 that $\underline{h}=\underline{\mathbb{E}}^{\mathrm{HM}}_{\mathcal{Q}}[\tau_{\mathbb{R}_{\geq 0}}\,|\,X_0]$. Hence by combining Equations (34), (36), (38), and (39), we now find that

$$\underline{\mathbb{E}}^{\mathrm{HM}}_{\mathcal{Q}}[\tau_{\mathbb{R}_{\geq 0}}\,|\,X_0]=\underline{h}\leq\underline{s}\leq\underline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau_{\mathbb{R}_{\geq 0}}\,|\,X_0]\,. \tag{40}$$

However, as noted in Section 3.1 we have the inclusion $\mathcal{P}^{\mathrm{HM}}_{\mathcal{Q}}\subseteq\mathcal{P}^{\mathrm{M}}_{\mathcal{Q}}\subseteq\mathcal{P}^{\mathrm{I}}_{\mathcal{Q}}$, so it immediately follows from the definition of the lower expectations that

$$\underline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau_{\mathbb{R}_{\geq 0}}\,|\,X_0]\leq\underline{\mathbb{E}}^{\mathrm{M}}_{\mathcal{Q}}[\tau_{\mathbb{R}_{\geq 0}}\,|\,X_0]\leq\underline{\mathbb{E}}^{\mathrm{HM}}_{\mathcal{Q}}[\tau_{\mathbb{R}_{\geq 0}}\,|\,X_0]\,.$$

Hence by Equation (40) we obtain the identity

$$\underline{\mathbb{E}}^{\mathrm{I}}_{\mathcal{Q}}[\tau_{\mathbb{R}_{\geq 0}}\,|\,X_0]=\underline{\mathbb{E}}^{\mathrm{M}}_{\mathcal{Q}}[\tau_{\mathbb{R}_{\geq 0}}\,|\,X_0]=\underline{\mathbb{E}}^{\mathrm{HM}}_{\mathcal{Q}}[\tau_{\mathbb{R}_{\geq 0}}\,|\,X_0]\,,$$

which concludes the proof that the lower expected hitting times are the same for all three types of continuous-time imprecise-Markov chains. We omit the proof for the upper expected hitting times; this is completely analogous, starting instead from Equation (35) and using the norm bound (26) to pass to the limit $\overline{h}=\overline{\mathbb{E}}^{\mathrm{HM}}_{\mathcal{Q}}[\tau_{\mathbb{R}_{\geq 0}}\,|\,X_0]$. ☐

Proof of Theorem 2. First fix any $\Delta>0$. For $\underline{h}_\Delta$ it then holds that

$$\underline{h}_\Delta=\Delta\mathbb{I}_{A^c}+\mathbb{I}_{A^c}e^{\underline{Q}\Delta}\underline{h}_\Delta\,.$$

Since $\underline{h}_\Delta(x)=0$ for all $x\in A$, we have $\underline{h}_\Delta=\mathbb{I}_{A^c}\underline{h}_\Delta$ and $\mathbb{I}_A\underline{h}_\Delta=0$. We can therefore rearrange terms and add $\mathbb{I}_A\underline{h}_\Delta$ to the above, to obtain

$$\mathbb{I}_A\underline{h}_\Delta=\mathbb{I}_{A^c}+\mathbb{I}_{A^c}\frac{e^{\underline{Q}\Delta}-I}{\Delta}\underline{h}_\Delta\,.$$

Because the individual limits exist by Proposition 16 and [De Bock, 2017, Prop 9], taking Δ to zero yields

$$\begin{aligned}\mathbb{I}_A\underline{h}&=\mathbb{I}_A\lim_{\Delta\to 0^+}\underline{h}_\Delta\\&=\mathbb{I}_{A^c}+\mathbb{I}_{A^c}\lim_{\Delta\to 0^+}\frac{e^{\underline{Q}\Delta}-I}{\Delta}\underline{h}_\Delta\\&=\mathbb{I}_{A^c}+\mathbb{I}_{A^c}\lim_{\Delta\to 0^+}\frac{e^{\underline{Q}\Delta}-I}{\Delta}\lim_{\Delta\to 0^+}\underline{h}_\Delta\\&=\mathbb{I}_{A^c}+\mathbb{I}_{A^c}\underline{Q}\,\underline{h}\,.\end{aligned}$$

So, $\underline{h}$ is indeed a solution to the system $\mathbb{I}_A\underline{h}=\mathbb{I}_{A^c}+\mathbb{I}_{A^c}\underline{Q}\,\underline{h}$. It follows from a completely analogous argument that also $\overline{h}$ is a solution to $\mathbb{I}_A\overline{h}=\mathbb{I}_{A^c}+\mathbb{I}_{A^c}\overline{Q}\,\overline{h}$.

That $\underline{h}$ and $\overline{h}$ are non-negative is clear. We now first show that $\underline{h}$ is the minimal non-negative solution to its corresponding system. To this end, suppose *ex absurdo* that there is some non-negative $h\in\mathbb{R}^{\mathcal{X}}$ such that $h(x)<\underline{h}(x)$ for some $x\in\mathcal{X}$, and $\mathbb{I}_A h=\mathbb{I}_{A^c}+\mathbb{I}_{A^c}\underline{Q}h$. Then clearly $x\in A^c$ since $\underline{h}(y)=0$ for all $y\in A$ and h is non-negative.

By Proposition 4, there is then some $Q\in\mathcal{Q}$ such that $Qh=\underline{Q}h$, and so also $\mathbb{I}_A h=\mathbb{I}_{A^c}+\mathbb{I}_{A^c}Qh$. By Proposition 2 there is some minimal non-negative solution h_* to the system $\mathbb{I}_A h_*=\mathbb{I}_{A^c}+\mathbb{I}_{A^c}Qh_*$, where the minimality implies in particular that $h_*\leq h$. Since $Q\in\mathcal{Q}$, we obtain

$$h_*(x)\leq h(x)<\underline{h}(x)=\inf_{Q'\in\mathcal{Q}}h^{Q'}(x)\leq h_*(x)\,,$$

which is a contradiction.

We next show that $\overline{h}$ is the minimal non-negative solution to its corresponding system; this will require a bit more effort and we need to start with some auxiliary constructions. By Proposition 4, there is some $Q\in\mathcal{Q}$ such that $Q\overline{h}=\overline{Q}\,\overline{h}$. Let G be the subgenerator of Q.

Consider the $\xi>0$ from Proposition 11, and let $\|\cdot\|_*$ be the norm from Section 4.1. Since $\mathbb{R}^{A^c}$ is finite-dimensional, the norms $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent, whence there is some $c\in\mathbb{R}_{>0}$ such that $\|f\|_*\leq c\,\|f\|$ for all $f\in\mathbb{R}^{A^c}$.

Now let $\Delta>0$ be such that $\Delta\,\|Q\|\leq 1$, $\Delta\,\|\underline{Q}\|\leq 1$, $\Delta\xi<{}^2/_3$, and $\Delta c\,\|Q\|^2<{}^\xi/_3$; this is clearly always possible. Define the map $H:\mathbb{R}^{A^c}\to\mathbb{R}^{A^c}$ for all $f\in\mathbb{R}^{A^c}$ as

$$H(f):=f+\Delta\mathbf{1}+\Delta Gf=\Delta\mathbf{1}+(I+\Delta G)f\,.$$

Let us show that H is a contraction on the Banach space $(\mathbb{R}^{A^c},\|\cdot\|_*)$, or in other words, that there is some $\alpha\in[0,1)$ such that $\|H(f)-H(g)\|_*\leq\alpha\,\|f-g\|_*$ for all $f,g\in\mathbb{R}^{A^c}$ [Renardy and Rogers, 2006, Sec 10.1.1]. So, fix any $f,g\in\mathbb{R}^{A^c}$. Then we have

$$\begin{aligned}\|H(f)-H(g)\|_*&=\|(I+\Delta G)f-(I+\Delta G)g\|_*\\&=\|(I+\Delta G)(f-g)\|_*\\&\leq\|I+\Delta G\|_*\,\|f-g\|_*\,,\end{aligned}$$

from which we find that

$$\begin{aligned}&\|H(f)-H(g)\|_*\\&\quad\leq\left(\left\|(I+\Delta G)-e^{G\Delta}\right\|_*+\left\|e^{G\Delta}\right\|_*\right)\|f-g\|_*\,.\end{aligned} \tag{41}$$

By Proposition 14 we have $\left\|e^{G\Delta}\right\|_*\leq e^{-\xi\Delta}$, and so using a standard quadratic bound on the negative scalar exponential,

$$\left\|e^{G\Delta}\right\|_*\leq 1-\Delta\xi+\frac{1}{2}\Delta^2\xi^2<1-\frac{2}{3}\Delta\xi\,, \tag{42}$$

where we used that $\Delta\xi<{}^2/_3$.

Moreover, we have that

$$\begin{aligned}\left\|(I+\Delta G)-e^{G\Delta}\right\|_*&\leq c\left\|(I+\Delta G)-e^{G\Delta}\right\|\\&\leq c\left\|(I+\Delta Q)-e^{Q\Delta}\right\|\\&\leq c\Delta^2\,\|Q\|^2\,,\end{aligned}$$

where the second inequality used Lemmas 3 and 4 and Corollary 3; and the final inequality used [Krak, 2021, Lemma B.8]. Since $\Delta c \|Q\|^2 < \xi/3$ we have

$$\left\|(I+\Delta G) - e^{G\Delta}\right\|_* < \frac{1}{3}\Delta\xi\,. \tag{43}$$

Combining Equations (41), (42), and (43) we obtain

$$\|H(f) - H(g)\|_* \leq \left(1 - \frac{1}{3}\Delta\xi\right)\|f-g\|_*\,.$$

Since $\Delta > 0$, $\xi > 0$, and $\Delta\xi < 2/3$, we conclude that H is indeed a contraction. Hence by the Banach fixed-point theorem [Renardy and Rogers, 2006, Thm 10.1], there is a unique fixed point $f \in \mathbb{R}^{A^c}$ such that $H(f) = f$ and, for any $g \in \mathbb{R}^{A^c}$, it holds that

$$\lim_{n\to+\infty} H^n(g) = f\,. \tag{44}$$

It is easy to see that this unique fixed point is given by $\overline{h}|_{A^c}$. Indeed, from the choice of Q and the fact that $\overline{h}$ satisfies $\mathbb{I}_A\overline{h} = \mathbb{I}_{A^c} + \mathbb{I}_{A^c}\overline{Q}\,\overline{h}$, we have

$$\mathbb{I}_A\overline{h} = \mathbb{I}_{A^c} + \mathbb{I}_{A^c}Q\overline{h} = \mathbb{I}_{A^c} + \mathbb{I}_{A^c}\overline{Q}\,\overline{h}\,.$$

Moreover, since $\overline{h}|_A = 0$ we have $\mathbb{I}_A\overline{h} = 0$ and $\overline{h} = (\overline{h}|_{A^c})\!\uparrow_{\mathcal{X}}$, whence

$$\big(Q\,\overline{h}\big)|_{A^c} = \big(Q\,(\overline{h}|_{A^c})\!\uparrow_{\mathcal{X}}\big)|_{A^c} = G\,\overline{h}|_{A^c}\,.$$

Noting that $\mathbb{I}_{A^c}|_{A^c} = \mathbf{1}$, after multiplying with Δ we find that $\Delta\mathbf{1} + \Delta G\,\overline{h}|_{A^c} = 0$. Comparing with the definition of H, we have

$$H(\overline{h}|_{A^c}) = \overline{h}|_{A^c} + \Delta\mathbf{1} + \Delta G\,\overline{h}|_{A^c} = \overline{h}|_{A^c} + 0 = \overline{h}|_{A^c}\,,$$

so $\overline{h}|_{A^c}$ is indeed a fixed-point of H. Since we already established that H has a unique fixed-point, we conclude from Equation (44) that

$$\overline{h}|_{A^c} = \lim_{n\to+\infty} H^n(g) \quad \text{for all } g \in \mathbb{R}^{A^c}. \tag{45}$$

Next let us show that H is monotone. To this end, fix any $f, g \in \mathbb{R}^{A^c}$ such that $f \leq g$; then clearly also $f\!\uparrow_{\mathcal{X}} \leq g\!\uparrow_{\mathcal{X}}$. Since $\Delta > 0$ is such that $\Delta\|Q\| \leq 1$, it follows from [Krak, 2021, Prop 4.9] that $(I + \Delta Q)$ is a transition matrix. By the monotonicity of transition matrices [Krak, 2021, Prop 3.32], we find that therefore

$$(I+\Delta Q)f\!\uparrow_{\mathcal{X}} \leq (I+\Delta Q)g\!\uparrow_{\mathcal{X}}\,,$$

which in turn implies that

$$\begin{aligned}(I+\Delta G)f &= \big((I+\Delta Q)f\!\uparrow_{\mathcal{X}}\big)|_{A^c}\\ &\leq \big((I+\Delta Q)g\!\uparrow_{\mathcal{X}}\big)|_{A^c} = (I+\Delta G)g\,.\end{aligned}$$

From the definition of H we therefore conclude that $H(f) \leq H(g)$. Since f, g with $f \leq g$ are arbitrary, this concludes the proof of the monotonicity of H.

Now, let us define $\overline{H} : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$ for all $f \in \mathbb{R}^{\mathcal{X}}$ as

$$\overline{H}(f) := \Delta\mathbb{I}_{A^c} + \mathbb{I}_{A^c}(I + \Delta\overline{Q})f\,.$$

We first note that, since $\Delta\left\|\underline{Q}\right\| \leq 1$, it follows from [De Bock, 2017, Prop 5] that $(I + \Delta\underline{Q})$ is a lower transition operator. From the conjugacy of $\underline{Q}$ and $\overline{Q}$, we have for any $f \in \mathbb{R}^{\mathcal{X}}$ that

$$\begin{aligned}-(I+\Delta\underline{Q})(-f) &= f + \Delta\cdot -\underline{Q}(-f)\\ &= f + \Delta\overline{Q}f = (I+\Delta\overline{Q})f\,,\end{aligned}$$

which implies that $(I+\Delta\overline{Q})$ is the upper transition operator that is conjugate to the lower transition operator $(I+\Delta\underline{Q})$. By the monotonicity of upper transition operators—see Section 3.2—this implies that $\overline{H}$ is monotone, or in other words that for all $f, g \in \mathbb{R}^{\mathcal{X}}$ with $f \leq g$ it holds that $\overline{H}(f) \leq \overline{H}(g)$.

Let us next show that

$$\overline{H}(f) \geq H(f|_{A^c})\!\uparrow_{\mathcal{X}} \quad \text{for all } f \in \mathbb{R}^{\mathcal{X}} \text{ with } f|_A = 0. \tag{46}$$

Indeed, if $f|_A = 0$ then $(f|_{A^c})\!\uparrow_{\mathcal{X}} = f$, and since $Q \in \mathcal{Q}$, it follows from the definitions of $\overline{Q}$ and G that then

$$(\overline{Q}\,f)|_{A^c} \geq (Q\,f)|_{A^c} = (Q\,(f|_{A^c})\!\uparrow_{\mathcal{X}})|_{A^c} = Gf|_{A^c}\,.$$

Hence we have

$$\begin{aligned}\overline{H}(f)|_{A^c} &= \Delta\mathbf{1} + f|_{A^c} + \Delta(\overline{Q}\,f)|_{A^c}\\ &\geq \Delta\mathbf{1} + f|_{A^c} + \Delta Gf|_{A^c} = H(f|_{A^c})\,.\end{aligned}$$

Moreover, we immediately have from the definition that $\overline{H}(f)|_A = 0 = \big(H(f|_{A^c})\!\uparrow_{\mathcal{X}}\big)|_A$, and so Equation (46) indeed holds.

Next, we note that for any $f \in \mathbb{R}^{\mathcal{X}}$ it holds that $\overline{H}(f)|_A = 0$, and so by Equation (46) we find that

$$\overline{H}(\overline{H}(f)) \geq H(\overline{H}(f)|_{A^c})\!\uparrow_{\mathcal{X}}\,.$$

Provided that also $f|_A = 0$, then using the previously established monotonicity of H we obtain

$$\begin{aligned}\overline{H}(\overline{H}(f)) &\geq H(\overline{H}(f)|_{A^c})\!\uparrow_{\mathcal{X}}\\ &\geq H(H(f|_{A^c})\!\uparrow_{\mathcal{X}}|_{A^c})\!\uparrow_{\mathcal{X}}\,.\end{aligned}$$

We clearly have $H(f|_{A^c})\!\uparrow_{\mathcal{X}}|_{A^c} = H(f|_{A^c})$, whence

$$\overline{H}^2(f) = \overline{H}(\overline{H}(f)) \geq H(H(f|_{A^c}))\!\uparrow_{\mathcal{X}} = (H^2(f|_{A^c}))\!\uparrow_{\mathcal{X}}\,.$$

Indeed, we can repeat this reasoning for $n \in \mathbb{N}$ steps, to conclude that

$$\overline{H}^n(f) \geq (H^n(f|_{A^c}))\!\uparrow_{\mathcal{X}} \quad \text{for all } f \in \mathbb{R}^{\mathcal{X}} \text{ with } f|_A = 0. \tag{47}$$

Now suppose *ex absurdo* that there is some non-negative $g \in \mathbb{R}^{\mathcal{X}}$, such that $g(x) < \overline{h}(x)$ for some $x \in \mathcal{X}$, and such that $\mathbb{I}_A g = \mathbb{I}_{A^c} + \mathbb{I}_{A^c} \overline{Q}\, g$. Since g is non-negative and $\overline{h}|_A = 0$, we must have that $x \in A^c$. Moreover, we clearly have $\mathbb{I}_A g = 0$, which implies that $g|_A = 0$ and so $g = \mathbb{I}_{A^c} g$. Hence it follows that $\Delta \mathbb{I}_{A^c} + \mathbb{I}_{A^c} \Delta \overline{Q} g = 0$, and we find that $\overline{H}(g) = \mathbb{I}_{A^c} g = g$. Hence g is a fixed point of $\overline{H}$. Since $g|_A = 0$, and using Equation (47), this implies that for any $n \in \mathbb{N}_0$ we have

$$g = \overline{H}^n(g) \geq (H^n(g|_{A^c}))\uparrow_{\mathcal{X}} \ .$$

Recalling that $x \in A^c$ is such that $g(x) < \overline{h}(x)$, we use Equation (45) to take limits in n and find that

$$g(x) \geq \lim_{n \to +\infty} H^n(g|_{A^c})(x) = \overline{h}(x) > g(x) \,,$$

which is a contradiction. □

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