A New Near-linear Time Algorithm For $k$-Nearest Neighbor Search Using a Compressed Cover Tree

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Abstract

Given a reference set $R$ of $n$ points and a query set $Q$ of $m$ points in a metric space, this paper studies an important problem of finding $k$-nearest neighbors of every point $q \in Q$ in the set $R$ in a near-linear time. In the paper at ICML 2006, Beygelzimer, Kakade, and Langford introduced a cover tree on $R$ and attempted to prove that this tree can be built in $O(n \log n)$ time while the nearest neighbor search can be done in $O(n \log m)$ time with a hidden dimensionality factor. This paper fills a substantial gap in the past proofs but not on their sizes.

1. The Neighbor Search, Overview Of Results

In the modern formulation, the $k$-nearest neighbor problem is to find all $k \geq 1$ nearest neighbors in a given reference set $R$ for all points from another given query set $Q$.

Both sets belong to a common ambient space $X$ with a distance metric $d$ satisfying all metric axioms. The simplest example of $X$ is $\mathbb{R}^n$ with the Euclidean metric. A query set $Q$ can be a point or a finite subset of a reference set $R$.

The exact $k$-nearest neighbor problem asks for all true (exact) $k$-nearest neighbors in $R$ for every point $q \in Q$.

Another (probabilistic) version of the $k$-nearest neighbor search [Har-Peled & Mendel (2006); Manocha & Girolami (2007)] aims to find exact $k$-nearest neighbors with a given probability. The approximate version [Arya & Mount (1993); Krauthgamer & Lee (2004); Andoni et al. (2018); Wang et al. (2021)] of the nearest neighbor search looks for an $\epsilon$-approximate neighbor $r \in R$ of every query point $q \in Q$ such that $d(q, r) \leq (1 + \epsilon)d(q, NN(q))$, where $\epsilon > 0$ is fixed and $NN(q)$ is the exact first nearest neighbor of $q$.

Definition 1.1 (diameter and aspect ratio). For any finite set $R$ with a metric $d$, the diameter is $\text{diam}(R) = \max_{p, q \in R} |d(p, q)|$, The aspect ratio is $\Delta(R) = \text{diam}(R)\max_{p \in R} \min_{q \in R} |d(p, q)|$, where $\min_{p \in R} \min_{q \in R} |d(p, q)|$ is the shortest distance between points of $R$.

Definition 1.2 ($k$-nearest neighbor set $NN_k$). For any point $q \in Q$, let $d_1 \leq \cdots \leq d_{|R|}$ be ordered distances from $q$ to all points of a reference set $R$ whose size (number of points) is denoted by $|R|$. For any $k \geq 1$, the $k$-nearest neighbor set of every query point $q \in Q$ consists of all $u \in R$ with $d(q, u) \leq d_k$.

For $Q = R = \{0, 1, 2, 3\}$, the point $q = 1$ has ordered distances $d_1 = 0 < d_2 = 1 = d_3 < d_4 = 2$. The nearest neighbor sets are $NN_1(1; R) = \{1\}$, $NN_2(1; R) = \{0, 1, 2\} = NN_3(1; R)$, $NN_4(1; R) = R$. So 0 can be a 2nd neighbor of 1, then 2 becomes a 3rd neighbor of 1, or these neighbors of 0 can be found in a different order.

Problem 1.3 (all $k$-neighbors search). Let $Q, R$ be finite subsets of query and reference sets in a metric space $(X, d)$. For any fixed $k \geq 1$, design an algorithm to exactly find $k$ distinct points from $NN_k(q; R)$ for all $q \in Q$ so that the parametrized worst-case time complexity is near-linear in time $\max\{|Q|, |R|\}$, where hidden constants may depend on structures of $Q, R$ but not on their sizes $|Q|, |R|$.

In a metric space, let $\tilde{B}(p, t)$ be the closed ball with a center $p$ and a radius $t \geq 0$. The notation $|\tilde{B}(p, t)|$ denotes the number (if finite) of points in the closed ball. Definition 1.4 recalls the expansion constant $c$ from [Beygelzimer et al. (2006a)] and introduces the new minimized expansion constant $c_m$, which is a discrete analog of the doubling dimension [Cole & Gottlieb (2006)].

Definition 1.4 (expansion constants $c$ and $c_m$). A subset $R$ of a metric space $(X, d)$ is called locally finite if the set
The constant $c(R)$ is the smallest $c(R) \geq 2$ such that $|B(p, 2t)| \leq c(R) \cdot |B(p, t)|$ for any point $p \in R$ and $t \geq 0$, see [Beygelzimer et al. (2006a)].

Introduce the new minimized expansion constant $c_m(R) = \lim_{\xi \to 0^+} \inf_{R \subseteq X} \sup_{p \in A, t \geq \xi} \frac{|B(p, 2t) \cap A|}{|B(p, t) \cap A|}$, where $A$ is a locally finite set which covers $R$.

Lemma 1.5. For any finite sets $R \subseteq U$ in a metric space, we have that $c_m(R) \leq c_m(U)$ and $c_m(R) \leq c(R)$.

Note that both $c(R)$, $c_m(R)$ are always defined when $R$ is finite. We show below that a single outlier can make the expansion constant $c(R)$ as large as $O(|R|)$.

In the Euclidean line $\mathbb{R}$, the set $R = \{1, 2, \ldots, n, 2n + 1\}$ of $|R| = n + 1$ points has $c(R) = n + 1$ because $B(2n + 1; n) = (2n + 1)$ is a single point, while $B(2n + 1; 2n) = R$ is the full set of $n + 1$ points. On the other hand, the same set $R$ can be extended to a larger uniform set $A = \{1, 2, \ldots, 2n - 1, 2n\}$ whose expansion constant is $c(A) = 2$. So the minimized constant of the original set $R$ is much smaller: $c_m(R) \leq c(A) = 2 < c(R) = n + 1$.

The constant $c$ from [Beygelzimer et al. (2006a)] equals $2^{|\text{dim}_K|}$ from [Krauthgamer & Lee (2004) Section 2.1].

In [Krauthgamer & Lee (2004) Section 1.1] the doubling dimension $2^{\text{dim}}$ is defined as a minimum value $\rho$ such that any set $X$ can be covered by $2^\rho$ sets whose diameters are half of the diameter of $X$. The past work [Krauthgamer & Lee (2004)] proves that $2^{\text{dim}} \leq 2^\rho$ for any subset of $\mathbb{R}^n$.

Theorem 1.15 in appendix C will prove that $c_m(R) \leq 2^n$ for any a finite subset $R \subseteq \mathbb{R}^n$, so $c_m(R)$ mimics $2^{\text{dim}}$.

Navigating nets. In 2004, [Krauthgamer & Lee (2004) Theorem 2.7] claimed that a navigating net can be constructed in time $O(2^{O(\text{dim}_K \cdot \log |R| \cdot \log(\log |R|))}$ and all $k$-nearest neighbors of a query point $q$ can be found in time $O(2^{O(\text{dim}_K \cdot (\log |R|)))}$, where $\text{dim}_K(R \cup \{q\})$ is the expansion constant defined above. The paper above sketched a proof of [Krauthgamer & Lee (2004) Theorem 2.7] in one sentence and skipped pseudo-codes. Unfortunately, the authors didn’t reply to our request for these details.

Modified navigating nets. In 2006, [Cole & Gottlieb (2006)] were used in 2006 to claim the time $O(\log(n) + \frac{1}{\epsilon}O(1))$ to find $(1 + \epsilon)$-approximate neighbors. The proof and pseudo-code were skipped for this claim and for the construction of the modified navigating net for the claimed time $O(|R| \cdot \log(|R|))$.

Cover trees. In 2006, [Beygelzimer et al. (2006a)] introduced a cover tree inspired by the navigating nets [Krauthgamer & Lee (2004)]. This cover tree was designed to prove a worst-case time for the nearest neighbor search in terms of the size $|R|$ of a reference set $R$ and the expansion constant $c(R)$ from Definition 1.4. Assume that a cover tree is already constructed on the set $R$. Then [Beygelzimer et al. (2006a) Theorem 5] claimed that a nearest neighbor of any query point $q \in Q$ could be found in time $O(c(R)_{12} \cdot \log |R|)$.

In 2015, [Curtin (2015) Section 5.3] pointed out that the proof of [Beygelzimer et al. (2006a) Theorem 5] contains a crucial gap, now also confirmed by a specific example in [Elkin & Kurlin (2022a) Counterexample 5.2].

The time complexity result of the cover tree construction algorithm [Beygelzimer et al. (2006a) Theorem 6] had a similar issue, the gap of which is exposed rigorously in [Elkin & Kurlin (2022a) Counterexample 4.2].

Further studies in cover trees. A noteworthy paper on cover trees [Kollar (2006)] introduced a new probabilistic algorithm for the nearest neighbor search, as well as corrected the pseudo-code of the cover tree construction algorithm of [Beygelzimer et al. (2006a) Algorithm 2]. Later in 2015, a new, more efficient implementation of cover tree was introduced in [Izbicki & Shelton (2015)]. However, no new time-complexity results were proven.

Another study [Jahanseir & Sheehy (2016)] explored connections between modified navigating nets [Cole & Gottlieb (2006)] and cover trees [Beygelzimer et al. (2006a)].

Several papers [Beygelzimer et al. (2006b); Ram et al. (2009); Curtin et al. (2015)] studied the possibility of solving $k$-nearest neighbor Problem 1.3 by using cover trees on both sets $Q, R$, see [Elkin & Kurlin (2022a) Section 6].

New contributions. This work corrects the past gaps of the single-tree approach [Beygelzimer et al. (2006a), which were discovered in [Elkin & Kurlin (2022a)] by using a new compressed cover tree $T(R)$ from Definition 2.1, which can be constructed on any finite reference set $R$ with a metric $d$.

- Definition 2.1 introduces a new compressed cover tree.
- Theorem 3.6 and Corollary 3.10 estimate the time to build a compressed cover tree, which corrects the proof of [Beygelzimer et al. (2006a) Theorem 6].
- Theorem 4.9 and Corollary 4.7 estimate the time to find all $k$-nearest neighbors as in Problem 1.3. These advances correct and generalize [Beygelzimer et al. (2006a) Theorem 5].
A new compressed cover tree for \( k \)-nearest neighbors

### Table 1. Building data structures with hidden \( c_m(R) \) or dimensionality constant \( 2^{\dim} \)

<table>
<thead>
<tr>
<th>Data structure</th>
<th>claimed time complexity</th>
<th>space</th>
<th>proofs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Navigating nets [Definition 2.1]</td>
<td>( O(2^{O(\dim)} \cdot</td>
<td>R</td>
<td>\cdot \log(\Delta) \cdot \log(</td>
</tr>
<tr>
<td>Compressed cover tree</td>
<td>( O(c_m(R)^{O(1)} \cdot</td>
<td>R</td>
<td>\cdot \log(</td>
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### Table 2. Results for building data structures with the hidden classical expansion constant \( c(R) \)

<table>
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</tbody>
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### Table 3. Results for exact \( k \)-nearest neighbors of one query point \( q \in X \)

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>Navigating nets [Definition 2.1]</td>
<td>( O(2^{O(\dim)} \cdot (\log(</td>
<td>R</td>
<td>) + k)) ) for ( k \geq 1 )</td>
</tr>
<tr>
<td>Cover tree [Definition 2.1]</td>
<td>( O(c(R)^{O(1)} \cdot \log(</td>
<td>R</td>
<td>)) ) for ( k = 1 )</td>
</tr>
<tr>
<td>Compressed cover tree, Definition 2.1</td>
<td>( O(c(R \cup {q})^{O(1)} \cdot \log(k) \cdot \log(</td>
<td>R</td>
<td>) + k) )</td>
</tr>
</tbody>
</table>

### Table 4. Results for exact \( k \)-nearest neighbors of one point \( q \) using hidden \( c_m(R) \)

<table>
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<th>proofs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Navigating nets [Definition 2.1]</td>
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<td>\Delta</td>
<td>) + \log(\log(\Delta)))) )</td>
</tr>
<tr>
<td>Compressed cover tree, Definition 2.1</td>
<td>( O(\log(k) \cdot (c_m(R)^{O(1)} \cdot \log(</td>
<td>\Delta</td>
<td>) + \log(\log(\log(\Delta)))) )</td>
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2. A New Compressed Cover Tree

This section introduces in Definition 2.1 a new compressed cover tree to solve Problem 1.3. We also prove relevant properties of the expansion constant $c(R)$ and minimized expansion constant $c_m(R)$ of Definition 1.4. All extra details and proofs of this section are in Appendices B, D.

A compressed cover tree in Definition 2.1 will be significantly simpler than an explicit cover tree [Elkin & Kurlin 2022]. Definition 2.2), where any given point $p$ can appear in many different nodes simultaneously.

To regain the functionality of the explicit cover tree, we introduce the new concept of a distinctive descendant set $S_i(p, T(R))$ in Definition 2.8. See Figure 1 for a comparison between implicit, explicit, and compressed cover trees.

**Definition 2.1** (a compressed cover tree $T(R)$). Let $R$ be a finite set in a metric space $(X, d)$. A compressed cover tree $T(R)$ has the vertex set $R$ with a root $r \in R$ and a level function $l : R \to \mathbb{Z}$ satisfying the conditions below:

- **2.7b) Root condition**: the level of the root node $r$ satisfies $l(r) \geq 1 + \max_{p \in R \setminus \{r\}} l(p)$.

- **2.7c) Cover condition**: for every node $q \in R \setminus \{r\}$, we select a unique parent $p$ and a level $l(q)$ such that $d(q, p) \leq q^{l(q) + 1}$ and $l(q) < l(p)$; this parent node $p$ has a single link to its child node $q$.

- **2.7d) Separation condition**: for $i \in \mathbb{Z}$, the cover set $C_i = \{ p \in R \mid l(p) \geq i \}$ has $d_{\min}(C_i) = \min_{p \in C_i} \min_{q \in C_i \setminus \{p\}} d(p, q) > 2^i$.

Since there is a 1-1 map between $R$ and all nodes of $T(R)$, the same notation $p$ can refer to a point in the set $R$ or to a node of the tree $T(R)$. Set $l_{\max} = 1 + \max_{p \in R \setminus \{r\}} l(p)$ and $l_{\min} = \min_{p \in R \setminus \{r\}} l(p)$. For any node $p \in T(R)$, $\text{Children}(p)$ denotes the set of all children of $p$, including $p$ itself. For any node $p \in T(R)$, define the node-to-root path as a unique sequence of nodes $w_0, \ldots, w_m$ such that $w_0 = p$, $w_m$ is the root and $w_{j+1}$ is the parent of $w_j$ for $j = 0, \ldots, m - 1$.

A node $q \in T(R)$ is a descendant of a node $p$ if $p$ is in the node-to-root path of $q$. A node $p$ is an ancestor of $q$ if $q$ is in the node-to-root path of $p$. Let $\text{Descendants}(p)$ be the set of all descendants of $p$, including itself $p$.

**Lemma 2.2** links the minimized expansion constant with the doubling dimension. This result is used in the proofs of the width bound of a compressed cover tree in Lemma 2.3 and for the time complexity of a compressed cover tree construction in Lemma 3.3 and for the $k$-nearest neighbor search in Lemma 4.5. All hyperlinks are clickable.

**Lemma 2.2** (packing). Let $S$ be a finite $\delta$-sparse set in a metric space $(X, d)$, so $d(a, b) > \delta$ for all $a, b \in S$. Then, for any point $p \in X$ and any radius $t > \delta$, we have $|B(p, t) \cap S| \leq (c_m(S))^\mu$, where $\mu = \lceil \log_2 (\frac{4t}{\delta} + 1) \rceil$.

Proof of Lemma 2.2 is in Appendix B.

**Lemma 2.3** (width bound). Let $R$ be a finite subset of a metric space $(X, d)$. For any compressed cover tree $T(R)$,
Definition 1.4. If there exists a point which is located in annulus $S$, any finite set ($the height of a compressed cover tree$).

Consider a compressed cover tree $T(R)$ that was built on set $R = \{1, 2, 3, 4, 5, 7, 8\}$. Let $S_i(p, T(R))$ be a distinctive descendant set of Definition 2.8. Then $V_2(1) = \emptyset$, $V_1(1) = \{5\}$ and $V_0(1) = \{3, 5, 7\}$. And also $S_2(1, T(R)) = \{1, 2, 3, 4, 5, 7, 8\}$, $S_1(1, T(R)) = \{1, 2, 3, 4\}$ and $S_0(1, T(R)) = \{1\}$.

**Lemma 2.4** (growth bound). Let $(A, d)$ be a finite metric space, let $q \in A$ be an arbitrary point and let $r \in \mathbb{R}$ be a real number. Let $c(A)$ be the expansion constant from Definition 1.4. If there exists a point $p \in A such that $2r < d(p, q) \leq 3r$, then $|B(q, 4r)| \geq (1 + \frac{1}{c(A)^2}) \cdot |B(q, r)|$.

Lemma 2.5 is a generalization of Lemma 2.4 and will be used to estimate the number of iterations in compressed cover tree construction algorithm, Lemma 3.8 and in the $k$-nearest neighbors algorithm, Lemma 4.8.

**Lemma 2.5** (extended growth bound). Let $(A, d)$ be a finite metric space, let $q \in A$ be an arbitrary point. Let $p_1, ..., p_n$ be a sequence of distinct points in $R$, in such a way that for all $i \in \{2, ..., n\}$ we have $4 \cdot d(p_i, q) \leq d(p_{i+1}, q)$. Then $|B(q, \frac{4}{3} \cdot d(q, p_n))| \geq (1 + \frac{1}{c(A)^2})^n \cdot |B(q, \frac{1}{3} \cdot d(q, p_1))|$.

**Definition 2.6** (the height of a compressed cover tree). For a compressed cover tree $T(R)$ on a finite set $R$, the height set is $H(T(R)) = \{i \mid C_{i-1} \neq C_i\} \cup \{l_{\text{max}}, l_{\text{min}}\}$. The size $|H(T(R))|$ of this set is called the height of $T(R)$.

**Lemma 2.7.** Any finite set $R$ has the upper bound $|H(T(R))| \leq 1 + \log_2(\Delta(R))$.

Intuitively $S_i(p, T(R))$ denotes all the descendants of pair $(p, i)$ in the explicit or implicit cover tree.

**Definition 2.8** (Distinctive descendant sets). Let $R \subseteq X$ be a finite reference set with a compressed cover tree $T(R)$. For any node $p \in T(R)$ and level $i \leq l(p) - 1$, set $V_i(p) = \{u \in \text{Descendants}(p) \mid i \leq l(u) \leq l(p) - 1\}$. If $i \geq l(p)$, then set $V_i(p) = \emptyset$. For any node $i \leq l(p)$, the distinctive descendant set is $S_i(p, T(R)) = \text{Descendants}(p) \setminus \bigcup_{u \in V_i(p)} \text{Descendants}(u)$ and has the size $|S_i(p, T(R))|$. Lemma 2.9 shows that if $q \in S_i(p, T(R))$ then there is a node-to-node path $q = a_0, ..., a_m = p$, so that $l(a_m - 1) \leq i - 1$.

**Lemma 2.9.** Let $R \subseteq X$ be a finite reference set with a cover tree $T(R)$. In the notations of Definition 2.8 let $p \in T(R)$ be any node. If $w \in S_i(p, T(R))$ then either $w = p$ or there exists $a \in \text{Children}(p) \setminus \{p\}$ such that $l(a) < i$ and $w \in \text{Descendants}(a)$.

**Definition 2.10** explains the concrete implementation of a compressed cover tree.

**Definition 2.10** (Children $(p, i)$ and Next $(p, i, T(R))$). In a compressed cover tree $T(R)$ on a set $R$, for any level $i$ and a node $p \in R$, set $\text{Children}(p, i) = \{a \in \text{Children}(p) \mid l(a) = i\}$. Let $\text{Next}(p, i, T(R))$ be the maximal level $j$ satisfying $j < i$ and $\text{Children}(p, j) \neq \emptyset$. If such level does not exist, we set $j = l_{\text{min}}(T(R)) - 1$. For every node $p$, we store its set of children in a linked hash map so that

(a) any key $i$ gives access to $\text{Children}(p, i)$, 
(b) $\text{Children}(p, i) \to \text{Children}(p, \text{Next}(p, i, T(R)))$, 
(c) we can directly access $\max\{j \mid \text{Children}(p, j) \neq \emptyset\}$.

### 3. Construction Of A Compressed Cover Tree

This section discusses a construction of a compressed cover tree. New Algorithm 3.4 builds a compressed cover tree by using the Insert() method from Beygelzimer et al. (2006a, Algorithm 2), which was specifically adapted for a compressed cover tree, see details in Appendix E.
The proof of [Beygelzimer et al., Theorem 6], which estimated the time complexity of Algorithm 2, was shown to be incorrect by [Elkin & Kurlin, Counterexample 4.2]. The main contribution of this section estimate the time complexity of Algorithm 3.4.

Theorem 3.6 bounds the time complexity as 
\[ O(c_m(R)^8 \cdot \max_{y=1, ..., |R|-1} L(T(W_y), p_y) \cdot |R|) \]
where \( c_m(R) \) is the minimized expansion constant from Definition 2.1.

Algorithm 3.4 Building a compressed cover tree \( T(R) \) from Definition 2.1.

1. **Input**: a finite subset \( R \) of \((X, d)\), root \( r \in R \)
2. **Output**: a compressed cover tree \( T(R) \)
3. Build the initial compressed cover tree \( T = T(\{r\}) \) consisting of the root node \( r \) by setting \( l(r) = +\infty \).
4. For \( p \in R \setminus \{r\} \) do:
   5. \( T \leftarrow \) run AddPoint(\(T, p\), Algorithm 3.5)
5. End for
6. For the root \( r \) of \( T \) set \( l(r) = 1 + \max_{p \in R \setminus \{r\}} l(p) \)

Theorem 3.6 (time complexity of \( T(R) \) via aspect ratio). Let \( R \) be a finite subset of a metric space \((X, d)\) having the aspect ratio \( \Delta(R) \). Algorithm 2.4 builds a compressed cover tree \( T(R) \) in time \( O((c_m(R))^8 \cdot \log_2(\Delta(R)) \cdot |R|) \), where \( c_m(R) \) is the minimized expansion constant from Definition 2.1.

Algorithm 3.5 Building \( T(W \cup \{p\}) \) in lines 4-6 of Algorithm 3.4.

1. **Function**: AddPoint(a compressed cover tree \( T(W) \) with a root \( r \), a point \( p \in X \))
2. **Output**: compressed cover tree \( T(W \cup \{p\}) \)
3. Set \( i \leftarrow l_{\text{max}}(T(W)) - 1 \) and \( \eta(l_{\text{max}} - 1) = l_{\text{max}} \)
   - If the root \( r \) has no children then \( i \leftarrow -\infty \)
4. Set \( R_i \leftarrow \{r\} \)
5. While \( i \geq l_{\text{min}} \) do:
   6. \( V = \bigcup_{q \in R_i} \{a \in \text{Children}(q) \mid l(a) = i\} \)
   7. Assign \( C_i(R_{(q(i)}) \leftarrow R_{(q(i))} \cup V \).
   8. Set \( R_i = \{a \in C_i(R_{(q(i)}) \mid d(p, a) \leq 2^i+1 \}
   9. If \( R_i \) is empty then 
      10. Move to line 15
12. \( t = \max_{a \in R_i} \) Next(a, i, \( T(W) \))
   - If \( R_i \) has no children, then we set \( t = l_{\text{min}} - 1 \)
13. \( \eta(i) \leftarrow i \) and \( i \leftarrow t \)
14. End while
15. Pick \( v \in R_{(q(i)} \) minimizing \( d(p, v) \). Set \( l(p) = \lceil \log_2(d(p, v)) \rceil - 1 \) and define \( v \) to be the parent of \( p \) and exit.

Proof. In Lemma 3.3, use the bounds from Lemma 2.7
\[
\max_{y=2, ..., |R|} \frac{L(T(W_{y-1}), p_y)}{H(T(R))} \leq 1 + \log_2(\Delta(R)).
\]

**Lemma 3.7.** Let \((X, d)\) be a metric space and let \( W \subseteq X \) be its finite subset. Let \( q \in X \setminus W \) be an arbitrary point. Let \( i \in L(T(W), q) \) be an arbitrarily iteration of Definition 3.4. Assume that \( t = \eta(\eta(i+1)) \) is defined. Then there exists \( p \in W \) satisfying \( 2^{i+1} < d(p, q) \leq 2^{i+1} \).

**Lemma 3.8** (Construction iteration bound). Let \( A, W \) be finite subsets of a metric space \((X, d)\) satisfying \( W \subseteq A \subseteq X \). Take a point \( q \in A \setminus W \). Given a compressed cover tree \( T(W) \) on \( W \), Algorithm 3.5 runs lines 3-14 this number of times: \( |L(T(W), q)| = O(c(A)^2 \cdot \log_2(|A|)) \).

**Outline Proof.** Assume that Algorithm 3.5 was launched with parameters \((q, T(W))\). Lemma 3.7 showed that for any iterations \( i \in L(T(W), q) \), if \( t = \eta(\eta(i+1)) \) exists, then there exists \( p \in W \) which belongs to annulus \( B(q, 2^{i+1}) \cap B(q, 2^{i+1}) \). We can select a subsequence \( S \) of iterations \( L(T(W), q) \), in such a way that for every \( i \in S \) there exists point \( p_i \in B(q, 2^{i+1}) \setminus B(q, 2^{i+1}) \). It can be shown that the size of \( S \) selected this way is \( 12 \cdot |S| \geq |L(T(W), q)| \)

Denote by \( P = (p_1, ..., p_n) \) the sequence of points \( p_i \) obtained from \( S \). Using Lemma 2.5, we obtain
\[
|\hat{B}(q, \frac{4}{3} d(q, p_n))| \geq (1 + \frac{1}{c(R)^2})^n \cdot |\hat{B}(q, \frac{1}{3} d(q, p_1))|
\]
which can be written as
\[ |A| \geq \frac{|B(q, \frac{4}{3} \cdot d(q, p_n))|}{|B(q, \frac{1}{3} \cdot d(q, p_1))|} \geq (1 + \frac{1}{c(A)^2})^{|S|} \]

Lemma 3.7 gives \( c(A)^2 \log(A) \geq |S| \). Combining this with the fact that \( 12 \cdot |S| \geq |L(T(W), q)| \) we finally conclude that \( |L(T(W), q)| \leq 12 \cdot c(A)^2 \cdot \log_2(|A|) \).

**Theorem 3.9** (time for \( T(R) \) via expansion constants). Let \( R \) be a finite subset of a metric space \((X, d)\). Let \( A \) be a finite subset of \( X \) satisfying \( R \subseteq A \subseteq X \). Then Algorithm 3.4 builds a compressed cover tree \( T(R) \) in time \( O((c_m(R))^5 \cdot c(A)^2 \cdot \log_2(|A|) \cdot |R|) \), where the constants \( c(R), c_m(R) \) appeared in Definition 1.4.

**Proof.** It follows from Lemmas 3.8 and 3.3.

**Corollary 3.10.** Let \( R \) be a finite subset of a metric space \((X, d)\). Then Algorithm 3.4 builds a compressed cover tree \( T(R) \) in time \( O((c_m(R))^5 \cdot c(R)^2 \cdot \log_2(|R|)) \), where the constants \( c_m(R), c(R) \) appeared in Definition 1.4.

**Proof.** In Theorem 3.9 set \( A = R \).

4. New \( k \)-nearest Neighbor Search Algorithm

This section is motivated by Elkin & Kurlin [2022a] Counterexample 5.2, which showed that the proof of past time complexity claim in Beygelzimer et al. [2006a] Theorem 5) for the nearest neighbor search algorithm contained gaps. For extra details and all proofs, see Appendix F.

The gaps are filled by new Algorithm 4.3 for all \( k \)-nearest neighbors, which generalizes and improves the original method in Beygelzimer et al. [2006a] Algorithm 1).

The first improvement is the \( \lambda \)-point in line 7 which helps find all \( k \)-nearest neighbors of a given query point for any \( k \geq 1 \). The second improvement is a new break condition for the loop in line 9. This condition is used in the proof of Lemma 4.8 to conclude that the total number of performed iterations is bounded by \( O(c(R)^2 \log(|R|)) \) during the whole run-time of the algorithm.

The latter improvement corrects the past gap in proof of Beygelzimer et al. [2006a] Theorem 5) by bounding the number of iterations independently from the explicit depth Elkin & Kurlin [2022a] Definition 3.2).

Assuming that we have already constructed a compressed cover tree on a reference set \( S \), the two main results estimate the time complexity of a new \( k \)-nearest neighbor method in Algorithm 4.3 in which finds all \( k \)-nearest neighbors of any query point \( q \in X \) in a reference set \( R \subseteq X \) as follows:

- Corollary 4.7 bounds the time complexity as \( \hat{O} \left( \log_2(k) \cdot \log_2(\Delta(R)) \right) \), where \( \Delta(R) \) is the aspect ratio and \( c_m(R) \) is considered fixed (hence hidden).

- Theorem 4.9 bounds the time complexity as \( \hat{O} \left( \log_2(k) \cdot \log_2(|R| + k) \right) \), where the expansion constant \( c(R \cup \{q \}) \) is considered fixed (hence hidden).

**Definition 4.1** (\( \lambda \)-point). Fix a query point \( q \) in a metric space \((X, d)\) and fix any level \( i \in \mathbb{Z} \). Let \( T(R) \) be its compressed cover tree on a finite reference set \( R \subseteq X \). Let \( C \) be a subset of a cover set \( C_i \) from Definition 2.7 satisfying \( \sum_{p \in C} |S_i(p, T(R))| \geq k \), where \( S_i(p, T(R)) \) is the distinctive descendant set from Definition 2.8. For any \( k \geq 1 \), define \( \lambda_k(q, C) \) as a point \( \lambda \in C \) that minimizes \( d(q, \lambda) \) subject to \( \sum_{p \in N(q, \lambda)} |S_i(p, T(R))| \geq k \).

**Definition 4.2.** Let \( R \) be a finite subset of a metric space \((X, d)\). Let \( T(R) \) be a cover tree of Definition 2.1 built on \( R \) and let \( q \in X \) be arbitrary point. Let \( L(T(R), q) \) be the set of all levels \( i \) during iterations of lines 7-10 of Algorithm 4.3 and \( \tilde{L} \) at a level \( q \in L(T(R), q) \), then we say that \( q \) is special. Set \( \eta(i) = \min \{ t \in L(T(R), q) \mid t > i \} \).

**Algorithm 4.3** \( k \)-nearest neighbor search by a compressed cover tree

1: **Input**: compressed cover tree \( T(R) \), a query point \( q \in X \), an integer \( k \in \mathbb{Z}_+ \)
2: Set \( i \leftarrow l_{\text{max}}(T(R)) - 1 \) and \( \eta(l_{\text{max}} - 1) = l_{\text{max}} \)
3: Let \( r \) be the root node of \( T(R) \). Set \( R_{l_{\text{max}}} = \{ r \} \)
4: while \( i \geq l_{\text{min}} \) do
5: \( V = \cup_{q \in R_{\eta(i)}} \{ a \in \text{Children}(q) \mid l(a) = i \} \)
6: Assign \( C_i(R_{\eta(i)}) \leftarrow R_{\eta(i)} \cup V \)
7: \( \lambda = \lambda_k(q, C_i(R_{\eta(i)})) \) by Algorithm 4.8
8: \( R_t = \{ p \in C_i(R_{\eta(i)}) \mid d(q, p) \leq d(q, \lambda) + 2^{i+2} \} \)
9: if \( d(q, \lambda) > 2^{i+2} \) then
10: Collect the distinctive descendants \( S_i(p, T(R)) \) of all points \( p \in R \) in set \( S \), see Algorithm F.3
11: Compute and output \( k \)-nearest neighbors of the query point \( q \) from set \( S \).
12: end if
13: Set \( j \leftarrow \max_{a \in R} \text{Next}(a, i, T(R)) \)
14: if such \( j \) is undefined, we set \( j = l_{\text{min}} - 1 \)
15: Set \( \eta(j) \leftarrow i \) and \( i \leftarrow j \)
16: Compute and output \( k \)-nearest neighbors of query point \( q \) from the set \( R_{l_{\text{min}}} \).

**Theorem 4.4** (correctness of Algorithm 4.3). Algorithm 4.3 correctly finds all \( k \)-nearest neighbors of query point \( q \) within the reference set \( R \).
Lemma 4.5. Algorithm 4.3 has the following time complexities of its lines

(a) \( \max \{ \text{Line 9}, \text{Line 12}, \text{Line 15}, \text{Line 16} \} = O(c_m(R)^{10} \cdot \log_2(k)) \);

(b) \( \text{Line 8} = O( |B(q, 5d_k(q, R))| \cdot \log_2(k)) \).

Theorem 4.6. Let \( R \) be a finite set in a metric space \((X, d)\), \( c_m(R) \) be the minimized constant from Definition 1.4. Given a compressed cover tree \( \mathcal{T}(R) \), Algorithm 4.3 finds all \( k \)-nearest neighbors of a query point \( q \in X \) in time

\[
O \left( \log_2(k) \cdot \left( (c_m(R))^{10} \cdot |L(q, \mathcal{T}(R))| + |B(q, 5d_k(q, R))| \right) \right),
\]

where \( L(q, \mathcal{T}(R)) \) is the set of all performer iterations (lines 4-15) of Algorithm 4.3.

**Proof.** Apply Lemma 4.5 to estimate the time complexity of Algorithm 4.3

\[
O \left( |L(T(R), q)| \cdot (\text{Line 4-9} + \text{Line 12-15} + \text{Line 16}) \right).
\]

Corollary 4.7. Given a compressed cover tree \( \mathcal{T}(R) \), Algorithm 4.3 finds all \( k \)-nearest neighbors of \( q \) in time

\[
O \left( (c_m(R))^{10} \cdot \log_2(k) \cdot \log_2(\Delta(R)) + |B(q, 5d_k(q, R))| \cdot \log_2(k) \right).
\]

**Proof.** Replace \( |L(q, \mathcal{T}(R))| \) in the time complexity of Theorem 4.6 by its upper bound in Lemma 2.7

\[
|L(q, \mathcal{T}(R))| \leq |H(T(R))| \leq \log_2(\Delta(R)).
\]

Lemma 4.8 is proved similarly to Lemma 3.8. For full details see Appendix C.

Lemma 4.8. Algorithm 4.3 executes lines 4-15 the following number of times:

\[
|L(T(R), q)| = O(c(R \cup \{q\})^2 \cdot \log_2(|R|)).
\]

Theorem 4.9. Let \( R \) be a finite reference set in a metric space \((X, d)\). Let \( q \in X \) be a query point, \( c(R \cup \{q\}) \) be the expansion constant of \( R \cup \{q\} \) and \( c_m(R) \) be the minimized expansion constant from Definition 1.4. Given a compressed cover tree \( \mathcal{T}(R) \), Algorithm 4.3 finds all \( k \)-nearest neighbors of \( q \) in time

\[
O \left( (c_m(R))^{10} \cdot \log_2(k) \cdot \left( (c_m(R))^{10} \cdot \log_2(|R|) + c(R \cup \{q\}) \cdot k \right) \right).
\]

**Proof.** By Theorem 4.6, the required time complexity is

\[
O \left( (c_m(R))^{10} \cdot \log_2(k) \cdot |L(q, \mathcal{T}(R))| + |B(q, 5d(q, \beta))| \cdot \log_2(k) \right),
\]

for some \( \beta \) among the first \( k \)-nearest neighbors of \( q \). Apply Definition 1.4 to get the upper bound

\[
|B(q, 5d(q, \beta))| \leq (c(R \cup \{q\})^3 \cdot |B(q, 5d(q, \beta))| (1)
\]

Since \( |B(q, 5d(q, \beta))| \leq k \), we have \( |B(q, 5d(q, \beta))| \leq (c(R \cup \{q\})^3 \cdot k \). It remains to apply Lemma 4.8.

\[
|L(q, \mathcal{T}(R))| = O(c(R \cup \{q\})^2 \cdot \log_2|R|).
\]

5. Discussion Of Contributions and Next Steps

This paper rigorously proved the time complexity of the exact \( k \)-nearest neighbor search. The submission to ICML is strongly motivated by the past gaps in the proofs of time complexities in the highly cited Beygelzimer et al. (2006a, Theorem 5) at ICML, Ram et al. (2009) Theorem 3.1 at NIPS, and March et al. (2010) Theorem 5.1) at KDD.

Though Elkin & Kurlin (2022a) provided concrete counterexamples, no corrections were published. Main Theorem 4.9 and Corollary 3.10 finally filled all the gaps.

Since the past obstacles were caused by unclear descriptions and missed proofs, often without pseudo-codes, this paper necessarily fills in all technical details. Otherwise, future generations would continue citing unreliable results.

To overcome the discovered challenges, first Definition 1.2 and Problem 1.3 rigorously dealt with a potential ambiguity of \( k \)-nearest neighbors at equal distances. This singular case was unfortunately not discussed in the past work at all.

A new compressed cover tree in Definition 2.1 substantially simplified the navigating net Krauthgamer & Lee (2004) and original cover tree Beygelzimer et al. (2006a) by avoiding repetitions of given data points. This compression clarified the construction and search in Algorithms 3.4 and 4.3.

Sections 3 and 4 corrected the approach of Beygelzimer et al. (2006a) as follows. Assuming that the expansion constants and aspect ratio of a reference set \( R \) are fixed, Corollaries 3.10 and 4.9 rigorously showed that the time complexities are linear in the maximum size of \( R, Q \) and near-linear \( O(k \log k) \) in the number \( k \) of neighbors.

The library MLpack (Curtin et al. 2013) implemented a version of an explicit cover tree, which was later defined in Elkin & Kurlin (2022a, Counterexample 4.2). The implementation of a compressed cover tree is similar but conceptually simpler due to its easier structure in Fig. 4.

The new results justify that the MLpack implementations of the \( k \)-nearest neighbors search now have proved theoretical guarantees for a near-linear time complexity, which was practically important for the recent advances below.
Main Theorem 4.9 helped justify a near-linear time complexity for several invariants based on computing $k$-nearest neighbors in a new area of Geometric Data Science, whose aim is to build continuous geographic-style maps for moduli space of real data objects parametrized by complete invariants under practically important equivalence relations.

The key example is a finite cloud of unlabeled points up to isometry maintaining all inter-point distances. The most general isometry invariant SDD (Simplexwise Distance Distribution [Kurlin 2023]) is conjectured to be complete for any finit point clouds in any metric space.

In a Euclidean space $\mathbb{R}^n$, the SDD was adapted to the stronger invariant SCD (Simplexwise Centered Distribution [Widdowson & Kurlin 2022]), whose completeness and polynomial complexity (in the number $m$ of points for a fixed dimension $n$) was proved in [Kurlin 2023b].

These density functions have efficient algorithms in the low dimensions $n = 2, 3$ through higher-degree Voronoi domains [Smith & Kurlin 2022] of periodic point sets.

The first continuous and complete invariant for periodic point sets in $\mathbb{R}^n$ is the isoset of local atomic environments up to a justified stable radius [Anosova & Kurlin 2021]. The first continuous metric on isosets was introduced in [Widdowson & Kurlin 2021] with an approximate algorithm that has a polynomial time complexity (for a fixed dimension $n$) and a small approximation factor (about $3$ in $\mathbb{R}^3$).

The much faster generically complete isometry invariant for both finite and periodic sets of points is the PDD (Pointwise Distance Distribution [Widdowson & Kurlin 2021]) consisting of distances to $k$ nearest neighbors per point.

The implemented search for atomic neighbors was so fast that all (more than 660 thousand) periodic crystals in the world’s largest database of real materials were hierarchically compared by the PDD and its simplified version AMD (Average Minimum Distance [Widdowson et al. 2022]).

Due to the ultra-fast running time, more than 200 billion pairwise comparisons were completed over two days on a modest desktop while past tools were estimated to require over 34 thousand years [Widdowson & Kurlin 2022].

The most important conclusion from the search results is the Crystal Isometry Principle saying that any real periodic crystal has a uniquely defined location in a single continuous space of all isometry classes of periodic point sets [Widdowson & Kurlin 2022].

This Crystal Isometry Space contains all known and not yet discovered crystals similar to the much simpler and discrete Mendeleev’s table of chemical elements.

The next step is to improve the complexity of the $k$-nearest neighbor search to a purely linear time $O(c(R)^{(1)}|R|)$ with no other extra hidden parameters by using a new compressed cover tree on both sets $Q, R$.

Since a similar approach Ram et al. (2009) was shown to have incorrect proof in Elkin & Kurlin (2022a Counterexample 6.5) and Curtin et al. (2015) used some additional parameters $I, \theta$, this goal will require significantly more effort to understand if $O(c(R)^{(1)}|R|)$ is achievable by using a compressed cover tree.

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A new compressed cover tree for \( k \)-nearest neighbors


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A new compressed cover tree for $k$-nearest neighbors

The appendices below contain the full version of the paper with detailed proofs and pseudo codes.

A. The $k$-nearest neighbor search and overview of results

In the modern formulation, $k$-nearest neighbors problem intends to discover all $k \geq 1$ nearest neighbors in a given reference set $R$ for all points from another given query set $Q$. Both sets belong to a common ambient space $X$ with a distance $d$ satisfying all metric axioms. The simplest example of $X$ is $\mathbb{R}^n$ with the Euclidean metric. A query set $Q$ can be a single point or a subset of a larger reference set $R$.

The exact $k$-nearest neighbor problem asks for all true (non-approximate) $k$-nearest neighbors in $R$ for every query point $q \in Q$. Another probabilistic version of the $k$-nearest neighbor search\cite{Har-Peled2006, Manocha2007} aims to find exact $k$-nearest neighbors with a given probability. The probabilistic $k$-nearest neighbor problem can be simplified to $k$ instances of 1-nearest-neighbors problem by splitting $R$ into $k$ subsets $R_1, ..., R_k$ and searching for nearest neighbors in each subset. The approximate version\cite{Arya1993, Krauthgamer2004, Andoni2018, Wang2021} of the nearest neighbor search looks for an $\epsilon$-approximate nearest neighbor $r \in R$ of every query point $q \in Q$ such that $d(q, r) \leq (1 + \epsilon)d(q, \text{NN}(q))$, where $\epsilon > 0$ is fixed and $\text{NN}(q)$ is the exact first nearest neighbor of $q$.

**Spacial data structures.** It is well known that the time complexity of a brute-force approach of finding all 1st nearest neighbors of points in the closed ball. Definition 1.4 recalls the expansion constant $\Delta(\cdot)$ of the nearest neighbor search looks for an $\epsilon$-approximate nearest neighbor $r \in R$ of every query point $q \in Q$ such that $d(q, r) \leq (1 + \epsilon)d(q, \text{NN}(q))$, where $\epsilon > 0$ is fixed and $\text{NN}(q)$ is the exact first nearest neighbor of $q$.

**Definition 1.1** (diameter and aspect ratio). For any finite set $R$ with a metric $d$, the diameter is $\text{diam}(R) = \max_{p \in R} \max_{q \in R} d(p, q)$. The aspect ratio is $\Delta(R) = \frac{\text{diam}(R)}{d_{\text{min}}(R)}$ where $d_{\text{min}}(R)$ is the shortest distance between points of $R$.

**Definition 1.2** ($k$-nearest neighbor set $\text{NN}_k$). For any point $q \in Q$, let $d_1 \leq \cdots \leq d_{|R|}$ be ordered distances from $q$ to all points of a reference set $R$ whose size (number of points) is denoted by $|R|$. For any $k \geq 1$, the $k$-nearest neighbor set $\text{NN}_k(q; R)$ consists of all $u \in R$ with $d(q, u) \leq d_k$.

For $Q = R = \{0, 1, 2, 3\}$, the point $q = 1$ has ordered distances $d_1 = 0 < d_2 = 1 = d_3 < d_4 = 2$. The nearest neighbor sets are $\text{NN}_1(1; R) = \{1\}$, $\text{NN}_2(1; R) = \{0, 1, 2\} = \text{NN}_3(1; R)$, $\text{NN}_4(1; R) = R$. So 0 can be a 2nd neighbor of 1, then 2 becomes a 3rd neighbor of 1, or these neighbors of 0 can be found in a different order.

**Problem 1.3** (all $k$-nearest neighbors search). Let $Q, R$ be finite subsets of query and reference sets in a metric space $(X, d)$. For any fixed $k \geq 1$, design an algorithm to exactly find $k$ distinct points from $\text{NN}_k(q; R)$ for all $q \in Q$ so that the parametrized worst-case time complexity is near-linear in time $\max(|Q|, |R|)$, where hidden constants may depend on structures of $Q, R$ but not on their sizes $|Q|, |R|$.

In a metric space, let $\bar{B}(p, t)$ be the closed ball with a center $p$ and a radius $t \geq 0$. The notation $|\bar{B}(p, t)|$ denotes the number (if finite) of points in the closed ball. Definition 1.4 recalls the expansion constant $c$ from\cite{Beygelzimer2006a} and introduces the new minimized expansion constant $c_m$, which is a discrete analog of the doubling dimension\cite{Cole2006}.
Definition 1.4 (expansion constants c and \(c_m\)). A subset \(R\) of a metric space \((X, d)\) is called locally finite if the set \(B(p, t) \cap R\) is finite for all \(p \in X\) and \(t \in \mathbb{R}^+\). Let \(R\) be a locally finite set in a metric space \(X\).

The expansion constant \(c(R)\) is the smallest \(c(R) \geq 2\) such that \(|\bar{B}(p, 2t)| \leq c(R) \cdot |\bar{B}(p, t)|\) for any point \(p \in R\) and \(t \geq 0\), see [Beygelzimer et al., 2006a].

Introduce the new minimized expansion constant \(c_m(R) = \lim_{\xi \to 0^+} \inf_{R \subseteq A \subseteq X} \sup_{p \in A, t > \xi} \frac{|\bar{B}(p, 2t) \cap A|}{|\bar{B}(p, t) \cap A|} \), where \(A\) is a locally finite set which covers \(R\).

Lemma 1.5. For any finite sets \(R \subseteq U\) in a metric space, we have that \(c_m(R) \leq c_m(U)\) and \(c_m(R) \leq c(R)\).

Proof. Let us first prove that \(c_m(R) \leq c_m(U)\). Let \(\epsilon > 0\) be arbitrary real number. By definition of \(c_m(U)\) there exists set \(\xi > 0\) and set \(A\) satisfying \(U \subseteq A\) for which

\[
\sup_{p \in A, t > \xi} \frac{|\bar{B}(p, 2t) \cap A|}{|\bar{B}(p, t) \cap A|} - c_m(U) \leq \epsilon
\]

(2)

Since \(R \subseteq U\) we have \(R \subseteq A\) therefore we can choose the same \(\xi\) and set \(U\) which satisfy inequality (2). Therefore it follows \(c_m(R) \leq c_m(U) + \epsilon\). Since \(\epsilon\) was chosen arbitrarily it follows that \(c_m(R) \leq c_m(U)\).

To prove that \(c_m(R) \leq c(R)\), note that \(\sup_{p \in A, t > \xi} \frac{|\bar{B}(p, 2t) \cap A|}{|\bar{B}(p, t) \cap A|} \leq \sup_{p \in A, t > 0} \frac{|\bar{B}(p, 2t) \cap A|}{|\bar{B}(p, t) \cap A|}\). Then by choosing \(\xi = \frac{d_{\min}(R)}{4}\) and \(A = R\) we have:

\[
c_m(R) \leq \sup_{p \in R, t > 0} \frac{|\bar{B}(p, 2t) \cap R|}{|\bar{B}(p, t) \cap R|} - c_m(U) = c(R)
\]

Note that both \(c(R), c_m(R)\) are always defined when \(R\) is finite. We will show that a single outlier can make the expansion constant \(c(R)\) as large as \(O(|R|)\). The set \(R = \{1, 2, \ldots, n, 2n + 1\}\) of \(|R| = n + 1\) points has \(c(R) = n + 1\) because \(\bar{B}(2n + 1; n) = \{2n + 1\}\) is a single point, while \(\bar{B}(2n + 1; 2n) = R\) is the full set of \(n + 1\) points. On the other hand the same set \(R\) can be extended to a larger uniform set \(A = \{1, 2, \ldots, 2n - 1, 2n\}\) whose expansion constant \(c(A) = 2\), therefore the minimized constant of the original set \(R\) becomes much smaller: \(c_m(R) \leq c(A) = 2 < c(R) = n + 1\).

The constant \(c\) from [Beygelzimer et al., 2006a] equals to \(2^{2\dim_K R}\) from [Krauthgamer & Lee, 2004 Section 2.1). In [Krauthgamer & Lee, 2004 Section 1.1) the doubling dimension \(2^{\dim}\) is defined as a minimum value \(\rho\) such that any set \(X\) can be covered by \(2^\rho\) sets whose diameters are half of the diameter of \(X\). The past work [Krauthgamer & Lee, 2004] proves that \(2^{\dim} \leq 2^n\) for any subset of \(\mathbb{R}^n\). Theorem C.15 will prove that \(c_m(R) \leq 2^n\) for any a finite subset \(R \subset \mathbb{R}^n\), so \(c_m(R)\) mimics \(2^{\dim}\).

Navigating nets. In 2004, [Krauthgamer & Lee, 2004 Theorem 2.7] claimed that a navigating net can be constructed in time \(O(2^{d_{\min_K R}} |R| \log |R| \log(\log |R|))\) and all \(k\)-nearest neighbors of a query point \(q\) can be found in time \(O(2^{d_{\min_K R}} (R \cup \{q\})) (k + \log |R|)\), where \(d_{\min_K R} (R \cup \{q\})\) is the expansion constant defined above. All proofs and pseudo-codes were omitted. The authors didn’t reply to our request for details.

Modified navigating nets [Cole & Gottlieb, 2006] were used in 2006 to claim the time \(O(\log(n) + (1/e)^{O(1)})\) for the \((1 + e)\)-approximate neighbors. All proofs and pseudo-codes were left out, also for the construction of the modified navigating net for the claimed time \(O(|R| \cdot \log(|R|))\).

Cover trees. In 2006, [Beygelzimer et al., 2006a] introduced a cover tree inspired by the navigating nets [Krauthgamer & Lee, 2004]. This cover tree was designed to prove a worst-case bound for the nearest neighbor search in terms of the size \(|R|\) of a reference set \(R\) and the expansion constant \(c(R)\) of Definition 1.4. Assume that a cover tree is already constructed on set \(R\). Then [Beygelzimer et al., 2006a Theorem 5] claims that nearest neighbor of any query point \(q\) could be found in time \(O(c(R)^{12} \cdot \log |R|)\). In 2015, [Curtin, 2015 Section 5.3] pointed out that the proof of [Beygelzimer et al., 2006a Theorem 5]
A new compressed cover tree for \(k\)-nearest neighbors

This work corrects the past gaps of the single-tree approach via an original cover tree \([\text{Beygelzimer et al.}(2006a)](\text{Beygelzimer et al.}(2006a))\) by using a new compressed cover tree \(T(R)\) from Definition 2.1 which can be constructed on any finite reference set \(R\) with a metric \(d\). Theorem 3.9 will prove that a compressed cover tree \(T(R)\) can be built in time \(O(c_m(R)^8 \cdot c(R)^2 \cdot \log_2(|R|) \cdot |R|)\).

The past gap in the proof of the time complexity \([\text{Beygelzimer et al.}(2006a)](\text{Beygelzimer et al.}(2006a))\) Theorem 1) for nearest neighbor search is tackled by new Algorithm F.2 which add an essential block to the original code in \([\text{Beygelzimer et al.}(2006a)](\text{Beygelzimer et al.}(2006a))\) Algorithm 1). The extra block eliminates the issue of having too many successive iterations when a query point \(q\) is disproportionately far away from the remaining candidate set \(R_i\) on some level \(i\). Then Lemma 4.8 shows that the number of iterations of Algorithm F.2 is bounded by \(O(c(R)^2 \log_2(|R|))\). This new lemma replaces the old result \([\text{Beygelzimer et al.}(2006a)](\text{Beygelzimer et al.}(2006a))\) Lemma 4.3), which had a similar bound for the number of explicit levels of a cover tree, for further information see \([\text{Elkin & Kurlin}(2022a)](\text{Elkin & Kurlin}(2022a))\) Definition 3.2) The old result cannot be used to estimate the number of iterations of \([\text{Beygelzimer et al.}(2006a)](\text{Beygelzimer et al.}(2006a))\) Algorithm 1) due to \([\text{Elkin & Kurlin}(2022a)](\text{Elkin & Kurlin}(2022a))\) Counterexample 5.2).

Figure 3. **Left:** an implicit cover tree from \([\text{Beygelzimer et al.}(2006a)](\text{Beygelzimer et al.}(2006a))\) Section 2) at ICML 2006 for a finite set of reference points \(R = \{1, 2, 3, 4, 5\}\) with the Euclidean distance \(d(x, y) = |x - y|\). **Right:** a new compressed cover tree in Definition 2.1 corrects the past worst-case complexity for \(k\)-nearest neighbors search in \(R\).

Further studies in cover trees. A noteworthy paper on cover trees \([\text{Kollar}(2006)](\text{Kollar}(2006))\) introduced a new probabilistic algorithm for the nearest neighbor search, as well as corrected the pseudo-code of the cover tree construction algorithm of \([\text{Beygelzimer et al.}(2006a)](\text{Beygelzimer et al.}(2006a))\) Algorithm 2). Later in 2015, a new, more efficient implementation of cover tree was introduced in \([\text{Izbicki & Shelton}(2015)](\text{Izbicki & Shelton}(2015))\). However, no new time-complexity results were proven. A study \([\text{Jahanseir & Sheehy}(2016)](\text{Jahanseir & Sheehy}(2016))\) explored connections between modified navigating nets \([\text{Cole & Gottlieb}(2006)](\text{Cole & Gottlieb}(2006))\) and cover trees \([\text{Beygelzimer et al.}(2006a)](\text{Beygelzimer et al.}(2006a))\). Multiple papers \([\text{Beygelzimer et al.}(2006b)](\text{Beygelzimer et al.}(2006b))\), \([\text{Ram et al.}(2009)](\text{Ram et al.}(2009))\), \([\text{Curtin et al.}(2015)](\text{Curtin et al.}(2015))\) studied possibility of solving \(k\)-nearest neighbor problem (Problem 1.3) by using cover tree on both, the query set and the reference set, for further details see \([\text{Elkin & Kurlin}(2022a)](\text{Elkin & Kurlin}(2022a))\) Section 6).

The main contributions are the following.

- Definition 2.1 introduces a compressed cover tree.
- Theorem 3.6 and Corollary 3.10 estimate the time to build a compressed cover tree.
- Theorem 4.9 and Corollary 4.7 estimate the time to find all \(k\)-nearest neighbors as in Problem 1.3
- Theorem G.6 estimates the time complexity of approximate \(k\)-nearest neighbor search.
A new compressed cover tree for $k$-nearest neighbors

Assume that a compressed cover tree $T(R)$ is already constructed on a reference set $R$. Our first main Theorem 4.9 shows that $k$-nearest neighbors of a query node $q$ can be found in time of

$$O\left((c(R \cup \{q\}))^2 \cdot \log_2(k) \cdot \left((c_m(R))^1 \cdot \log_2(|R|) + c(R \cup \{q\}) \cdot k\right) \right).$$

Recall that $c(R)$ can potentially become as large as $O(|R|)$ when $R$ is not uniformly distributed. Our second main Corollary 4.7 estimates the time complexity of the new $k$-nearest neighbor search by using only the minimized expansion constant $c_m(R)$ of Definition 1.4 and the aspect ratio $\Delta(R)$ of Definition 1.1 as parameters. These parameters are less dependent on the point distribution (or noise) in the sets $R, Q$. In many cases, $\Delta(R)$ is relatively small and $c_m(R)$ depends mostly on the dimension of the ambient space $X$. It is shown that $k$-nearest neighbors of $q$ in a reference set $R$ can be found in time

$$O\left((c_m(R))^1 \cdot \log_2(k) \cdot \log_2(\Delta(R)) + |\hat{B}(q, 5d_k(q, R))| \cdot \log_2(k) \right),$$

where $d_k(q, R)$ is the distance from $q$ to its $k$th nearest neighbor. Tables 7-8 summarize past and new results.

**Table 5. Results for building data structures with hidden classic expansion constant $c(R)$ of Definition 1.4 or KR-type constant $2^{\dim KR}$**

<table>
<thead>
<tr>
<th>Data structure</th>
<th>claimed time complexity</th>
<th>space</th>
<th>proofs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Navigating nets</td>
<td>$O(2^O(dim KR \cdot</td>
<td>R</td>
<td>\log(</td>
</tr>
<tr>
<td>Cover tree Beygelzimer et al. (2006a)</td>
<td>$O(c(R)^O(1) \cdot</td>
<td>R</td>
<td>\cdot \log</td>
</tr>
<tr>
<td>Compressed cover tree</td>
<td>$O(c(R)^O(1) \cdot</td>
<td>R</td>
<td>\cdot \log</td>
</tr>
<tr>
<td>[dfn 2.1]</td>
<td></td>
<td>Lemma B.1</td>
<td></td>
</tr>
</tbody>
</table>

**Table 6. Results for exact $k$-nearest neighbors of one query point $q \in X$ using hidden classic expansion constant $c(R)$ of Definition 1.4 or KR-type constant $2^{\dim KR}$**

<table>
<thead>
<tr>
<th>Data structure</th>
<th>claimed time complexity</th>
<th>space</th>
<th>proofs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Navigating nets</td>
<td>$O(2^O(dim KR \cdot (\log(</td>
<td>R</td>
<td>) + k)))$ for $k \geq 1$ Krauthgamer &amp; Lee (2004, Theorem 2.7)</td>
</tr>
<tr>
<td>Krauthgamer &amp; Lee (2004)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cover tree Beygelzimer et al. (2006a)</td>
<td>$O(c(R)^O(1) \log</td>
<td>R</td>
<td>)$ for $k = 1$ Beygelzimer et al. (2006a, Theorem 5)</td>
</tr>
<tr>
<td>Compressed cover tree, Definition 2.1</td>
<td>$O(c(R)^O(1) \cdot \log(k) \cdot (\log(</td>
<td>R</td>
<td>) + k))$</td>
</tr>
</tbody>
</table>
### B. Compressed cover tree

This section introduces in Definition 2.1 a new compressed cover tree, which will be used to solve Problem 1.3. Other important results are Lemmas 2.2 and 2.4. Given a \( \delta \)-sparse finite metric space \( R \), Lemma 2.2 shows that the number of points of \( R \) in the closed ball \( B(p, t) \) has the upper bound \( c_m(S)^\mu \), where \( \mu \) depends on \( \delta \). Lemma 2.4 will imply that if there are points \( p, q \) in a finite metric space \( R \) satisfying \( 2r < d(p, q) \leq 3r \) for some \( r \in \mathbb{R} \), then \( |B(q, 4r)| \geq (1 + \frac{1}{4r^\delta})|B(q, r)| \).

**Definition 2.1** (a compressed cover tree \( T(R) \)). Let \( R \) be a finite set in a metric space \( (X, d) \). A compressed cover tree \( T(R) \) has the vertex set \( R \) with a root \( r \in R \) and a level function \( l : R \to \mathbb{Z} \) satisfying the conditions below.  

\[ \text{Root condition: the level of the root node } r \text{ satisfies } l(r) \geq 1 + \max_{p \in R \setminus \{r\}} l(p). \]

\[ \text{Cover condition: for every node } q \in R \setminus \{r\}, \text{ we select a unique parent } p \text{ and a level } l(q) \text{ such that } d(q, p) \leq 2^{(q) + 1} \text{ and } l(q) < l(p); \text{ this parent node } p \text{ has a single link to its child node } q. \]

\[ \text{Separation condition: for } i \in \mathbb{Z}, \text{ the cover set } C_i = \{ p \in R \mid l(p) \geq i \} \text{ has } d_{\min}(C_i) = \min_{p \in C_i} \min_{q \in C_i \setminus \{p\}} d(p, q) > 2^i. \]

Since there is a \( 1-1 \) map between \( R \) and all nodes of \( T(R) \), the same notation \( p \) can refer to a point in the set \( R \) or to a node of the tree \( T(R) \). Set \( l_{\max} = 1 + \max_{p \in R \setminus \{r\}} l(p) \) and \( l_{\min} = \min_{p \in R} l(p) \). For any node \( p \in T(R) \), \( \text{Children}(p) \) denotes the set of all children of \( p \), including \( p \) itself. For any node \( p \in T(R) \), define the node-to-root path as a unique sequence of nodes \( w_0, \ldots, w_m \) such that \( w_0 = p \) and \( w_m \) is the root and \( w_{j+1} \) is the parent of \( w_j \) for \( j = 0, \ldots, m - 1 \).

A node \( q \in T(R) \) is a descendant of a node \( p \) if \( p \) is in the node-to-root path of \( q \). A node \( p \) is an ancestor of \( q \) if \( q \) is in the node-to-root path of \( p \). Let \( \text{Descendants}(p) \) be the set of all descendants of \( p \), including itself \( p \).

**Lemma B.1** (Linear space of \( T(R) \)). Let \( (R, d) \) be a finite metric space. Then any cover tree \( T(R) \) from Definition 2.1 takes \( O(|R|) \) space.

**Proof.** Since \( T(R) \) is a tree, both its vertex set and its edge set contain at most \( |R| \) nodes. Therefore \( T(R) \) takes at most \( O(|R|) \) space. \( \square \)
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Example B.2 ($\mathcal{T}(R)$ in Fig. 5). Let $(\mathbb{R},d=|x-y|)$ be the real line with euclidean metric. Let $R=\{1,2,3,\ldots,15\}$ be its finite subset. Fig. 5 shows a compressed cover tree on the set $R$ with the root $r=8$. The cover sets of $\mathcal{T}(R)$ are $C_{-1}=\{1,2,3,\ldots,15\}$, $C_0=\{2,4,6,8,10,12,14\}$, $C_1=\{4,8,12\}$ and $C_2=\{8\}$. We check the conditions of Definition 2.1:

- Root condition (2.1a): since $\max_{p\in R\setminus\{8\}}d(p,8)=7$ and $\left\lceil \log_2(7) \right\rceil - 1 = 2$, the root can have the level $l(8)=2$.
- Covering condition (2.1b): for any $i \in (-1,0,1,2)$, let $p_i$ be arbitrary point having $l(p_i)=i$. Then we have $d(p_{-1},p_0)=1 \leq 2^0$, $d(p_0,p_1)=2 \leq 2^1$ and $d(p_1,p_2)=4 \leq 2^2$.
- Separation condition (2.1c): $d_{\min}(C_{-1})=1 > \frac{1}{2} = 2^{-1}$, $d_{\min}(C_0)=2 > 1 = 2^0$, $d_{\min}(C_1)=4 > 2 = 2^1$.

A cover tree was defined in Beygelzimer et al. (2006a, Section 2) as a tree version of a navigating net from Krauthgamer & Lee (2004, Section 2.1). For any index $i \in \{0,1,2\}$, the level $i$ set of this cover tree coincides with the cover set $C_i$ above, which can have nodes at different levels in Definition 2.1. Any point $p \in C_i$ has a single parent in the set $C_{i+1}$, which satisfies conditions (2.1b,c). Beygelzimer et al. (2006a, Section 2) referred to this original tree as an implicit representation of a cover tree. Such a tree in Figure 6 (left) contains infinitely many repetitions of every point $p \in R$ in long branches and will be called an implicit cover tree.

Since an implicit cover tree is formally infinite, for practical implementations, the authors of Beygelzimer et al. (2006a) had to use another version that they named an explicit representation of a cover tree. We call this version an explicit cover tree. Here is the full defining quote at the end of Beygelzimer et al. (2006a, Section 2): "The explicit representation of the tree coalesces all nodes in which the only child is a self-child". In an explicit cover tree, if a subpath of every node-to-root path consists of all identical nodes without other children, all these identical nodes collapse to a single node, see Figure 6 (middle).

Since an explicit cover tree still contains repeated points, Definition 2.1 is well-motivated by the aim to include every point only once, which saves memory and simplifies all subsequent algorithms, see Fig. 6 (right).

Example B.3 (a short train line tree). Let $G$ be the unoriented metric graph consisting of two vertices $r,q$ connected by three different edges $e,h,g$ of lengths $|e|=2^0$, $|h|=2^1$, $|g|=1$. Let $p_4$ be the middle point of the edge $e$. Let $p_1$ be the middle point of the subedge $(p_4,r)$. Let $p_2$ be the middle point of the edge $h$. Let $p_3$ be the middle point of the subedge $(p_3,q)$. We construct a compressed cover tree $\mathcal{T}(R)$ by choosing the level $l(p_i)=i$ and by setting the root $r$ to be the parent of both $p_2$ and $p_4$, $p_4$ to be the parent of $p_3$, and $p_2$ to be the parent of $p_1$. Then $\mathcal{T}(R)$ satisfies all the conditions of Definition 2.1: see a comparison of the three cover trees in Fig. 6. 

![Figure 4. Compressed cover trees $\mathcal{T}(R)$ from Definition 2.1 for $R=\{0,1,2^i\}$.](image)

![Figure 5. Compressed cover tree $\mathcal{T}(R)$ on the set $R$ in Example B.2 with root 16.](image)
Proof. Let \( \{w_0, ..., w_m\} \) be a subpath of the node-to-root path for \( w_0 = q, w_{m-1} = u, w_m = p \). Then \( d(w_i, w_{i+1}) \leq 2^{l(p) + 1} \).
Figure 8. This volume argument proves Lemma 2.2. By using an expansion constant, we can find an upper bound for the number of smaller balls of radius $\frac{\delta}{2}$ that can fit inside a larger $B(p, t)$.

$2^{l(u)+1}$ for any $i$. The first required inequality follows from the triangle inequality below:

\[
d(p, q) \leq \sum_{j=0}^{m-1} d(w_j, w_{j+1}) \leq \sum_{j=0}^{m-1} 2^{l(w_j)+1} \leq \sum_{t=t_{\min}}^{l(u)+1} 2^t \leq 2^{l(p)+1}
\]

Finally, $l(u) \leq l(p) - 1$ implies that $d(p, q) \leq 2^{l(p)+1}$.

Lemma 2.2 uses the idea of Curtin et al. (2015, Lemma 1) to show that if $S$ is a $\delta$-sparse subset of a metric space $X$, then $S$ has at most $(c_m(S))^\mu$ points in the ball $B(p, r)$, where $c_m(S)$ is the minimized expansion constant of $S$, while $\mu$ depends on $\delta, r$.

**Lemma 2.2** (packing). Let $S$ be a finite $\delta$-sparse set in a metric space $(X, d)$, so $d(a, b) > \delta$ for all $a, b \in S$. Then, for any point $p \in X$ and any radius $t > \delta$, we have $|B(p, t) \cap S| \leq (c_m(S))^\mu$, where $\mu = \lceil \log_2 \left( \frac{2t+\delta}{\delta} \right) \rceil$.

**Proof.** Assume that $d(p, q) > t$ for any point $q \in S$. Then $B(p, t) \cap S = \emptyset$ and the lemma holds trivially. Otherwise $B(p, t) \cap S$ is non-empty. By Definition 1.4 of a minimized expansion constant, for any small enough $\epsilon > 0$, we can always find $\xi \leq \frac{2t+\delta}{\delta \mu}$ and a set $A$ such that $S \subseteq A \subseteq X$ for which:

\[
|B(q, 2s) \cap A| \leq (c_m(S) + \epsilon) \cdot |B(q, s) \cap A|
\]

for any $q \in A$ and $s > \xi$. Note that for any $u \in B(p, t) \cap S$ we have $\bar{B}(u, \frac{\delta}{2}) \subseteq \bar{B}(p, t + \frac{\delta}{2})$. Therefore, for any point $q \in B(p, t) \cap S$, we get

\[
\bigcup_{u \in B(p, t) \cap S} \bar{B}(u, \frac{\delta}{2}) \subseteq \bar{B}(p, t + \frac{\delta}{2}) \subseteq \bar{B}(q, 2t + \frac{\delta}{2})
\]

Since all the points of $S$ were separated by $\delta$, we have

\[
|B(p, t) \cap S| \cdot \min_{u \in B(p, t) \cap S} |\bar{B}(u, \frac{\delta}{2}) \cap A| \leq \sum_{u \in B(p, t) \cap S} |\bar{B}(u, \frac{\delta}{2}) \cap A| \leq |\bar{B}(q, 2t + \frac{\delta}{2}) \cap A|
\]

In particular, by setting $q = \operatorname{arg\,min}_{a \in S \cap B(p, t)} |\bar{B}(a, \frac{\delta}{2})|$, we get:

\[
|B(p, t) \cap S| \cdot |\bar{B}(q, \frac{\delta}{2}) \cap A| \leq |\bar{B}(q, 2t + \frac{\delta}{2}) \cap A|
\]

Inequality 3 applied $\mu$ times for the radii $s_i = \frac{2t + \frac{\delta}{2}}{2^i}$, $i = 1, \ldots, \mu$, implies that:

\[
|\bar{B}(q, 2t + \frac{\delta}{2}) \cap A| \leq (c_m(S) + \epsilon)^\mu |\bar{B}(q, \frac{2t + \frac{\delta}{2}}{2^\mu}) \cap A| \leq (c_m(S) + \epsilon)^\mu |\bar{B}(q, \frac{\delta}{2}) \cap A|.
\]
By combining inequalities \((4)\) and \((5)\), we get
\[
|B(p, t) \cap S| \leq \frac{|\bar{B}(q, 2t + \frac{a}{2}) \cap A|}{|B(q, \frac{a}{2}) \cap A|} \leq (c_m(S) + \epsilon)^n.
\]

The required inequality is obtained by letting \(\epsilon \to 0\).

**Krauthgamer & Lee** (2004 Section 1.1) defined \(\text{dim}(X)\) of a space \((X, d)\) as the minimum number \(m\) such that every set \(U \subseteq X\) can be covered by \(2^m\) sets whose diameter is a half of the diameter of \(U\). If \(U\) is finite, an easy application of Lemma 2.2 for \(\delta = \frac{a}{2}\) shows that \(\text{dim}(X) \leq \sup_{A \subseteq X} (c_m(A))^4 \leq \sup_{A \subseteq X} \inf_{B \subseteq X} (c(B))^4\), where \(A\) and \(B\) are finite subsets of \(X\).

Let \(T(R)\) be an implicit cover tree of **Beygelzimer et al.** (2006a) on a finite set \(R\). **Beygelzimer et al.** (2006a Lemma 4.1) showed that the number of children of any node \(p \in T(R)\) has the upper bound \((c(R))^q\). Lemma 2.3 generalizes **Beygelzimer et al.** (2006a Lemma 4.1) for a compressed cover tree.

**Lemma 2.3** (width bound). Let \(R\) be a finite subset of a metric space \((X, d)\). For any compressed cover tree \(T(R)\), any node \(p\) and any level \(i \leq l(p)\) we have
\[
\{q \in \text{Children}(p) \mid l(q) = i\} \cup \{p\} \leq (c_m(R))^4,
\]
where \(c_m(R)\) is the minimized expansion constant of \(R\).

**Proof.** By the covering condition of \(T(R)\), any child \(q\) of \(p\) located on the level \(i\) has \(d(q, p) \leq 2^{i+1}\). Then the number of children of the node \(p\) at level \(i\) at most \(|\bar{B}(p, 2^{i+1})|\). The separation condition in Definition 2.1 implies that the set \(C_i\) is a \(2^i\)-sparse subset of \(X\). We apply Lemma 2.2 for \(t = 2^{i+1}\) and \(\delta = 2^i\). Since \(4 \cdot \frac{a}{2} + 1 \leq 4 \cdot 2 + 1 \leq 2^4\), we get \(|\bar{B}(q, 2^{i+1}) \cap C_i| \leq (c_m(C_i))^4\). Lemma 1.5 implies that \((c_m(C_i))^4 \leq (c_m(R))^4\), so the upper bound is proved.

**Lemma 2.4** (growth bound). Let \((A, d)\) be a finite metric space, let \(q \in A\) be an arbitrary point and let \(r \in \mathbb{R}\) be a real number. Let \(c(A)\) be the expansion constant from Definition 2.1. If there exists a point \(p \in A\) such that \(2r < d(p, q) \leq 3r\), then \(|\bar{B}(q, 4r)| \geq (1 + \frac{1}{c(A)^2}) \cdot |\bar{B}(q, r)|\).

**Proof.** Since \(\bar{B}(q, r) \subset \bar{B}(p, 3r + r)\), we have \(|\bar{B}(q, r)| \leq |\bar{B}(q, 4r)| \leq c(A)^2 \cdot |\bar{B}(p, r)|\). And since \(\bar{B}(p, r)\) and \(\bar{B}(q, r)\) are disjoint and are subsets of \(\bar{B}(q, 4r)\), we have
\[
|\bar{B}(q, 4r)| \geq |\bar{B}(q, r)| + |\bar{B}(p, r)| \geq |\bar{B}(q, r)| + \frac{|\bar{B}(q, r)|}{c(A)^2} \geq (1 + \frac{1}{c(A)^2}) \cdot |\bar{B}(q, r)|,
\]
which proves the claim.

**Lemma 2.5** (extended growth bound). Let \((A, d)\) be a finite metric space, let \(q \in A\) be an arbitrary point. Let \(p_1, \ldots, p_n\) be a sequence of distinct points in \(R\), in such a way that for all \(i \in \{2, \ldots, n\}\) we have \(4 \cdot d(p_i, q) \leq d(p_{i+1}, q)\). Then
\[
|\bar{B}(q, \frac{4}{3} \cdot d(q, p_n))| \geq (1 + \frac{1}{c(A)^2})^n \cdot |\bar{B}(q, \frac{1}{3} \cdot d(q, p_1))|.
\]

**Proof.** Let us prove this by induction. In basecase \(n = 1\) define \(r = \frac{d(q, p_n)}{3}\). Now by Lemma 2.4 we have
\[
|\bar{B}(q, \frac{4}{3} d(q, p_1))| = |\bar{B}(q, 4r)| \geq (1 + \frac{1}{c(A)^2}) \cdot |\bar{B}(q, r)| = |\bar{B}(q, \frac{1}{3} d(q, p_1))|.
\]
Assume now that the claim holds for some \(i = m\) and let \(p_1, \ldots, p_{m+1}\) be a sequence satisfying the condition of Lemma 2.5. By induction assumption we have \(|\bar{B}(q, \frac{4}{3} d(q, p_m))| \geq (1 + \frac{1}{c(A)^2})^m \cdot |\bar{B}(q, \frac{1}{3} d(q, p_1))|\). Let us pick \(r = \frac{d(q, p_{m+1})}{3}\). Then
we have:

\[ |\Bar{B}(q, \frac{4}{3} \cdot d(q, p_{m+1}))| \geq (1 + \frac{1}{c(A)^2}) \cdot |\Bar{B}(q, \frac{1}{3} \cdot d(q, p_{m+1}))| \]
\[ \geq (1 + \frac{1}{c(A)^2}) \cdot |\Bar{B}(q, \frac{4}{3} \cdot d(q, p_m))| \]
\[ \geq (1 + \frac{1}{c(A)^2}) \cdot (1 + \frac{1}{c(A)^2})^{m} \cdot |\Bar{B}(q, \frac{1}{3} \cdot d(q, p_1))| \]
\[ \geq (1 + \frac{1}{c(A)^2})^{m+1} \cdot |\Bar{B}(q, \frac{1}{3} \cdot d(q, p_1))| \]

which proves the claim.

Lemma B.7. For every \( x \in \mathbb{R} \) satisfying \( x \geq 2 \), the following inequality holds:

\[ x^2 \geq \frac{1}{\log_2(1 + \frac{1}{x^2})}. \]

Proof. Let \( \ln \) be natural logarithm. Note first that for any \( u > 0 \) we have:

\[ \frac{u}{u + 1} = \int_0^u \frac{dt}{u + 1} \leq \int_0^u \frac{dt}{t + 1} = \ln(u + 1). \]

By setting \( u = \frac{1}{x^2} > 0 \) we get:

\[ \log_2(1 + \frac{1}{x^2}) = \frac{\ln(\frac{1}{x^2})}{\ln(2)} \geq \frac{1}{\ln(2)} \cdot \frac{1}{\frac{1}{x^2} + 1} = \frac{1}{\ln(2)} \cdot 1 + \frac{1}{x^2}. \]

Let us now show that for \( x \geq 2 \) we have: \( \frac{1}{\ln(2)} \cdot \frac{1}{\frac{1}{x^2} + 1} \geq \frac{1}{x^2} \). Note first that \( 4 \geq \frac{\ln(2)}{1 - \ln(2)} \). Since \( x \geq 2 \) we have \( x^2 \geq \frac{\ln(2)}{1 - \ln(2)} \).

Therefore \( x^2 - \ln(2) \cdot x^2 \geq \ln(2) \) and \( x^2 \geq \ln(2) \cdot (1 + x^2) \). It follows that \( \frac{1}{\ln(2)} \cdot \frac{1}{x^2} \geq \frac{1}{x^2} \), which proves the claim.

Definition 2.6 (the height of a compressed cover tree). For a compressed cover tree \( T(R) \) on a finite set \( R \), the height set is \( H(T(R)) = \{ i | C_{i-1} \neq C_i \} \cup \{ l_{\max}, l_{\min} \}. \) The size \( |H(T(R))| \) of this set is called the height of \( T(R) \).

The new concept of the height \( |H(T)| \) will justify a near-linear parameterized worst-case complexity in Theorem 4.9. By condition \([2.1]\), the height \( |H(T(R))| \) counts the number of levels \( i \) whose cover sets \( C_i \) include new points that were absent on higher levels. Then \( |H(T)| \leq |R| \) as any point can be alone at its own level.

Lemma B.8. Any finite set \( R \) has the bound \( |H(T(R))| \leq 1 + \log_2(\Delta(R)). \)

Proof. We have \( |H(T(R))| \leq l_{\max} - l_{\min} + 1 \) by Definition 2.6. We estimate \( l_{\max} - l_{\min} \) as follows.

Let \( p \in R \) be a point such that \( \text{diam}(R) = \max_{q \in R} d(p, q) \). Then \( R \) is covered by the closed ball \( \Bar{B}(p; \text{diam}(R)) \). Hence the cover set \( C_i \) at the level \( i = \log_2(\text{diam}(R)) \) consists of a single point \( p \). The separation condition in Definition 2.1 implies that \( l_{\max} \leq \log_2(d_{\max}(R)) \). Since any distinct points \( p, q \in R \) have \( d(p, q) \geq d_{\min}(R) \), the covering condition implies that no new points can enter the cover set \( C_i \) at the level \( i = \log_2(d_{\min}(R)) \), so \( l_{\min} \geq \log_2(d_{\min}(R)) \). Then \( |H(T(R))| \leq 1 + l_{\max} - l_{\min} \leq 1 + \log_2(d_{\text{diam}(R)}^{\text{diam}(R)}). \)

If the aspect ratio \( \Delta(R) = O(\text{Poly}(|R|)) \) polynomially depends on the size \( |R| \), then \( |H(T(R))| \leq O(\log(|R|)) \). Lemma 2.4 corresponds to Beygelzimer et al. (2006a, Lemma 4.2) with slightly modified notation.
C. The minimized expansion constant in a normed vector space on $\mathbb{R}$

In this section, main Theorem [C.15] will show that, for any finite subset $R$ of a normed vector space $(\mathbb{R}^n, \| \cdot \|)$, the minimized expansion constant from Definition [1.4] has the upper bound $2^n$, so

$$c_m(R) = \inf_{0 < \delta} \inf_{R \subseteq \mathbb{R}^n} \sup_{p \in R, t > \delta} \frac{|\bar{B}(p, 2t) \cap A|}{|B(p, t) \cap A|} \leq 2^n.$$  

The proof of Theorem [C.15] is based on the volume argument. We briefly explain the idea before giving the proof later. For this purpose, we recall the definition of the Lebesgue measure in Definition [C.2].

In Definition [C.5] we define concepts of grid $G(\delta) = \delta \cdot \mathbb{Z}^n$ and cubic regions $\bar{V}(p, \delta) = p + [-\frac{\delta}{2}, \frac{\delta}{2}]$. For every $\delta > 0$ we define grid extension $U(\delta)$ of $R$ as set $U(\delta) = (G(\delta) \setminus f(R)) \cup R$, where $f : R \to G(\delta)$ is used to replace points of $R$ with their nearest neighbors in grid $G(\delta)$.

Note that $\xi$ in the definition of $c_m(R)$ acts as a low bound for radius $t > \xi$. Let $\gamma > 0$ be a constant, that depends on dimension $n$ and norm $\| \cdot \|$. In Lemma [C.13] it is shown that if $\delta$ satisfies $0 < \delta < \frac{\xi}{\gamma}$, then for any $p \in U(\delta)$ and $t > \xi$ we can bound $|\bar{B}(p, t) \cap U(\delta)|$ as follows:

$$\frac{\mu(\bar{B}(p, t - \delta \cdot \gamma))}{\delta^n} \leq |\bar{B}(p, t) \cap U(\delta)| \leq \frac{\mu(\bar{B}(p, t + \delta \cdot \gamma))}{\delta^n}.$$  

Therefore

$$\frac{|\bar{B}(p, 2t) \cap U(\delta)|}{|B(p, t) \cap U(\delta)|} \leq \frac{\mu(\bar{B}(p, 2t + \delta \cdot \gamma))}{\mu(\bar{B}(p, t - \delta \cdot \gamma))}.$$  

Now since this inequality is satisfied for any $\delta > 0$, we can choose arbitrary dense grid extension $U(\delta)$. It will be shown that when $\delta \to 0$, then

$$\frac{\mu(\bar{B}(p, 2t + \delta \cdot \gamma))}{\mu(\bar{B}(p, t - \delta \cdot \gamma))} \to 2^n.$$  

Then we can conclude that $c_m(R) \leq 2^n$.

**Definition C.1** (Normed vector space $(\mathbb{R}^n, \| \cdot \|)$ on real numbers $\mathbb{R}$ [Rudin 1990]). Consider $\mathbb{R}^n$ as a vector space. A norm is a function $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ satisfying the properties below.

1. **Non-negativity**: $\| x \| \geq 0$.
2. The norm is positive on nonzero vectors, so $\| x \| = 0$ implies that $x = 0$.
3. For every vector $x \in \mathbb{R}^n$, and every scalar $a \in \mathbb{R} : \| a \cdot x \| = |a| \cdot \| x \|$.
4. The triangle inequality holds for every $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n, \| x + y \| \leq \| x \| + \| y \|$.

A norm induces a metric by the formula $d(x, y) = \| x - y \|$. For every $i \in \{1, ..., n\}$ let $e_i$ be a unit vector of $\mathbb{R}^n$ i.e. $e_i(i) = 1$ and $e_i(j) = 0$ for all $j \in \{1, ..., n\} \setminus \{i\}$. Define $\rho = \max_{i \in \{1, ..., n\}} |e_i|$.

**Definition C.2** (Lebesgue outer measure, Jones 2000 Section 2.A)). Let $\mathbb{R}^n$ be an $n$-dimensional space. Define $n$-dimensional interval as

$I = \{ x \in \mathbb{R}^n | a_i \leq x_i \leq b_i, i = 1, ..., n \} = [a_1, b_1] \times ... \times [a_n, b_n],$

with sides parallel to the coordinate axes. Define Lebesgue outer measure $\mu^* : \{ A | A \subseteq \mathbb{R}^n \} \to [0, \infty) \cup \{ \infty \}$ of interval $I$ as $\mu^*(I) = (b_1 - a_1) \cdot ... \cdot (b_n - a_n)$. The Lebesgue $\mu$ measure of a set $A \subseteq \mathbb{R}^n$ is defined as:

$$\mu^*(A) = \inf \{ \sum_{i=0}^{\infty} \mu^*(I_i) | A \subseteq \bigcup_{i=0}^{\infty} I_i \},$$

where the infimum is taken over all covering of $A$ by countably many intervals $I_i, i = 1, 2, ...$. If set $E \subseteq \mathbb{R}^n$ satisfies $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$ for all $A \subseteq \mathbb{R}^n$, then $E$ is lebesgue-measurable and we set $\mu(E) = \mu^*(E)$. ■
It should be noted that all open sets and closed sets, as well as compact sets are Lebesgue-measurable.

**Lemma C.3** (Basic properties of Lebesgue measure. [Jones (2000) Section 2.A]). A Lebesgue outer measure \( \mu^* \) of Definition C.2 satisfies the following conditions:

1. \( \mu^*(\emptyset) = 0 \),
2. \( \mu^*(A) \leq \mu^*(B) \) whenever \( A \subseteq B \subseteq \mathbb{R}^n \) and
3. \( \mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i) \).

**Lemma C.4** (Lebesgue measure scale property. [Jones (2000) Section 3.B]). Let \( \mu \) be Lebesgue measure on a normed vector space \((\mathbb{R}^n, \| \cdot \|)\). Then, for any \( p \in \mathbb{R}^n \) and \( t \in \mathbb{R}_+ \), we have: \( \mu(B(p, t)) = t^n \cdot \mu(B(p, 1)) \).

**Definition C.5** (Grid and Cubic region). Let \( \mathbb{R}^n \) be a normed vector space and let \( \delta \in \mathbb{R} \). Define \( \delta \)-grid on \( \mathbb{R}^n \) as the set \( G(\delta) = \{ t \cdot \delta \mid t \in \mathbb{Z}^n \} \). For any \( p \in \mathbb{R}^n \) define its open cubic region \( V(p, \delta) \subseteq \mathbb{R}^n \) as the set \( \{ p + u \mid u \in (-\frac{\delta}{2}, \frac{\delta}{2})^n \} \) and closed cubic region \( \bar{V}(p, \delta) \subseteq \mathbb{R}^n \) as \( \{ p + u \mid u \in [-\frac{\delta}{2}, \frac{\delta}{2}]^n \} \).

Note that the union \( \bigcup_{p \in G(\delta)} V(p, \delta) \) covers whole set \( \mathbb{R}^n \).

**Lemma C.6** (Cubic regions are separate). In conditions of Definition C.5 let \( p, q \in G(\delta) \) be distinct points. Then their cubic regions are separate i.e. \( V(p, \delta) \cap V(q, \delta) = \emptyset \).

**Proof.** Assume contrary that there exists \( a \in V(p, \delta) \cap V(q, \delta) \), then \( |a(i) - p(i)| < \frac{\delta}{2} \) and \( |a(i) - q(i)| < \frac{\delta}{2} \) for all \( i \in \{1, ..., n\} \). Since \( p \neq q \), there exists index \( j \), such that \( p(j) \neq q(j) \). By definition of grid \( G(\delta) \) it follows that \( |p(j) - q(j)| \geq \delta \). However, by triangle inequality we have

\[
|p(j) - q(j)| \leq |p(j) - a(j)| + |q(j) - a(j)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\]

which is a contradiction. Therefore \( V(p, \delta) \cap V(q, \delta) = \emptyset \).

**Lemma C.7.** Let \( \mathbb{R}^n \) be a normed vector space of Definition C.1. Let \( \delta \in \mathbb{R} \) and let \( G(\delta) \) be a grid of Definition C.5. Let \( p \in G(\delta) \) and let \( q \in V(p, \delta) \), then \( d(p, q) \leq \frac{n \cdot \delta \cdot \rho}{2} \).

**Proof.** Let \( \gamma \in \mathbb{R} \) be such that \( q = p + \gamma \). By condition (3) of Definition C.1 we have \( \|\gamma(i)\| \leq \|e_i\| \cdot \frac{\delta}{2} \leq \frac{\delta \cdot \rho}{2} \) for all \( i \in \{1, ..., n\} \). By the definition of norm and triangle inequality we have:

\[
d(p, q) = \|p - q\| = \|\gamma\| \leq \sum_{i=1}^{n} \|\gamma(i)\| \leq \frac{n \cdot \delta \cdot \rho}{2}.
\]

Any normed vector space \((\mathbb{R}^n, \| \cdot \|)\) has the metric \( d(x, y) = \|x - y\| \).

**Lemma C.8** (Existence of covering grid). Let \( R \) be a finite subset of a normed vector space \((\mathbb{R}^n, \| \cdot \|)\). Then for any \( \delta \in \mathbb{R} \) having \( \delta < \frac{d_{\text{min}}(R)}{n \cdot \rho} \), then any map \( f : R \rightarrow G(\delta) \) which maps \( p \in R \) to one of its nearest neighbor in \( G(\delta) \) is a well-defined injection.

**Proof.** Let \( f \) be an arbitrary map \( f : R \rightarrow G(\delta) \) mapping point \( p \in R \) to one of its nearest neighbors. This map is clearly well-defined. Let us now show that it is injective. Assume that \( x, y \in R \) are such that \( f(x) = f(y) \). Then by triangle inequality and Lemma C.7 we have:

\[
d(x, y) \leq d(x, p) + d(p, y) \leq n \cdot \delta \cdot \rho < d_{\text{min}}(R),
\]

it follows that \( x = y \). Therefore map \( f \) is injective.
Lemma C.9. Let $R$ be a finite subset of normed space $(\mathbb{R}^n, \rho)$, let $\rho$ be as in Definition C.1, and let $\delta \in \mathbb{R}$ be such that $0 < \delta < \frac{d_{\min}(R)}{n\rho}$. Let $p \in R$ be arbitrary point and let $t > \frac{n\cdot \delta \cdot \rho}{2}$ be a real number. Then there exists a set $U(\delta)$ satisfying $R \subseteq U(\delta)$ and

$$|G(\delta) \cap \bar{B}(p, t - \frac{n\cdot \delta \cdot \rho}{2})| \leq |U(\delta) \cap \bar{B}(p, t)| \leq |G(\delta) \cap \bar{B}(p, t + \frac{n\cdot \delta \cdot \rho}{2})|$$

Proof. Let $f : R \rightarrow G(\delta)$ be an injection from Lemma C.8, which maps every $q \in R$ to one of its nearest neighbors in $G(\delta)$. Define $U(\delta) = (G(\delta) \setminus f(R)) \cup R$. Let us first show that

$$g : U(\delta) \cap \bar{B}(p, t) \rightarrow G(\delta) \cap \bar{B}(p, t + \frac{n\cdot \delta \cdot \rho}{2})$$

defined by $g(q) = f(q)$, if $q \in R$ and $g(q) = q$, if $q \notin R$, is an injection. Let us show first that the map $g$ is well-defined, if $q \notin R$, the claim is trivial. Let $q \notin R$, then by triangle inequality $d(g(q), p) \leq d(q, p) + d(g(q), q) \leq t + \frac{n\cdot \delta \cdot \rho}{2}$. Assume now that $g(a) = g(b)$ for some $a, b \in U(\delta) \cap \bar{B}(p, t)$. Let us first show that either $a, b$ both belong to $R$ or neither of $a, b$ belong to $R$. Assume contrary that $a \in R$ and $b \notin R$. Since $b \notin R$ we have $b \in G(\delta) \setminus f(R)$. On the other hand since $h(a) = h(b)$ we have $f(a) = b$, therefore $b \in f(R)$, which is a contradiction. If both, $a$ and $b$ belong to $R$ we have $a = b$, similarly if $a, b \notin R$ we have $a = b$ by injectivity of function $f$. Therefore we have now shown that $g$ is well-defined injection. It follows $|U(\delta) \cap \bar{B}(p, t)| \leq |G(\delta) \cap \bar{B}(p, t + \frac{n\cdot \delta \cdot \rho}{2})|$. Let us now show that map

$$h : G(\delta) \cap \bar{B}(p, t - \frac{n\cdot \delta \cdot \rho}{2}) \rightarrow U(\delta) \cap \bar{B}(p, t),$$

defined by $h(q) = f^{-1}(q)$, if $q \in f(R)$ and $h(q) = q$, if $q \notin f(R)$ is well-defined injection. Let us first show that the map is well-defined. Let $q \in G(\delta) \cap \bar{B}(p, t - \frac{n\cdot \delta \cdot \rho}{2})$, if $q \notin f(R)$ the claim is satisfied trivially. If $q \in f(R)$, then by definition $d(h(q), q) \leq \frac{n\cdot \delta \cdot \rho}{2}$. By using triangle inequality we obtain:

$$d(p, h(q)) \leq d(p, q) + d(q, h(q)) \leq t - \frac{n\cdot \delta \cdot \rho}{2} + \frac{n\cdot \delta \cdot \rho}{2} \leq t.$$

Therefore $h(q) \in U(\delta) \cap \bar{B}(p, t)$.

Let us now show that $h$ is an injection. Let $a, b \in G(\delta) \cap \bar{B}(p, t - \frac{n\cdot \delta \cdot \rho}{2})$ assume that $h(a) = h(b)$, let us show that $a = b$. Let us first show that either $a, b \in f(R)$ or neither of $a, b$ belong to $f(R)$. Assume contrary that $a \in f(R)$ and $b \notin f(R)$, then $h(a) = h(b)$ implies that $f^{-1}(a) = b$. Since $f^{-1}(a) \in R$, we have $b \in R$. Since $b \in G(\delta)$, it follows that $f(b) = b$, which is a contradiction since $b \notin f(R)$. Assume now that $a, b \notin f(R)$, then the claim follows by noting that $f^{-1}$ is injection. If $a, b \notin f(R)$, then claim follows by noting that $h(a) = a$ and $h(b) = b$. Therefore map $h$ is injection. It follows that $|G(\delta) \cap \bar{B}(p, t - \frac{n\cdot \delta \cdot \rho}{2})| \leq |U(\delta) \cap \bar{B}(p, t)|$.

Lemma C.10. Let $R$ be a finite subset of normed vector space $\mathbb{R}^n$ and let $\delta \in \mathbb{R}$. For any $p \in G(\delta)$ recall that $V(p, \delta)$ is Minkowski sum $p + (-\frac{1}{2}, \frac{1}{2})^n$. Define

$$W(p, t, \delta) = \bigcup_{q \in \bar{B}(p, t) \cap G(\delta)} V(q, \delta).$$

Then for any $p \in R$ and $t > \frac{n\cdot \delta \cdot \rho}{2}$ we have:

$$B(p, t - \frac{n\cdot \delta \cdot \rho}{2}) \subseteq W(p, t, \delta) \subseteq B(p, t + \frac{n\cdot \delta \cdot \rho}{2}).$$

Proof. Let $u \in \bar{B}(p, t - \frac{n\cdot \delta \cdot \rho}{2})$ be an arbitrary point. Since $\{V(q, \delta) \mid q \in G(\delta)\}$ covers $R$ it follows that there exists $a \in G(\delta)$ such that $u \in V(a, \delta)$. By triangle inequality we obtain:

$$d(a, p) \leq d(a, u) + d(u, p) \leq \frac{n\cdot \delta \cdot \rho}{n} + t - \frac{n\cdot \delta \cdot \rho}{n} \leq t.$$
It follows that $V(w, \delta) \subseteq W(p, t)$, therefore $p \in W(p, t, \delta)$. We have $\bar{B}(p, t - \frac{n \cdot \delta \cdot \rho}{2}) \subseteq W(p, t, \delta)$. Let $u \in W(p, t, \delta)$, then there exists $a \in G(\delta)$ such that $u \in V(a, \delta)$ and $V(a, \delta) \subseteq W(p, t)$. By triangle inequality we obtain:

$$d(u, p) \leq d(u, a) + d(a, p) \leq \frac{n \cdot \delta \cdot \rho}{n} + t.$$ 

It follows that $u \in \bar{B}(p, t + \frac{n \cdot \delta \cdot \rho}{2})$. Therefore $\bar{W}(p, t, \delta) \subseteq \bar{B}(p, t + \frac{n \cdot \delta \cdot \rho}{2})$ which proves the claim.

\[\Box\]

**Lemma C.11 (Countable additivity, \cite{Jones2000} Section 2.A).** Assume that $A_i \subseteq \mathbb{R}^n$, $i = 1, 2, \ldots$, are pairwise disjoint i.e. $A_i \cap A_j = \emptyset$ for all $i \neq j$ Lebesgue-measurable sets. Then

$$\mu(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

**Lemma C.12 (Lebesgue measure of $\bar{W}(p, t, \delta)$).** In notations of Lemma C.10 let $\mu$ be a Lebesgue measure on $R$ from Definition C.2 then $\mu(\bar{W}(p, t, \delta)) = \delta^n \cdot |\bar{B}(p, t) \cap G(\delta)|$.

**Proof.** Define $W(p, t, \delta) = \bigcup_{q \in \bar{B}(p, t) \cap G(\delta)} V(q, \delta)$. Recall that for all $p \in \mathbb{R}^n$ and $\delta > 0$ set $\bar{V}(p, t)$ is a closed $n-$dimensional interval and $V(p, t)$ is an open $n$-dimensional interval. Therefore we have $\mu(\bar{V}(p, t)) = \mu(V(p, t))$. Since $\bar{V}(p, t)$ is a closed interval, it follows that $\mu(\bar{V}(p, t)) = \delta^n$. Since all the sets of $W$ are separate we can use Lemma C.11 to obtain:

$$\mu(W(p, t, \delta)) = \sum_{A \in W(p, t)} \mu(A) = \sum_{A \in W(p, t)} \mu(A) = \delta^n \cdot |\bar{B}(p, t) \cap G(\delta)|$$

By Lemma C.3(2), since $W(p, t, \delta) \subseteq \bar{W}(p, t, \delta)$ we obtain $\mu(\cup \bar{W}(p, t)) \geq \delta^n \cdot |\bar{B}(p, t) \cap G(\delta)|$. On the other hand, by Lemma C.3(3) we obtain

$$\mu(\bar{W}(p, t, \delta)) \leq \sum_{A \in \bar{W}(p, t)} \mu(A) = \delta^n \cdot |\bar{B}(p, t) \cap G(\delta)|$$

Therefore we have shown that $\mu(\bar{W}(p, t, \delta)) = \delta^n \cdot |\bar{B}(p, t) \cap G(\delta)|$.

\[\Box\]

**Lemma C.13 (Set $U(\delta)$ bounds).** Let $\mathbb{R}^n$ be a normed vector space. Let $R \subseteq \mathbb{R}^n$ be its finite subset. Then any set $U(\delta)$ of Lemma C.9 satisfies the following inequalities:

$$\frac{\mu(\bar{B}(p, t - \delta \cdot n \cdot \rho))}{\delta^n} \leq |\bar{B}(p, t) \cap U(\delta)| \leq \frac{\mu(\bar{B}(p, t + \delta \cdot \gamma \cdot n \cdot \rho))}{\delta^n},$$

for all $p \in R$ and $t > n \cdot \delta \cdot \rho$.

**Proof.** Let $p \in \mathbb{R}^n$ be an arbitrary point and let $t > n \cdot \delta \cdot \rho$ be an arbitrary real number. By Lemma C.9 it follows:

$$|G(\delta) \cap \bar{B}(p, t + \frac{n \cdot \delta \cdot \rho}{2})| \leq |\bar{B}(p, t) \cap U(\delta)| \leq |G(\delta) \cap \bar{B}(p, t + \frac{n \cdot \delta \cdot \rho}{2})|.$$ 

Let $\bar{W}(p, t + \frac{n \cdot \delta \cdot \rho}{2}, \delta) = \cup_q \{\bar{V}(q, \delta) \mid q \in \bar{B}(p, t + \frac{n \cdot \delta \cdot \rho}{2})\}$. By Lemma C.10 we have:

$$\bar{B}(p, t - n \cdot \delta \cdot \rho) \subseteq W(p, t - \frac{n \cdot \delta \cdot \rho}{2}, \delta)$$

and $W(p, t + \frac{n \cdot \delta \cdot \rho}{2}, \delta) \subseteq \bar{B}(p, t + n \cdot \delta \cdot \rho)$.

By Lemma C.3 we have $\mu(\bar{W}(p, t + n \cdot \delta \cdot \rho, \delta)) \leq \mu(\bar{B}(p, t + n \cdot \delta \cdot \rho))$. By Lemma C.12 we have:

$$\mu(\bar{W}(p, t + \frac{n \cdot \delta \cdot \rho}{2}, \delta)) = \delta^n \cdot |\bar{B}(p, t + \frac{n \cdot \delta \cdot \rho}{2}) \cap G(\delta)|.$$ 

By combining the facts we obtain:

$$|\bar{B}(p, t) \cap U(\delta)| \leq |G(\delta) \cap \bar{B}(p, t + \frac{n \cdot \delta \cdot \rho}{2})| \leq \frac{\mu(\bar{W}(p, t + n \cdot \delta \cdot \rho, \delta))}{\delta^n} \leq \frac{\mu(\bar{B}(p, t + n \cdot \delta \cdot \rho))}{\delta^n}.$$
For any node which concludes the proofs.

Lemma C.14 (Set $U(\delta)$ is locally finite). Let $\mathbb{R}^n$ be a normed vector space. Let $R \subseteq \mathbb{R}^n$ be its finite subset Then any set $U(\delta)$ from Lemma C.9 is locally finite.

Proof. With the exact same proof of Lemma C.13 it can be shown that

$$|\bar{B}(p,t) \cap U(\delta)| \leq \frac{\mu(\bar{B}(p,t + \delta \cdot n \cdot \rho))}{\delta^n}$$

is satisfied for all $p \in R$ and $t > 0$. Therefore $|\bar{B}(p,t) \cap U(\delta)|$ is finite as well.

Recall that minimized expansion constant of Definition 1.4 of a finite subset $R$ of a metric space $(X,d)$ was defined as $c_m(R) = \lim_{\xi \to 0^+} \inf_{R \subseteq \mathbb{R}^n \subseteq X} \sup_{\xi \in A, t > \xi} \frac{|\bar{B}(p,t) \cap A|}{|\bar{B}(p,t) \cap U(\delta)|}$ where $A$ is a locally finite set which covers $R$.

Theorem C.15 (The minimized expansion constant of a finite subset $R$ of $\mathbb{R}^n$ is at most $2^n$). Let $R$ be a finite subset of a normed Euclidean space $\mathbb{R}^n$. Let $c_m(R)$ be the minimized expansion constant of Definition 1.4 then $c_m(R) \leq 2^n$.

Proof. Let $0 < \xi < \frac{d_{\min}(R)}{2^n}$ be an arbitrary real number. Let $0 < \delta < \frac{\xi}{n \cdot \rho}$ be a real number. Since $\delta < \frac{d_{\min}(R)}{2^n \cdot \rho}$ by Lemma C.13 we have:

$$\frac{\mu(\bar{B}(p,t - \delta \cdot n \cdot \rho))}{\delta^n} \leq |\bar{B}(p,t) \cap U(\delta)| \leq \frac{\mu(\bar{B}(p,t + \delta \cdot \gamma \cdot n \cdot \rho))}{\delta^n}$$

Note that by Lemma C.4 we have: $\mu(\bar{B}(q,y)) = y^n \cdot \mu(\bar{B}(q,1))$ for any $q \in \mathbb{R}^n$ and $y \in \mathbb{R}_+$. Therefore

$$\frac{|\bar{B}(p,2t) \cap U(\delta)|}{|\bar{B}(p,t) \cap U(\delta)|} \leq \frac{\mu(\bar{B}(p,2t + n\delta \rho)) \cdot \delta^2}{\mu(\bar{B}(p,t - n\delta \rho)) \cdot \delta^2} = \frac{(2t + n\delta \rho)^n \cdot \mu(\bar{B}(p,1))}{(t - n\delta \rho)^n \cdot \mu(\bar{B}(p,1))} = \frac{(2t + n\rho)^n}{(t - n\rho)^n}$$

is satisfied for all $t > \xi$. Since $0 < \xi < \frac{d_{\min}(R)}{2^n}$ was chosen arbitrarily, we conclude that:

$$c_m(R) = \lim_{\xi \to 0^+} \inf_{R \subseteq \mathbb{R}^n \subseteq X} \sup_{\xi \in A, t > \xi} \frac{|\bar{B}(p,t) \cap A|}{|\bar{B}(p,t) \cap U(\delta)|} \leq \lim_{\delta \to 0} \frac{|\bar{B}(p,t) \cap U(\delta)|}{|\bar{B}(p,t) \cap U(\delta)|} = \lim_{\delta \to 0} \frac{(2t + \delta \cdot n \rho)^n}{(t - \delta \cdot n \rho)^n} = \frac{2^n \cdot t^n}{t^n} = 2^n.$$

D. Distinctive descendant sets

This section introduces auxiliary concepts for future proofs. The main concept is a distinctive descendant set in Definition 2.8. The distinctive descendant set at a level $i$ of a node $p \in T(R)$ in a compressed cover tree corresponds to the set of descendants of a copy of node $p$ at level $i$ in the original implicit cover tree $T(R)$. Other important concepts are $\lambda$-point of Definition 2.6 that is used in Algorithm F.2 as an approximation for $k$-nearest neighboring point. The $\beta$-point property of $\lambda$-point defined in Lemma D.15 plays a major role in the proof of the main worst-case time complexity result Theorem 4.9.

Definition 2.8 (Distinctive descendant sets). Let $R \subseteq X$ be a finite reference set with a compressed cover tree $T(R)$. For any node $p \in T(R)$ and level $i \leq l(p)-1$, set $V_i(p) = \{ u \in \text{Descendants}(p) \mid i \leq l(u) \leq l(p)-1 \}$. If $i \geq l(p)$, then set $V_i(p) = \emptyset$. For any level $i \leq l(p)$, the distinctive descendant set is $S_i(p, T(R)) = \text{Descendants}(p) \setminus \bigcup_{u \in V_i(p)} \text{Descendants}(u)$ and has the size $|S_i(p, T(R))|$. 

Lemma D.1 (Distinctive descendant set inclusion property). In conditions of Definition 2.8 let $p \in R$ and let $i, j$ be integers satisfying $l_{\min}(T(R)) \leq i \leq j \leq l(p)-1$. Then $S_i(p, T(R)) \subseteq S_j(p, T(R))$. 

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Let us prove this claim by induction on size $|R|$. Assume now that the claim holds for any tree $T(R)$, where $|R| = m$ and let us prove that if we add any node $v \in X \setminus R$ to tree $T(R)$, then $\sum_{p \in R} |\mathcal{E}(p, T(R) \cup \{v\})| = 2 \cdot |R| + 2$. Assume that we have added $v$ to $T(R)$, in such a way that $v$ is its new parent. Then $|\mathcal{E}(p, T(R) \cup \{v\})| = |\mathcal{E}(p, T(R))| + 1$ and $|\mathcal{E}(v, T(R) \cup \{v\})| = 1$. We have:

$$\sum_{p \in R \cup \{v\}} |\mathcal{E}(p, T(R))| = \sum_{p \in R} |\mathcal{E}(p, T(R))| + 1 + |\mathcal{E}(v, T(R) \cup \{v\})| \leq 2 \cdot |R| + 2 \leq 2(|R| \cup \{v\})$$

which completes the induction step. \hfill $\square$

**Algorithm D.4** This algorithm returns sizes of distinctive descendant set $\mathcal{S}(p, T(R))$ for all essential levels $i \in \mathcal{E}(p, T(R))$

1. **Function**: CountDistinctiveDescendants(Node $p$, a level $i$ of $T(R)$)
2. **Output**: an integer
3. **if** $i > l_{\text{min}}(T(Q))$ **then**
4. **for** $q \in \text{Children}(p)$ having $l(p) = i - 1$ or $q = p$ **do**
5. **Set** $s = 0$
6. $j \leftarrow 1 + \text{Next}(q, i - 1, T(R))$
7. $s \leftarrow s + \text{CountDistinctiveDescendants}(q, j)$
8. **end for**
9. **else**
10. **Set** $s = 1$
11. **end if**
12. **Set** $|\mathcal{S}(p)| = s$ and **return** $s$

**Lemma D.5.** Let $R$ be a finite subset of a metric space. Let $T(R)$ be a compressed cover tree on $R$. Then, Algorithm D.4 computes the sizes $|\mathcal{S}(p, T(R))|$ for all $p \in R$ and essential levels $i \in \mathcal{E}(p, T(R))$ in time $O(|R|)$. \hfill $\square$

**Proof.** By Lemma D.3 we have $\sum_{p \in R} |\mathcal{E}(p, T(R))| \leq 2 \cdot |R|$. Since CountDistinctiveDescendants is called once for every any combination $p \in R$ and $i \in \mathcal{E}(p, T(R))$ it follows that the time complexity of Algorithm D.4 is $O(R)$. \hfill $\square$
A new compressed cover tree for \(k\)-nearest neighbors

Recall that the neighbor set \(N(q; r) = \{p \in C \mid d(q, p) \leq d(q, r)\}\) was introduced in Definition 1.2.

**Definition D.6** (\(\lambda\)-point). Fix a query point \(q\) in a metric space \((X, d)\) and fix any level \(i \in \mathbb{Z}\). Let \(\mathcal{T}(R)\) be its compressed cover tree on a finite reference set \(R \subseteq X\). Let \(C\) be a subset of a cover set \(C_i\) from Definition 2.7 satisfying

\[
\sum_{p \in C} |S_i(p, \mathcal{T}(R))| \geq k,
\]

where \(S_i(p, \mathcal{T}(R))\) is the distinctive descendant set from Definition 2.8. For any \(k \geq 1\), define \(\lambda_k(q, C)\) as a point \(\lambda \in C\) that minimizes \(d(q, \lambda)\) subject to \(\sum_{p \in N(q, \lambda)} |S_i(p, \mathcal{T}(R))| \geq k\). \(\blacksquare\)

**Algorithm D.7** Finding \(k\)-lowest element of a finite subset \(A \subseteq R\) with priority function \(f : A \to \mathbb{R}\)

1. **Input**: Ordered subset \(A \subseteq R\), priority function \(f : A \to \mathbb{R}\), an integer \(k \in \mathbb{Z}\)
2. Initialize an empty max-binary heap \(B\) and an empty array \(D\) on points \(A\).
3. for \(p \in A\) do
   4. add \(p\) to \(B\) with priority \(f(p)\)
   5. if \(|H| \geq k\) then
      6. remove the point with a maximal value from \(B\)
   7. end if
8. end for
9. Transfer points from the binary heap \(B\) to the array \(D\) in reverse order.
10. return \(D\).

**Algorithm D.8** Computation of a \(\lambda\)-point of Definition D.6 in line 6 of Algorithm F.2

1. **Input**: A point \(q \in X\), a subset \(C\) of a level set \(C_i\) of a compressed cover tree \(\mathcal{T}(R)\), an integer \(k \in \mathbb{Z}\)
2. Define \(f : C \to \mathbb{R}\) by setting \(f(p) = d(p, q)\).
3. Run Algorithm D.7 on inputs \((C, f, k)\) and retrieve array \(D\).
4. Find the smallest index \(j\) such that \(\sum_{t=0}^{j} |S_i(D[t], \mathcal{T}(R))| \geq k\).
5. return \(\lambda = D[j]\).

**Lemma D.9.** Let \(A \subseteq R\) be a finite subset and let \(f : A \to \mathbb{R}\) be a priority function and let \(k \in \mathbb{Z}_+\). Then Algorithm D.7 finds \(k\)-smallest elements of \(A\) in time \(|A| \cdot \log_2(k)\)

**Proof.** Adding and removing element from binary heap data structure [Cormen (1990) section 6.5] takes at most \(O(\log(n))\) time, where \(n\) is the size of binary heap. Since the size of our binary heap is capped at \(k\) and we add/remove at most \(|A|\) elements, the total time complexity is \(O(|A| \cdot \log_2(k))\). \(\blacksquare\)

**Lemma D.10** (time complexity of a \(\lambda\)-point). In the conditions of Definition D.6, the time complexity of Algorithm D.8 is \(O(|C| \cdot \log_2(k))\).

**Proof.** Note that in line 4 we have \(|S_i(D[t], \mathcal{T}(R))| \geq 1\) for all \(t = 0, \ldots, j\). Therefore the time complexity of line 4 is \(O(k)\). By Lemma D.9, the time complexity of line 3 is \(O(|C| \cdot \log_2(k))\), which proves the claim. \(\blacksquare\)

**Lemma D.11** (separation). In the conditions of Definition 2.8, let \(p \neq q\) be nodes of \(\mathcal{T}(R)\) with \(l(p) \geq i\), \(l(q) \geq i\). Then \(S_i(p, \mathcal{T}(R)) \cap S_i(q, \mathcal{T}(R)) = \emptyset\).

**Proof.** Without loss of generality assume that \(l(p) \geq l(q)\). If \(q\) is not a descendant of \(p\), the lemma holds trivially due to \(\text{Descendants}(q) \cap \text{Descendants}(p) = \emptyset\). If \(q\) is a descendant of \(p\), then \(l(q) \leq l(p) - 1\) and therefore \(q \in V_i(p)\). It follows that \(S_i(p, \mathcal{T}(R)) \cap \text{Descendants}(q) = \emptyset\) and therefore

\[
S_i(p, \mathcal{T}(R)) \cap S_i(q, \mathcal{T}(R)) \subseteq S_i(p, \mathcal{T}(R)) \cap \text{Descendants}(q) = \emptyset.
\]

**Lemma D.12** (Sum lemma). In the notations of Definition 2.8, assume that \(i\) is arbitrarily index and a subset \(V \subseteq R\) satisfies \(l(p) \geq i\) for all \(p \in V\). Then

\[
|\bigcup_{p \in V} S_i(p, \mathcal{T}(R))| = \sum_{p \in V} |S_i(p, \mathcal{T}(R))|.
\]
Proof. Proof follows from Lemma \[{}^{D.11}\].

By Lemma \[{}^{D.12}\] in Definition \[{}^{D.6}\] one can assume that \(|\bigcup_{p \in C} S_i(p, T(R))| \geq k\).

**Lemma 2.9.** Let \(R \subseteq X\) be a finite reference set with a cover tree \(T(R)\). In the notations of Definition 2.8 let \(p \in T(R)\) be any node. If \(w \in S_i(p, T(R))\) then either \(w = p\) or there exists \(a \in \text{Children}(p) \setminus \{p\}\) such that \(l(a) < i\) and \(w \in \text{Descendants}(a)\).

Proof. Let \(w \in S_i(p)\) be an arbitrary node satisfying \(w \neq p\). Let \(s\) be the node-to-root path of \(w\). The inclusion \(S_i(p) \subseteq \text{Descendants}(p)\) implies that \(w \in \text{Descendants}(p)\). Let \(a \in \text{Children}(p) \setminus \{p\}\) be a child on the path \(s\). If \(l(a) \geq i\) then \(a \in V_i(p)\). Note that \(w \in \text{Descendants}(a)\). Therefore \(w \notin S_i(p)\), which is a contradiction. Hence \(l(a) < i\).

**Lemma 2.13.** In the notations of Definition 2.8 let \(p \in T(R)\) be any node. If \(w \in S_i(p, T(R))\) then \(d(w, p) \leq 2^{i+1}\).

Proof. By Lemma 2.9 either \(w = \gamma\) or \(w \in \text{Descendants}(a)\) for some \(a \in \text{Children}(\gamma) \setminus \{\gamma\}\) for which \(l(a) < i\). If \(w = \gamma\), then trivially \(d(\gamma, w) \leq 2^i\). Else \(w\) is a descendant of \(a\), which is a child of node \(\gamma\) on level \(i - 1\) or below, therefore by Lemma B.6 we have \(d(\gamma, w) \leq 2^i\) anyway.

**Lemma 2.14.** Let \(R\) be a finite subset of a metric pace. Let \(T(R)\) be a compressed cover tree on \(R\). Let \(R_j \subseteq C_j\), where \(C_j\) is the \(i\)th cover set of \(T(R)\). Let \(i = \max_{p \in R} \text{Next}(p, j, T(R))\). Set \(C_i(R_j) = R_j \cup \{a \in \text{Children}(p)\text{ for some } p \in R_i | l(a) = i\}\). Then

\[
\bigcup_{p \in C_i(R_j)} S_i(p, T(R)) = \bigcup_{p \in R_j} S_j(p, T(R)).
\]

Proof. Let \(a \in \bigcup_{p \in C_i(R_j)} S_i(p, T(R))\) be an arbitrary node. Then there exits \(u \in C_i(R_j)\) having \(a \in S_i(u, T(R))\). By definition of index \(i\), either \(u \in R_j\) or \(u\) has a parent in \(R_j\). If \(u \in R_j\) then we note that \(V_j(u) \subseteq V_i(u)\). Since \(a \notin V_i(u)\), we also have \(a \notin V_j(u)\).

Otherwise let \(w\) be a parent of \(u\). Therefore there are no descendants of \(w\) in having level in interval \([l(u) + 1, l(p) - 1]\). Since \(l(u) = i\) and \(j > i\) it follows that \(V_j(u) = \emptyset\). Denote \(w\) to be the lowest level ancestor of \(u\) on level \(j\). By cases above we have \(a \notin V_j(w)\). Therefore it follows that

\[
a \in S_j(w, T(R)) \subseteq \bigcup_{p \in R_j} S_j(p, T(R)).
\]

To prove the converse inclusion assume now that \(a \in \bigcup_{p \in R_j} S_j(p, T(R))\). Then \(a \in S_j(v, T(R))\) for some \(w \in R_j\). Assume that \(w\) has no children at the level \(i\). Then \(V_j(w) = V_i(w)\) and

\[
a \in S_i(w, T(R)) \subseteq \bigcup_{p \in C_i(R_j)} S_i(p, T(R)).
\]

Assume now that \(w\) has children at the level \(i\). If there exists \(b \in \text{Children}(w)\) for which \(a \in \text{Descendants}(b)\). Since \(V_i(b) = \emptyset\), we conclude that

\[
a \in S_i(b, T(R)) \subseteq \bigcup_{p \in C_i(R_j)} S_i(p, T(R)).
\]

Assume that \(a \notin \text{Descendants}(b)\) for all \(b \in \text{Children}(w)\) with \(l(b) = i\). Then \(a \notin \text{Descendants}(w)\) and \(a \notin \text{Descendants}(b')\) for any \(b' \in V_j(w)\). Then \(a \in S_i(w, T(R))\) and the proof finishes:

\[
\bigcup_{p \in R_j} S_j(p, T(R)) \subseteq \bigcup_{p \in C_i(R_j)} S_i(p, T(R)).
\]
Lemma D.15 (β-point). In the notations of Definition D.6 let \( C \subseteq C_t \) so that \( \cup_{p \in C} S_t(p, T(R)) \) contains all \( k \)-nearest neighbors of \( q \). Set \( \lambda = \lambda_k(q, C) \). Then \( R \) has a point \( \beta \) among the first \( k \) nearest neighbors of \( q \) such that \( d(q, \lambda) \leq d(q, \beta) + 2^{i+1} \).

Proof. We show that \( R \) has a point \( \beta \) among the first \( k \) nearest neighbors of \( q \) such that

\[
\beta \in \bigcup_{p \in C} S_t(p, T(R)) \setminus \bigcup_{p \in N(q, \lambda) \setminus \{\lambda\}} S_t(p, T(R)).
\]

Lemma D.12 and Definition D.6 imply that

\[
| \bigcup_{p \in N(q, \lambda) \setminus \{\lambda\}} S_t(p, T(R)) | = \sum_{p \in N(q, \lambda) \setminus \{\lambda\}} |S_t(p, T(R))| < k.
\]

Since \( \cup_{p \in C} S_t(p, T(R)) \) contains all \( k \)-nearest neighbors of \( q \), a required point \( \beta \in R \) exists.

Let us now show that \( \beta \) satisfies \( d(q, \lambda) \leq d(q, \beta) + 2^{i+1} \). Let \( \gamma \in C \setminus (N(q, \lambda) \cup \{\lambda\}) \) be such that \( \beta \in S_t(\gamma, T(R)) \). Since \( \gamma \notin N(q, \lambda) \setminus \{\lambda\} \), we get \( d(\gamma, q) \geq d(q, \lambda) \). The triangle inequality says that \( d(q, \gamma) \leq d(q, \beta) + d(\gamma, \beta) \). Finally Lemma D.15 implies that \( d(\gamma, \beta) \leq 2^{i+1} \). Then

\[
d(q, \lambda) \leq d(q, \gamma) \leq d(q, \beta) + d(\gamma, \beta) \leq d(q, \beta) + 2^{i+1}
\]

So \( \beta \) is a desired \( k \)-nearest neighbor satisfying \( d(q, \lambda) \leq d(q, \beta) + 2^{i+1} \).

\[\square\]

E. Construction of a compressed cover tree

This section introduces a new method Algorithm E.2 for construction of a compressed cover tree, which is based on Insert() method [Beygelzimer et al. (2006a, Algorithm 2) that was specifically adapted for compressed cover tree. The proof of Beygelzimer et al. (2006a, Theorem 6), which estimated the time complexity of Beygelzimer et al. (2006a, Algorithm 2) was shown to be incorrect [Elkin & Kurlin (2022a, Counterexample 4.2). The main contribution of this section are two new time complexity results that bound the time complexity of Algorithm E.2.

- Theorem 3.6 bounds the time complexity as \( O(c_m(R)^{10} \cdot \log_2(\Delta(R)) \cdot |R|) \) by using minimized expansion constant \( c_m(R) \) and aspect ratio \( \Delta(R) \) as parameters.
- Theorem 3.9 bounds the time complexity as \( O(c(R)^{12} \cdot \log_2 |R| \cdot |R|) \) by using expansion constant \( c(R) \) as parameter.

Definition 2.10 explains the concrete implementation of compressed cover tree.

Definition 2.10 (Children(\(p, i\) and Next(\(p, i, T(R))\)). In a compressed cover tree \( T(R) \) on a set \( R \), for any level \( i \) and a node \( p \in R \), set \( \text{Children}(p, i) = \{ a \in \text{Children}(p) \mid l(a) = i \} \). Let \( \text{Next}(p, i, T(R)) \) be the maximal level \( j \) satisfying \( j < i \) and \( \text{Children}(p, i) \neq \emptyset \). If such level does not exist, we set \( j = l_{\text{min}}(T(R)) - 1 \). For every node \( p \), we store its set of children in a linked hash map so that

(a) any key \( i \) gives access to \( \text{Children}(p, i) \).

(b) \( \text{Children}(p, i) \rightarrow \text{Children}(p, \text{Next}(p, i, T(R))) \).

(c) we can directly access \( \max \{ j \mid \text{Children}(p, j) \neq \emptyset \} \).

Definition E.1 (construction iteration set \( L(T(W), p) \)). Let \( W \) be a finite subset of a metric space \((X, d)\). Let \( T(W) \) be a cover tree of \( W \) built on \( W \) and let \( p \in X \setminus W \) be an arbitrary point. Let \( L(T(W), p) \subseteq H(T(R)) \) be the set of all levels \( i \) during iterations 3-13 of Algorithm E.3 launched with inputs \( T(W), p \). Set \( \eta(i) = \min\{t \in L(T(W), p) \mid t > i\} \).

Let \( R \) be a finite subset of a metric space \((X, d)\). A compressed cover tree \( T(R) \) will be incrementally constructed by adding points one by one as summarized in Algorithm E.2. First we select a root node \( r \in R \) and form a tree \( T(\{r\}) \) of a
**Algorithm E.2** Building a compressed cover tree $T(R)$ from Definition 2.1

1: **Input**: a finite subset $R$ of a metric space $(X, d)$
2: **Output**: a compressed cover tree $T(R)$.
3: Choose a random point $r \in R$ to be a root of $T(R)$
4: Build the initial compressed cover tree $T = T(\{r\})$ by making $l(r) = +\infty$.
5: **for** $p \in R \setminus \{r\}$ **do**
6: \hspace{1em} $T \leftarrow \text{run AddPoint}(T, p)$ described in Algorithm E.3
7: **end for**
8: For root $r$ of $T$ set $l(r) = 1 + \max_{p \in R \setminus \{r\}} l(p)$

**Algorithm E.3** Building $T(W \cup \{p\})$ in lines 5-7 of Algorithm E.2

1: **Function** AddPoint(a compressed cover tree $T(W)$ with a root $r$, a point $p \in X$)
2: **Output**: compressed cover tree $T(W \cup \{p\})$.
3: Set $i \leftarrow l_{\max}(T(W)) - 1$ and $\eta(l_{\max} - 1) = l_{\max}$ \{If the root $r$ has no children then $i \leftarrow -\infty\}$
4: Set $R_{l_{\max}} \leftarrow \{r\}$.
5: **while** $i \geq l_{\min}$ **do**
6: \hspace{1em} Assign $C_i(R_{\eta(i)}) \leftarrow R_{\eta(i)} \cup \{a \in \text{Children}(q) \mid l(a) = i\}$
7: \hspace{1em} Set $R_i = \{a \in C_i(R_{\eta(i)}) \mid d(p, a) \leq 2^i\}$
8: \hspace{1em} **if** $R_i$ is empty **then**
9: \hspace{2em} Launch Algorithm E.4 with parameters $(p, R_{\eta(i)})$.
10: **end if**
11: \hspace{1em} $t = \max_{a \in R_i} \text{Next}(a, i, T(W))$ \{If $R_i$ has no children we set $t = l_{\min} - 1\}$
12: \hspace{1em} $\eta(i) \leftarrow i$ and $i \leftarrow t$
13: **end while**
14: Launch Algorithm E.4 with parameters $(p, R_{\eta(i)})$.

**Algorithm E.4** Assign node subprocedure

1: **Function** AssignParent(Point $p$, subset of nodes $U \subseteq T(W)$)
2: **Output**: Compressed cover tree $T(W \cup \{p\})$.
3: Pick $v \in U$ minimizing $d(v, p)$.
4: Set $l(p) = \lfloor \log_2(d(p, v)) \rfloor - 1$ and let $v$ be a parent of $p$. 
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We first note that the parent which proves the claim. Assume now (2.1c) to check. Consider arbitrary cover set $C$ for which the covering condition (2.1b) after adding point $p$ to $T(W)$ follows from the following inequality:

$$
d(p, \theta) \geq d(p, \gamma) - d(\gamma, \theta) > 2^{l(\theta)} - (2^{l(\theta)} + 1) > 2^{l(\theta)}.\,$$

which proves the claim.

**Theorem 3.2** (correctness of Algorithm 3.4). Algorithm 3.4 builds a compressed cover tree in Definition 2.1.

**Proof.** It suffices to prove that Algorithm 3.4 correctly extends a compressed cover tree $T(W)$ for any finite subset $W \subseteq X$ by adding a point $p$. Let us prove that $T(W \cup \{p\})$ satisfies Definition 2.1.

We first note that the parent $v$ of $p$ is always assigned in Algorithm 3.4 by setting $l(p) = \lfloor \log_2(d(p,v)) \rfloor - 1$. Note that the set $U$ is never empty, when Algorithm 3.4 is launched. The covering condition (2.1b) after adding point $p$ to $T(W)$ follows from the following inequality:

$$d(p, \theta) \leq 2^{\log_2(d(p,v))} \leq 2^{l(p)+1}.$$

To check (2.1c) Consider arbitrary cover set $C_h = \{q \in T(W \cup \{p\}) \mid l(q) \geq h\}$. Since we have assumed that $T(W)$ is a valid cover tree, all the cover sets $C_h$ for $h > l(p)$ satisfy the condition. Let us consider cover sets having $h \leq l(p)$. Let $\theta \in C_h$ be an arbitrary node. Consider a sequence of iterations $l_{\min}(T(W)) \leq a(0) < a(1) < ... < a(l) = l_{\max}(T(W))$ that were considered during run-time of the algorithm. Note that the parent of $p$ was assigned at $i = a(0)$. Since $\theta \in W = S_{l_{\max}(W),T(W)}$, either (a) $\theta \in \bigcup_{q \in R_{a(0)}} S_{a(0)}(q, R_{a(0)})$ or (b) there exists index $j$ satisfying

$$\theta \in \bigcup_{q \in R_{a(0)}(j+1)} S_{a(j+1)}(q, T(W)) \setminus \bigcup_{q \in R_i} S_{a(j)}(q, T(W)).$$

Let us first consider case (a). Let $v$ be a parent of $p$ in $T(W \cup \{p\})$. Recall that the parent $v$ of $p$ was assigned in line 4 of Algorithm 3.4. Therefore we have $d(v,p) \leq d(p,\theta)$ and by line 4 we have:

$$d(p, \theta) \geq d(p,v) \geq 2^{l(p)+1} > 2^{l(p)} \geq 2^h,$$

which proves the claim.

Assume now (b) holds. Denote $i = a(j+1)$, since $a(j)$ was previous level, it follows $\eta(i) = a(j)$. By Lemma D.14 we have:

$$\bigcup_{q \in C_i(R_{\eta(i)})} S_i(q, T(W)) = \bigcup_{q \in R_{\eta(i)}} S_{\eta(i)}(q, T(W)).$$

Therefore there exists a node $u \in C_i(R_{\eta(i)}) \setminus R_i$ for which $\theta \in S_i(u, T(W))$. By line 5 of Algorithm 3.4 we have $d(u, p) > 2^{l+1}$. If $u = \theta$, then the parent of $p$ was selected from set $R_{\eta(i)}$ and the proof is similar to (a). Else by Lemma E.5 it follows that $d(p, \theta) > 2^{l(\theta)} \geq 2^h$ which proves the claim.

\qed
Lemma 3.3 (time complexity of a key step for \( T(R) \)). Arbitrarily order all points of a finite reference set \( R \) in a metric space \( (X, d) \) starting from the root: \( r = p_1, p_2, \ldots, p_{|R|} \). Set \( W_1 = \{ r \} \) and \( W_{y+1} = W_y \cup \{ p_y \} \) for \( y = 1, \ldots, |R| - 1 \). Then Algorithm E.4 builds a compressed cover tree \( T(R) \) in time

\[
O \left( (c_m(R))^8 \cdot \max_{y=1,\ldots,|R|-1} L(T(W_y), p_y) \cdot |R| \right),
\]

where \( c_m(R) \) is the minimized expansion constant from Definition 1.4.

Proof. The worst-case time complexity of Algorithm E.2 is dominated by lines 6,7 which call Algorithm E.3 \( O(|R|) \) times in total.

Assume that we have already constructed a cover tree on set \( T(W_y) \), the goal Algorithm E.3 is to construct tree \( T(W_y \cup \{ p_{y+1} \}) \). By Definition E.1 loop on lines 5 - 13 is performed \( L(T(W_y), p_{y+1}) \) times. Let \( R_s \) be the maximal size of set \( R_i \) during all iterations \( i \in L(T(W_y), p_{y+1}) \). By Lemma 2.3 since \( W_y \subseteq R \subseteq X \) we have

\[
|C_i(R_{\eta(y)})| \leq c_m(W_{y+1})^4 \cdot |R_s| \leq c_m(R)^4 \cdot |R_s|
\]

nodes, where \( C_{\eta(i)}(R_{\eta(i)}) \) is defined in line 6. Therefore both, lines 7 and 6 take at most \( c_m(R)^4 |R_s| \) time. In line 11 we handle \( |R_s| \) elements, for each of them we can retrieve index \( \text{Next}(a, i, T(W_y)) \) in \( O(1) \) time, since for every \( a \in T(R) \) we can update the last index \( j \), when \( a \) had children on level \( j \) in line 6. Therefore line 11 takes at most \( O(|R_s|) \) time. Algorithm E.4 takes at most \( O(|R_s|) \) time. Therefore line 9 and line 14 take at most \( O(|R_s|) \) time. Let us now bound \( |R_s| \) during the whole run-time of the algorithm.

Let \( i \) be an arbitrary level. Note that \( R_i \subseteq B(p, 2^{i+1}) \cap C_i \) where \( C_i \) is a \( i \) th cover set of \( T(R) \). Since \( C_i \) is \( 2^i \)-spares subset of \( R \) we can apply packing Lemma 2.2 with \( r = 2^i+1 \) and \( \delta = 2^i \) to obtain \( |B(p, 2^{i+1}) \cap C_i| \leq (c_m(W))^4 \). Lemma 1.5 implies \( (c_m(W))^4 \leq (c_m(R))^4 \), therefore \( |B(p, 2^i) \cap C_i| \leq (c_m(R))^4 \).

The time complexity of loop 6,13 in Algorithm E.3 is dominated by line 6 that has time \( O((c_m(R))^4 |R_s|) \leq O((c_m(R))^8) \). The whole Algorithm E.2 has time

\[
O((c_m(R))^8 \cdot \max_{y=1,\ldots,|R|-1} L(T(W_{y-1}), p_y) \cdot |R|)
\]
as desired. \( \square \)

Theorem 3.6 (time complexity of \( T(R) \) via aspect ratio). Let \( R \) be a finite subset of a metric space \( (X, d) \) having the aspect ratio \( \Delta(R) \). Algorithm E.4 builds a compressed cover tree \( T(R) \) in time \( O((c_m(R))^8 \cdot \log_2(\Delta(R)) \cdot |R|) \), where \( c_m(R) \) is the minimized expansion constant from Definition 1.4.

Proof. In Lemma 3.3 use the upper bounds due to Lemma 5.8, as follows:

\[
\max_{y=1,\ldots,|R|-1} L(T(W_{y-1}), p_y) \leq H(T(R)) \leq 1 + \log_2(\Delta(R)).
\]

\( \square \)

Lemma 3.7. Let \( (X, d) \) be a metric space and let \( W \subseteq X \) be its finite subset. Let \( q \in X \setminus W \) be an arbitrary point. Let \( i \in L(T(W), q) \) be arbitrarily iteration of Definition 3.1. Assume that \( t = \eta(i+1) \) is defined. Then there exists \( p \in W \) satisfying \( 2^{i+1} < d(p, q) \leq 2^{i+1} \).

Proof. Note first that since \( \eta(i+3) \in L(T(R), q) \), there exists distinct \( u \in R_{\eta(\eta(i+3))} \) and \( v \in C_{\eta(\eta(i+1))}(R_{\eta(\eta(i+1))}) \), in such a way that \( u \) is the parent of \( v \). Let us show that both of \( u, v \) can belong to set \( R_i \). Assume contrary that both \( u, v \) \( \in \) \( R_i \). Then by line 7 of Algorithm E.3 we have \( d(v, q) \leq 2^{i+1} \) and \( d(u, q) \leq 2^{i+1} \). By triangle inequality \( d(u, v) \leq d(u, q) + d(q, v) \leq 2^{i+2} \leq 2^{\eta(i+1)} \). Recall that we denote a level of a node by \( l \). On the other hand we have \( l(u) \geq \eta(i+1) \) and \( l(v) \geq \eta(i+1) \), by separation condition of Definition 2.1 we have \( d(u, v) > 2^{\eta(i+1)} \), which is a contradiction. Therefore only one of \( \{ u, v \} \) can belong to \( R_i \). It suffices two consider the two cases below:

Assume that \( v \notin R_i \). Since \( v \) is children of \( u \) we have \( d(u, v) \leq 2^{\eta(i+1)+1} \). By line 7 of Algorithm E.3 we have \( d(u, q) \leq 2^{\eta(i+1)+1} \). By triangle inequality

\[
d(v, q) \leq d(v, u) + d(u, q) \leq 2^{\eta(i+1)+1} + 2^{\eta(i+1)+1} \leq 2^{\eta(i+1)+2} \leq 2^{\eta(i+1)+1}
\]
Since \( v \notin R_t \) there exists level \( t \) having \( \eta(i+1) \geq t \geq i \) and \( v \in C_t(R_\eta(t)) \setminus R_t \). Therefore by line 7 of Algorithm E.3 we have \( d(q, v) > 2^{t+1} \geq 2^{i+1} \). It follows that we have found point \( v \in R \) satisfying \( 2^{i+1} < v \leq 2^{\eta(i+1)+1} \). Therefore \( p = v \), is the desired point.

Assume that \( u \notin R_t \). Since \( u \in R_\eta(i+1) \), by line 7 of Algorithm E.3 we have \( d(u, q) \leq 2^{\eta(i+1)+1} \). On the other hand since \( u \notin R_t \), there exists level \( t \) having \( \eta(i+3) \geq t \geq i \) and \( u \in C_t(R_\eta(t)) \setminus R_t \). Therefore by line 7 of Algorithm E.3 we have \( d(q, u) > 2^{t+2} \geq 2^{i+2} \). It follows that we have found point \( u \in R \) satisfying \( 2^{i+1} < u \leq 2^{\eta(i+1)+1} \). Therefore \( p = u \), is the desired point.

Lemma 3.8 (Construction iteration bound). Let \( A, W \) be finite subsets of a metric space \( X \) satisfying \( W \subseteq A \subseteq X \). Take a point \( q \in A \setminus W \). Given a compressed cover tree \( T(W) \) on \( W \), Algorithm 3.5 runs lines 3–14 this number of times:

\[
|L(T(W), q)| = O(c(A)^2 \cdot \log_2(|A|)).
\]

Proof. Let \( x \in L(T(R), q) \) be the lowest level of \( L(T(R), q) \). Define \( s_1 = \eta(\eta(x)+1) \) and let \( s_i = \eta(\eta(s_{i-1}+1)+1) \), if it exists. Assume that \( s_{n+1} \) is the last sequence element for which \( \eta(\eta(s_{n-1}+1)+1) \) is defined. Define \( S = \{s_1, \ldots, s_n\} \). For every \( i \in \{1, \ldots, n\} \) let \( p_i \) be the point provided by Lemma 3.7 that satisfies

\[
2^{s_i+1} < d(p_i, q) \leq 2^{\eta(s_{i+1})+1}.
\]

Let \( P \) be the sequence of points \( p_i \). Denote \( n = |P| = |S| \). Let us show that \( S \) satisfies the conditions of Lemma 2.5. Note that:

\[
4 \cdot d(p_i, q) \leq 2^{\eta(s_{i+1})+1} \leq 2^{\eta(s_{i+1}+3)} \leq 2^{\eta(\eta(s_{i+1})+1)+1} \leq 2^{s_{i+1}+1} \leq d(p_{i+1}, q).
\]

By Lemma 2.5 applied for set \( A \) and sequence \( P \) we get:

\[
|\tilde{B}(q, \frac{4}{3}d(q, p_n))| \geq (1 + \frac{1}{c(R)^2})^n \cdot |\tilde{B}(q, \frac{1}{3}(d(q, p_1)))|.
\]

Since \( \eta(x) \in L(T(R), q) \), there exists some point \( u \in R_{\eta(x)} \). By definition of \( R_t \) we have \( d(u, q) \leq 2^{\eta(x)+1} \). Also

\[
2^{\eta(s_{x+1})+1} - 2^{\eta(\eta(s_{x+1})+1)+1} \leq \frac{d(q, p_1)}{3}.
\]

It follows that:

\[
1 \leq |\tilde{B}(q, 2^{\eta(s_{x+1})+1})| \leq |\tilde{B}(q, 2^{\eta(\eta(s_{x+1})+1)+1})| \leq |\tilde{B}(q, \frac{d(q, p_1)}{3})|.
\]

Therefore we have

\[
|A| \geq \frac{|\tilde{B}(q, \frac{4}{3}d(q, p_n))|}{|\tilde{B}(q, \frac{1}{3}(d(q, p_1)))|} \geq (1 + \frac{1}{c(A)^2})^n
\]

Note that \( c(A) \geq 2 \) by definition of expansion constant. Then by applying log and by using Lemma B.7 we obtain: \( c(A)^2 \log(A) \geq n = |S| \). Let \( x \) be minimal level of \( L(T(W), q) \) and let \( y \) be the maximal level of \( L(T(W), q) \) Note that \( S \) is a sub sequence of \( L \) in such a way that:

- \([x, s_1] \cap L(T(R), q) \leq 3,
- \) for all \( i \in 1, \ldots, n \) we have \([s_i, s_{i+1}] \cap L(T(R), q) \leq 6
- \) \([s_n, y] \cap L(T(R), q) < 12

Since segments \([x, s_1], [s_1, s_2], \ldots, [s_2, s_n], [s_n, y]\) cover \([L(T(R), q)]\), it follows that \(|S| \geq \frac{|L(T(R), q)|}{12} \). We obtain that

\[
|L(T(R), q)| \leq 12 \cdot c(A)^2 \cdot \log_2(|A|),
\]

which proves the claim.
**Theorem 3.9** (time for $\mathcal{T}(R)$ via expansion constants). Let $R$ be a finite subset of a metric space $(X, d)$. Let $A$ be a finite subset of $X$ satisfying $R \subseteq A \subseteq X$. Then Algorithm F.2 launches $\mathcal{T}(R)$ in time $O((c(R))^{8} \cdot c(A)^{2} \cdot \log_{2}(|A|) \cdot |R|)$, see the expansion constants $c(A), c_{m}(R)$ in Definition F.4.

**Proof.** It follows from Lemmas 3.8 and 3.3.

**Corollary 3.10.** Let $R$ be a finite subset of a metric space $(X, d)$. Then Algorithm 3.4 builds a compressed cover tree $\mathcal{T}(R)$ in time $O((c(R))^{8} \cdot c(R)^{2} \cdot \log_{2}(|R|))$, where the constants $c(R)$, $c_{m}(R)$ appeared in Definition F.4.

**Proof.** The proof follows from Theorem 3.9 by setting $A = R$.

### F. $k$-nearest neighbor search algorithm

This section is motivated by Beygelzimer et al. [2006a] Counterexample 5.2, which showed that the proof of past time complexity claim in Beygelzimer et al. [2006a] Theorem 5) for the nearest neighbor search algorithm contained gaps. The two main results of this sections are Corollary F.7 and Theorem F.9 which provide new time complexity results for $k$-nearest neighbor problem, assuming that a compressed cover tree was already constructed for the reference set $R$. For the construction algorithm of compressed cover tree and its time complexity, we refer to Section E.

The past mistakes are resolved by introducing a new Algorithm F.2 for finding $k$-nearest neighbors that generalize and improves the original method for finding nearest neighbors using an implicit cover, Beygelzimer et al. [2006a] Algorithm 1). The first improvement is $\lambda$-point of line 6 which allows us to search for all $k$-nearest neighbors of a given query point for any $k \geq 1$. The second improvement is a new loop break condition on line 8. The new loop break condition is utilized in the proof of Lemma 4.8 to conclude that the total number of performed iterations is bounded by $O((c(R)^{2}\log(|R|))$ during whole run-time of the algorithm. The latter improvement closes the past gap in proof Beygelzimer et al. [2006a] Theorem 5) by bounding the number of iterations independently from the explicit depth Elkin & Kurlin (2022a, Definition 3.2), that generated the past confusion.

Recall from Definition D.2 that an essential set $E(p, \mathcal{T}(R)) \subseteq H(\mathcal{T}(R))$ consists of all levels $i \in H(\mathcal{T}(R))$ for which $p$ has non-trivial children in $\mathcal{T}(R)$ at level $i$. By Lemma D.5 the sizes of distinctive descendants $|S_{p}(p, \mathcal{T}(R))|$ can be precomputed in a linear time $O(|R|)$ for all $p \in R$ and $i \in E(p, \mathcal{T}(R))$. Since the size of distinctive descendant set $|S_{p}(p, \mathcal{T}(R))|$ can only change at indices $i \in E(p, \mathcal{T}(R))$, we assume that the sizes of $|S_{p}(p, \mathcal{T}(R))|$ can be retrieved in a constant time $O(1)$ for any $p \in R$ and $i \in H(\mathcal{T}(R))$ during the run-time of Algorithm F.2.

**Definition F.1.** Let $R$ be a finite subset of a metric space $(X, d)$. Let $\mathcal{T}(R)$ be a cover tree of Definition 2.1 built on $R$ and let $q \in X$ be arbitrary point. Let $L(\mathcal{T}(R), q) \subseteq H(\mathcal{T}(R))$ be the set of all levels $i$ during iterations of lines 17 of Algorithm F.2 launched with inputs $\mathcal{T}(R), q$. If Algorithm F.2 reaches line 13 at level $q \in L(\mathcal{T}(R), q)$, then we say that $q$ is special. We denote $\eta(i) = \min\{t \in L(\mathcal{T}(R), q) \mid t > i\}$.

Note that $\eta(i)$ of Definition F.1 may be undefined. If $\eta(i)$ is defined, then by definition we have $\eta(i) \geq i + 1$. Let $d_{k}(q, R)$ be the distance of $q$ to its $k$th nearest neighbor in $R$.

**Example F.4** (Simulated run of Algorithm F.2). Let $R$ and $\mathcal{T}(R)$ be as in Example B.2. Let $q = 0$ and $k = 5$. Figures 10, 11, 12 and 13 illustrate simulated run of Algorithm F.2 on input $(T(R), q, k)$. Recall that $t_{max} = 2$ and $t_{min} = -1$. During the iteration $i$ of Algorithm F.2 we maintain the following coloring: Points in $R_{i}$ are colored orange. Points $C_{\eta(i)}(R_{\eta(i)})$ (of line 5) that are not contained in $R_{i}$ are colored yellow. The $\lambda$-point of line 6 is denoted by using purple color. All the nodes that were present in $R_{\eta(i)}$, but are no longer included in $R_{i}$ will be colored red. Finally all the points that are selected as $k$-nearest neighbors of $q$ are colored green in the final iteration. Nodes that haven't been yet visited or that will never be visited are colored white. Let $R_{2} = \{8\}$. Consider the following steps:

**Iteration $i = 1$:** Figure 10 illustrates iteration $i = 1$ of the Algorithm F.2. In line 5 we find $C_{1}(R_{2}) = \{4, 8, 12\}$. Since node 4 minimizes distance $d(C_{1}(R_{2}), 0)$ and distinctive descendant set $S_{2}(4, \mathcal{T}(R))$ consists of 7 elements we get $\lambda = 4$ and therefore $d(q, \lambda) = 4 < 2^{4+2} = 8$. In line 7 we find $R_{1} = \{r \in C \mid d(0, r) \leq d(q, \lambda) + 2^{4} = 12\} = \{4, 8, 12\}$.

**Iteration $i = 0$:** Figure 11 illustrates iteration $i = 0$ of the Algorithm F.2. In line 5 we find $C_{0}(R_{1}) = \{2, 4, 6, 8, 10, 12, 14\}$. Since $|S_{1}(2, \mathcal{T}(R))| = 3$, $|S_{1}(4, \mathcal{T}(R))| = 1$ and $|T_{1}(6)| = 3$ and 6 is the node with smallest to distance 0 satisfying...
A new compressed cover tree for $k$-nearest neighbors

Algorithm F.2 $k$-nearest neighbor search by a compressed cover tree

1: **Input**: compressed cover tree $T(R)$, a query point $q \in X$, an integer $k \in \mathbb{Z}_+$
2: Set $i \leftarrow l_{\text{max}}(T(R)) - 1$ and $\eta(l_{\text{max}} - 1) = l_{\text{max}}$
3: Let $r$ be the root node of $T(R)$. Set $R_{l_{\text{max}}} = \{ r \}$.
4: while $i \geq l_{\text{min}}$ do
5: Assign $C_i(R_{\eta(i)}) \leftarrow R_{\eta(i)} \cup \{ a \in \text{Children}(p) \text{ for some } p \in R_{\eta(i)} | l(a) = i \}$
   \{Recall that \text{Children}(p) contains node $p$ \}
6: Compute $\lambda = \lambda_k(q, C_i(R_{\eta(i)}))$ from Definition\textbf{D.6} by Algorithm\textbf{D.8}.
7: Find $R_i = \{ p \in C_i(R_{\eta(i)}) | d(q, p) \leq d(q, \lambda) + 2^{i+2} \}$
8: if $d(q, \lambda) > 2^{i+2}$ then
9: Define list $S = \emptyset$
10: for $p \in R_i$ do
11: Update $S$ by running Algorithm\textbf{F.3} on $(p, i)$
12: end for
13: Compute and output $k$-nearest neighbors of the query point $q$ from set $S$.
14: end if
15: Set $j \leftarrow \max_{a \in R_i} \text{Next}(a, i, T(R))$ \{If such $j$ is undefined, we set $j = l_{\text{min}} - 1$ \}
16: Set $\eta(j) \leftarrow i$ and $i \leftarrow j$.
17: end while
18: Compute and output $k$-nearest neighbors of query point $q$ from the set $R_{l_{\text{min}}}$.

Algorithm F.3 The node collector called in line\textbf{11} of Algorithm\textbf{F.2}

1: **Input**: $p \in R$, index $i$.
2: **Output**: a list $S \subseteq R$ containing all nodes of $S_i(p, T(R))$.
3: Add $p$ to list $S$.
4: if $i > l_{\text{min}}(T(R))$ then
5: Set $j = \text{Next}(p, i, T(R))$
6: Set $C = \{ a \in \text{Children}(p) | l(a) = j \}$
7: for $u \in C$ do
8: Call Algorithm\textbf{F.3} with $(u, j)$.
9: end for
10: end if
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$$\sum_{p \in \mathcal{N}(0, 6) = \{2, 4, 6\}} |S_1(p, \mathcal{T}(R))| \geq 5 = k.$$ It follows that $\lambda = 6$. In line 7 we find $R_0 = \{r \in C(R_1) \mid d(0, r) \leq d(q, \lambda) + 2 = 10\} = \{2, 4, 6, 8, 10\}$. Since $d(q, \lambda) > 2^{i+2} = 4$. We proceed into lines 8 - 14.

**Final block** lines 8 - 14 for $i = 0$: Figure 12 marks all the points $S$ discovered by line 11 as orange. Figure 13 illustrates the final selection of $k$ points from set $S$ that are selected as the final output $\{1, 2, 3, 4, 5\}$.

---

**Figure 10.** Iteration $i = 1$ of simulation in Example F.4 of Algorithm F.2

**Figure 11.** Iteration $i = 0$ of simulation in Example F.4 of Algorithm F.2

**Figure 12.** Line 11 of Iteration $i = 0$ of simulation in Example F.4 of Algorithm F.2

**Figure 13.** Line 13 of iteration $i = 0$ of simulation in Example F.4 of Algorithm F.2

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Note that $\bigcup_{p \in R_i} S_i(p, T(R))$ is decreasing set for which $\bigcup_{p \in R_{i\text{max}}} S_{i\text{max}}(p, T(R)) = R$ and

$$\bigcup_{p \in R_{i\text{min}}} S_{i\text{min}}(p, T(R)) = R_{i\text{min}}.$$  

**Lemma F.5** ($k$-nearest neighbors in the candidate set for all $i$). Let $R$ be a finite subset of an ambient metric space $(X, d)$, let $q \in X$ be a query point and let $k \in \mathbb{Z} \cap [1, \infty)$ be a parameter. Let $T(R)$ be a compressed cover tree of $R$. Assume that $|R| \geq k$. Then for any iteration $i \in L(T(R), q)$ of Definition F.1 the candidate set $\bigcup_{p \in R_i} S_i(p, T(R))$ contains all $k$-nearest neighbors of $q$.

**Proof.** Since $R_{i\text{max}} = \{r\}$, where $r$ is the root $T(R)$ we have $S_{i\text{max}}(r, T(R)) = R$ and therefore any point among $k$-nearest neighbors of $q$ is contained in $R_{i\text{max}}$. Let $i$ be the largest index for which there exists a point among $k$-nearest neighbor of $q$ that doesn’t belong to $\bigcup_{p \in R_i} S_i(p, T(R))$. Let us denote such point by $\beta$, then:

$$\beta \in \bigcup_{p \in R_{\eta(i)}} S_{\eta(i)}(p, T(R)) \setminus \bigcup_{p \in R_i} S_i(p, T(R)).$$

By Lemma D.14 we have

$$\bigcup_{p \in C_{\eta(i)}(R_{\eta(i)})} S_i(p, T(R)) = \bigcup_{p \in R_{\eta(i)}} S_{\eta(i)}(p, T(R))$$  

(6)

Let $\lambda$ be as in line 6 of Algorithm F.2. By Equation (6) we have

$$| \bigcup_{p \in C_{\eta(i)}(R_{\eta(i)})} S_i(p, T(R)) | \geq k,$$

therefore by Definition D.6 such $\lambda$ exists. Since $\beta \in \bigcup_{p \in C_{\eta(i)}(R_{\eta(i)})} S_i(p, T(R))$, there exists $\alpha \in C_{\eta(i)}(R_{\eta(i)})$ satisfying $\beta \in S_i(\alpha, T(R))$. By assumption it follows $\alpha \notin R_i$. By line 7 of the algorithm we have

$$d(\alpha, q) > d(q, \lambda) + 2^{i+2}.$$  

(7)

Let $w$ be arbitrary point in set $\bigcup_{p \in N(q; \lambda)} S_i(p, T(R))$. Therefore $w \in S_i(\gamma, T(R))$ for some $\gamma \in N(q; \lambda)$. By Lemma D.13 applied on $i$ we have $d(\gamma, w) \leq 2^{i+1}$. By Definition D.6 since $\gamma \in N(q; \lambda)$ we have $d(q, \gamma) \leq d(q, \lambda)$. By (7) and the triangle inequality we obtain:

$$d(q, w) \leq d(q, \gamma) + d(\gamma, w) \leq d(q, \lambda) + 2^{i+1} < d(\alpha, q) - 2^{i+1}$$  

(8)

On the other hand $\beta$ is a descendant of $\alpha$ thus we can estimate:

$$d(q, \beta) \geq d(q, \alpha) - d(\alpha, \beta) \geq d(\alpha, q) - 2^{i+1}$$  

(9)

By combining Inequality (8) with Inequality (9) we obtain $d(q, w) < d(q, \beta)$. Since $w$ was arbitrary point from $\bigcup_{p \in N(q; \lambda)} S_i(p, T(R))$, that contains at least $k$ points, $\beta$ cannot be any $k$-nearest neighbor of $q$, which is a contradiction. □

**Theorem 4.4** (correctness of Algorithm F.3). Algorithm F.3 correctly finds all $k$-nearest neighbors of query point $q$ within the reference set $R$.

**Proof.** Note that Algorithm F.2 is terminated by either reaching line 18 or by going inside block 10-12.

Assume first that Algorithm F.2 is terminated by reaching line 18. Claim follows directly from Lemma F.5 by noting that since $i = l_{\text{min}}$ all the nodes $p \in R_{l_{\text{min}}}$ do not have any children. Therefore it follows $\bigcup_{p \in R_{l_{\text{min}}}} S_i(p, T(R)) = R_{l_{\text{min}}}$. Thus all the $k$-nearest neighbors of $q$ are contained in the set $R_{l_{\text{min}}}$. 

Assume then that block 10-12 is reached during some iteration $i \in L(T(R), q)$. By Lemma F.5 set $\bigcup_{p \in R_i} S_i(p, T(R))$ contains all $k$-nearest neighbors of $q$. Note that in line 11 we collect all nodes of $\bigcup_{p \in R_i} S_i(p, T(R))$ into single array $S$. Therefore in line 13 we correctly select $k$ nearest neighbors of $q$ from array $S$, which proves the claim. □
Lemma 4.5. Algorithm 4.3 has the following time complexities of its lines

\[ (a) \max\{\#\text{Line }4-10, \#\text{Line }12-15, \#\text{Line }16\} = O(c_m(R)^{10} \cdot \log_2(k)); \]
\[ (b) \#\text{Line }8-14 = O(\bar{B}(q, 5d_k(q, R)) \cdot \log_2(k)). \]

Proof. (a) Let \( q \in L(T(R), q) \) be as in Definition 2.1. Note that if iteration \( q \) is encountered, it becomes the last iteration of \( L(T(R), q) \). The total number of children encountered in line 5 during single iteration \( 4-17 \) is at most is at most \( (c_m(R))^4 \cdot \max_{i \in L(T(R), q) \setminus q} |R_i| \). From Lemma D.10 we obtain that line 6 which launches Algorithm D.8 takes at most
\[ |C(R_i)| \cdot \log_2(k) = (c_m(R))^4 \cdot \max_{i \in L(q, T(R)) \setminus q} |R_i| \cdot \log_2(k) \]
time. Line 7 never does more work than line 5 since in the worst case scenario \( R_{q(i)} \) is copied to \( R_i \) in its current form. Line 15 handles \( |R_i| \) nodes, since we can keep track of value of \( \text{Next}(a, i, T(R)) \) of Definition 2.10 by updating it when necessary in line 5 we can retrieve its value in \( O(1) \) time. Therefore maximal run-time of line 15 is \( \max_{i \in L(q, T(R)) \setminus q} |R_i| \).

Final line 18 picks lowest \( k \)-elements \( R_{q(i)} \) ranked by function \( f(p) = d(p, q) \). By Lemma D.9 it can be computed in time
\[ O(\log_2(k) \cdot \max_{i \in L(q, T(R)) \setminus q} |R_i|) \]
adjunction (10). Let us now bound \( \max_{i \in L(q, T(R)) \setminus q} |R_i| \), by showing \( |R_i| \leq c_m(R)^6 \). Let \( C_i \) be the 4th level of \( T(R) \) as in Definition 2.1. For all \( i \in L(T(R), q) \setminus q \) we have:
\[ R_i = \{ r \in C_i(R_{q(i)}) \mid d(q, \lambda) = d(q, \lambda) + 2^i + 2 \} \]
\[ \subseteq B(q, 2^i) \cap C_i \]
From cover-tree condition we know that all the points in \( C_i \) are separated by \( 2^i \). We will now apply Lemma 2.2 with \( t = 2^i + 3 \) and \( \delta = 2^i \). Since \( 4 \cdot 2^5 + 1 = 2^6 + 1 \leq 2^6 \) we obtain
\[ \max_{i \in L(q, T(R)) \setminus q} |R_i| \leq |B(q, 2^i) \cap C_i| \leq c_m(R)^6. \]
The claim follows by replacing \( \max_{i \in L(q, T(R)) \setminus q} |R_i| \) with \( c_m(R)^6 \) in (10).

(b) Let us now bound the run-time of \#Line 8-17, which runs Algorithm F.3 for all \( \langle p, i \rangle \), where \( p \in R_i \). Let \( S \) be a distinctive descendant set from Definition 2.8. Algorithm F.3 visits every node \( u \in \cup_{p \in R_i} S_i(p, T(R)) \) once, therefore its running time is \( O(\cup_{p \in R_i} |S_i(p, T(R))|) \). Let us now show that
\[ \cup_{p \in R_i} S_i(p, T(R)) \subseteq \bar{B}(q, 5d_k(q, R)) \]
Note first that by Lemma 2.5 set \( \cup_{p \in R_i} S_i(p, T(R)) \) contains all \( k \)-nearest neighbors of \( q \). Using Lemma D.15 we find \( \beta \) among \( k \)-nearest neighbors of \( q \) satisfying \( d(q, \lambda) \leq d(q, \beta) + 2^{i+1} \). From assumption It follows \( 2^{i+1} \leq d(q, \beta) \).

By line 8 we have \( d(q, \lambda) \leq 2^{i+1} \). By line 13 we perform depth-first traversal on
\[ A = \cup_{p \in R_i} S_i(p, T(R)). \]
Let \( u \in \cup_{p \in R_i} S_i(p, T(R)) \) be arbitrary node and let \( v \in R_i \) be such that \( u \in S_i(v, T(R)) \). By Lemma D.13 we have
\[ d(u, v) \leq 2^{i+1} \]. Since \( v \in R_i \) we have \( d(q, v) \leq d(q, \lambda) + 2^{i+2} \). By triangle inequality
\[ d(u, q) \leq d(u, v) + d(v, q) \leq 2^{i+1} + d(\lambda, v) + 2^{i+2} \leq 2^{i+1} + 2^{i+1} + d(q, \beta) + 2^{i+2} \leq 5 \cdot d(q, \beta) \]
It follows that \( \cup_{p \in R_i} S_i(p, T(R)) \subseteq \bar{B}(q, 5 \cdot d(q, \beta)) \).

Let us now bound the time complexity of line 13. By Lemma D.9 for any set \( A \) is takes \( \log(k) \cdot |A| \) time to select \( k \)-lowest elements. We have:
\[ \#\text{Line }8-17 = O(\bar{B}(q, 5 \cdot d_k(q, R))) \cdot \log(k). \]
Theorem 4.6. Let $R$ be a finite set in a metric space $(X, d)$, $c_m(R)$ be the minimized constant from Definition 1.4. Given a compressed cover tree $T(R)$, Algorithm 4.3 finds all $k$-nearest neighbors of a query point $q \in X$ in time

$$O \left( \log_2(k) \cdot (c_m(R))^{10} \cdot |L(q, T(R))| + |B(q, 5d_k(q, R))| \right),$$

where $L(T(R), q)$ is the set of all performer iterations (lines 13-15) of Algorithm 4.3.

Proof. Apply Lemma 4.5 to estimate the time complexity of Algorithm F.2:

$$O(\text{Line[8]} + \text{Line[14]} - \text{Line[17]} + \text{Line[18]} + \text{Line}[8] + \text{Line}[14]).$$

Corollary 4.7 gives a run-time bound using only minimized expansion constant $c_m(R)$, where if $R \subseteq \mathbb{R}^m$, then $c_m(R) \leq 2^m$. Recall that $\Delta(R)$ is aspect ratio of $R$ introduced in Definition 1.4.

Corollary 4.7. Let $R$ be a finite set in a metric space $(X, d)$. Given a compressed cover tree $T(R)$, Algorithm 4.3 finds all $k$-nearest neighbors of $q$ in time $O \left( (c_m(R))^{10} \cdot \log_2(k) \cdot \log_2(\Delta(R)) \right)$.

Proof. Replace $|L(q, T(R))|$ in the time complexity of Theorem 4.6 by its upper bound from Lemma B.8 $|L(q, T(R))| \leq |H(T(R))| \leq \log_2(\Delta(R))$. \hfill \Box

If we are allowed to use the standard expansion constant, which corresponds to KR-dimension of Krauthgamer & Lee (2004), then we obtain a stronger result, Theorem 4.9.

Lemma F.6. Let $R$ be a finite reference set in a metric space $(X, d)$ and let $q \in X$ be a query point. Let $\varrho$ be the special level of $L(T(R), q)$. Let $i \in L(T(R), q) \setminus \varrho$ be any level. Then if $p \in R_i$ we have $d(p, q) \leq 2^{i+3}$.

Proof. By assumption in this part of the algorithm we have $d(q, \varrho) \leq 2^{i+2}$. By line 7 of Algorithm F.2 since $p \in R_i$ we have $d(p, q) \leq d(q, \varrho) + 2^{i+2} \leq 2^{i+2} + 2^{i+2} \leq 2^{i+3}$, which proves the claim. \hfill \Box

Lemma F.7. Let $R$ be a finite reference set in a metric space $(X, d)$ and let $q \in X$ be a query point. Let $\varrho$ be the special level of $L(T(R), q)$. Let $i \in L(T(R), q) \setminus \varrho$ be any level. Then if $p \in C_t(R_{\eta(i)}) \setminus R_i$, we have $d(p, q) > 2^{i+2}$.

Proof. By assumption $p \in C_t(R_{\eta(i)}) \setminus R_i$. By line 7 of Algorithm F.2 it follows that $d(q, p) > 2^{i+2} + d(q, \varrho) \geq 2^{i+2}$. Therefore $d(p, q) > 2^{i+2}$, which proves the claim. \hfill \Box

Lemma F.8. Let $i$ be a non-minimal level of $L(T(R), q)$ of Definition F.1. Assume that $t = \eta(\eta(i + 3))$ is defined. Then there exists $p \in R$ satisfying $2t^{i+2} < d(p, q) \leq 2t^{i+4}$.

Proof. Note first that since $\eta(i + 3) \in L(T(R), q)$, there exists distinct $u, v \in R_{\eta(i + 3)}$ and $v \in C_{\eta(i + 3)}(R_{\eta(i + 3)})$, in such a way that $u$ is the parent of $v$. Let us show that both $u, v$ cant belong to set $R_t$. Assume contrary that both $u, v \in R_t$. Then by Lemma F.6 we have $d(v, q) \leq 2t^{i+3}$ and $d(u, q) \leq 2t^{i+3}$. By triangle inequality $d(u, v) \leq d(u, q) + d(q, v) \leq 2t^{i+4} \leq 2\eta(i + 3)$. Recall that we denote a level of a node by $l$. On the other hand we have $l(u) \geq \eta(i + 3)$ and $l(v) \geq \eta(i + 3)$, by separation condition of Definition 2.5 we have $d(u, v) > 2\eta(i + 3)$, which is a contradiction. Therefore only one of $\{u, v\}$ can belong to $R_t$. It suffices two consider the two cases below:

Assume that $v \notin R_t$. Since $v$ is children of $u$ we have $d(u, v) \leq 2\eta(i + 3) + 1$. By Lemma F.6 we have $d(u, q) \leq 2\eta(\eta(i + 3) + 3)$. By triangle inequality

$$d(v, q) \leq d(v, u) + d(u, q) \leq 2\eta(\eta(i + 3) + 3) + 2\eta(i + 3) + 1 \leq 2\eta(\eta(i + 3) + 4).$$

Since $v \notin R_t$ there exists level $t$ having $\eta(i + 3) \geq t \geq i$ and $v \in C_t(R_{\eta(i)}) \setminus R_t$. Therefore by Lemma F.7 we have $d(v, q) > 2^{t^{i+2}} \geq 2^{i+2}$. It follows that we have found point $v \in R$ satisfying $2^{i+2} < v \leq 2\eta(i + 3) + 4$. Therefore $p = v$, is the desired point.

Assume that $u \notin R_t$. Since $u \in R_{\eta(i + 3)}$, by Lemma F.6 we have $d(u, q) \leq 2\eta(i + 3) + 3$. On the other hand since $u \notin R_t$, there exists level $t$ having $\eta(i + 3) \geq t \geq i$ and $u \in C_t(R_{\eta(i)}) \setminus R_t$. Therefore by Lemma F.7 we have
\[ d(q, u) > 2^{i+2} \geq 2^{i+2}. \] It follows that we have found point \( u \in R \) satisfying \( 2^{i+2} < u \leq 2^{\eta(\eta(i+3)+4}. \) Therefore \( p = u, \) is the desired point.

**Lemma 4.8.** Algorithm 4.3 executes lines 4-15 the following number of times: \( |L(T(R), q)| = O(c(R \cup \{ q \})^2 \cdot \log_2(|R|)). \)

**Proof.** Let \( x \in L(T(R), q) \) be the lowest level of \( L(T(R), q). \) Define \( s_1 = \eta(\eta(x)+1) \) and let \( s_i = \eta(\eta(s_{i-1}+3)+3), \) if it exists. Assume that \( s_{n+1} \) is the last sequence element for which \( \eta(\eta(s_{n-1}+3)+3) \) is defined. Define \( S = \{ s_1, \ldots, s_n \}. \) For every \( i \in \{ 1, \ldots, n \} \) let \( p_i \) be the point provided by Lemma 2.5 that satisfies
\[ 2^{s_i+2} < d(p_i, q) \leq 2^{\eta(\eta(s_i+3)+4}. \]

Let \( P \) be the sequence of points \( p_i. \) Denote \( n = |P| = |S|. \) Let us show that \( S \) satisfies the conditions of Lemma 2.5. Note that:
\[ 4 \cdot d(p_i, q) \leq 4 \cdot 2^{\eta(\eta(s_i+3)+4} \leq 2^{\eta(\eta(s_i+3)+6} \leq 2^{\eta(\eta(s_i+3)+3)+2} \leq 2^{s_{i+1}+2} \leq d(p_{i+1}, q) \]

By Lemma 2.5 applied for \( A = R \cup q \) and sequence \( P \) we get:
\[ |B(q, \frac{4}{3} d(p_i, p_n))| \geq (1 + \frac{1}{c(R)^2})^n \cdot |B(q, \frac{1}{3} d(q, p_1))| \]

Since \( \eta(x) \in L(T(R), q) \), there exists some point \( u \in R_\eta(x). \) By Lemma F.6 we have \( d(u, q) \leq 2^{\eta(x)+3}. \) Also \n\[ 2^{\eta(\eta(s_i+3)+3} \leq \frac{2^{\eta(s_i+3)+2}}{3} < \frac{d(q, p_1)}{3}. \] It follows that:
\[ 1 \leq |B(q, 2^{\eta(x)+3})| \leq |B(q, 2^{\eta(s_i+3)} + 1)| \leq |B(q, \frac{d(q, p_1)}{3})| \]

Therefore we have
\[ |R| \geq \frac{|B(q, \frac{4}{3} d(p_i, p_n))|}{|B(q, \frac{1}{3} d(q, p_1))|} \geq (1 + \frac{1}{c(R \cup \{ q \})^2})^n \]

Note that \( c(R \cup \{ q \}) \geq 2 \) by definition of expansion constant. Then by applying \( \log \) and by using Lemma B.7 we obtain:
\[ c(R \cup \{ q \})^2 \log(|R|) \geq n = |S|. \] Let \( x \) be minimal level of \( L(T(R), q) \) and let \( y \) be the maximal level of \( L(T(R), q) \) Note that \( S \) is a subsequence of \( L \) in such a way that:

- \([x, s_1] \cap L(T(R), q) \leq 3,\]
- for all \( i \in 1, \ldots, n \) we have \([s_i, s_{i+1}] \cap L(T(R), q) \leq 10,\]
- \([s_n, y] \cap L(T(R), q) \leq 20\]

Since segments \([x, s_1], [s_1, s_2], \ldots, [s_2, s_n], [s_n, y]\) cover \( L(T(R), q), \) it follows that \( |S| \geq \frac{L(T(R), q)}{20}. \) We obtain that
\[ |L(T(R), q)| \leq 20 \cdot c(R \cup \{ q \})^2 \cdot \log_2(|R|), \]

which proves the claim. □

**Theorem 4.9.** Let \( R \) be a finite reference set in a metric space \((X, d)\). Let \( q \in X \) be a query point, \( c(R \cup \{ q \}) \) be the expansion constant of \( R \cup \{ q \} \) and \( c_m(R) \) be the minimized expansion constant from Definition 1.4. Given a compressed cover tree \( T(R), \) Algorithm 4.3 finds all \( k \)-nearest neighbors of \( q \) in time \( O\left( c(R \cup \{ q \})^2 \cdot \log_2(k) \cdot \left( (c_m(R))^{10} \cdot \log_2(|R|) + c(R \cup \{ q \}) \cdot k \right) \). \]

**Proof.** By Theorem 4.6, the required time complexity is
\[ O\left( (c_m(R))^{10} \cdot \log_2(k) \cdot |L(q, T(R))| + |B(q, 5d(q, \beta))| \cdot \log_2(k) \right) \]
for some point $\beta$ among the first $k$-nearest neighbors of $q$. Apply Definition 1.4

$$|B(q, 5d(q, \beta))| \leq (c(R \cup \{q\}))^3 \cdot |B(q, \frac{5}{8}d(q, \beta))|$$

(14)

Since $|B(q, \frac{5}{8}d(q, \beta))| \leq k$, we have $|B(q, 5d(q, \beta))| \leq (c(R \cup \{q\}))^3 \cdot k$. It remains to apply Lemma 4.8 $|L(q, T(R))| = O(c(R \cup \{q\})^2 \cdot \log_2 |R|)$.

Corollary F.9 combines Theorem 3.9 with Theorem 4.9 to show that Problem 1.3 can be solved in $O(\epsilon^{O(1)} \cdot \log(k) \cdot \max(|Q|, |R|) \cdot \log(|R|) + k)$ time.

**Corollary F.9** (solution to Problem 1.3). In the notations of Theorem 4.9 set $c = \max_{q \in Q} c(R \cup \{q\})$. Algorithms E.2 and F.2 solve Problem 1.3 in time

$$O\left(\max(|Q|, |R|) \cdot c^2 \cdot \log_2(k) \cdot ((c_m(R))^{10} \cdot \log_2(|R|) + c \cdot k)\right).$$

**Proof.** For any $q \in Q$, since $\log_2 |R \cup \{q\}| \leq 2 \log_2 |R|$, a tree $T(R)$ can be built in time

$$O(c^2 \cdot c_m(R)^8 \cdot \log |R|)$$

by Theorem 3.9. Therefore the time complexity is dominated by running Algorithm E.2 on all points $q \in Q$. The final complexity is obtained by multiplying the time from Theorem 4.9 by $|Q|$.

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**G. Approximate $k$-nearest neighbor search**

The original navigating nets and cover trees were used in Krauthgamer & Lee (2004, Theorem 2.2) and Beygelzimer et al. (2006a, Section 3.2) to solve the $(1 + \epsilon)$-approximate nearest neighbor problem for $k = 1$. The main result, Theorem G.6 justifies a near linear parameterized complexity to find approximate a $k$-nearest neighbor set $P$ formalized in Definition G.1

**Definition G.1** (approximate $k$-nearest neighbor set $P$). Let $R$ be a finite reference set and let $Q$ be a finite query set of a metric space $(X, d)$. Let $q \in Q \subseteq X$ be a query point, $k \geq 1$ be an integer and $\epsilon > 0$ be a real number. Let $N_k = \bigcup_{i=1}^k NN_i(q)$ be the union of neighbor sets from Definition 1.2. A set $P \subseteq R$ is called an approximate $k$-nearest neighbors set, if $|P| = k$ and there is an injection $f : P \rightarrow N_k$ satisfying $d(q, p) \leq (1 + \epsilon) \cdot d(q, f(p))$ for all $p \in P$.

**Definition G.3** (Iteration set of approximate $k$-nearest neighbor search). Let $R$ be a finite subset of a metric space $(X, d)$. Let $T(R)$ be a cover tree of Definition 2.2 built on $R$ and let $q \in X$ be an arbitrary point. Let $L(T(R), q) \subseteq H(T(R))$ be the set of all levels $i$ during iterations of lines 3/19 of Algorithm G.2 launched with inputs $(T(R), q)$. We denote $\eta(i) = \min_{t \in L(T(R), q)} \{t \mid t > i\}$.

**Lemma G.4** (k-nearest neighbors in the candidate set for all $i$). Let $R$ be a finite subset of an ambient metric space $(X, d)$, let $q \in X$ be a query point, let $k \in \mathbb{Z} \cap [1, \infty)$ and $\epsilon \in \mathbb{R}_+$ be parameters. Let $T(R)$ be a compressed cover tree of $R$. Assume that $|R| \geq k$. Then for any iteration $i \in L(T(R), q)$ of Algorithm G.2 the candidate set $\bigcup_{p \in R_i} S_i(p, T(R))$ contains all $k$-nearest neighbors of $q$.

**Proof.** Proof of this lemma is similar to Lemma G.4 and is therefore omitted.

**Lemma G.5** shows that Algorithm G.2 correctly returns an Approximate $k$-nearest neighbor set of Definition G.1.

**Lemma G.5** (Correctness of Algorithm G.2). Algorithm G.2 finds an approximate $k$-nearest neighbors set of any query point $q \in X$.

**Proof.** Assume first that condition on line 7 of Algorithm G.2 is satisfied during some iteration $i \in H(T(R))$ of Algorithm G.2. Let us denote

$$A = \bigcup_{p \in C_i(R_\eta(i))} \{S_i(p, T(R)) \mid d(p, q) < d(q, \lambda)\}, B = \bigcup_{p \in C_i(R_\eta(i))} \{S_i(p, T(R)) \mid d(p, q) = d(q, \lambda)\}.$$

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Algorithm G.2 This algorithm finds approximate $k$-nearest neighbor of Definition [G.1]

1: **Input**: compressed cover tree $T(R)$, a query point $q \in X$, an integer $k \in \mathbb{Z}_+$, real $\epsilon \in \mathbb{R}_+$.
2: Set $i \leftarrow l_{\max}(T(R)) - 1$ and $\eta(l_{\max} - 1) = l_{\max}$. Set $R_{l_{\max}} = \{\text{root}(T(R))\}$.
3: while $i \geq l_{\min}$ do
4:   Assign $C_i(R_{\eta(i)}) \leftarrow R_{\eta(i)} \cup \{a \in \text{Children}(p) \mid p \in R_{\eta(i)} \mid \lambda(a) = i\}$.
5:   Compute $\lambda = \lambda_k(q, C_i(R_{\eta(i)}))$ from Definition [D.6] by Algorithm [D.8].
6:   Find $R_i = \{p \in C_i(R_{\eta(i)}) \mid d(q, p) \leq d(q, \lambda) + 2^{i+1}\}$.
7:   if $\frac{2^{i+2} + 2^{i+1}}{\epsilon} \leq d(q, \lambda)$ then
8:      Let $\mathcal{P} = \emptyset$.
9:      for $p \in C_i(R_{\eta(i)})$ do
10:         if $d(p, q) < d(q, \lambda)$ then
11:            $\mathcal{P} = \mathcal{P} \cup S_i(p, T(R))$
12:         end if
13:      end for
14:      Fill $\mathcal{P}$ until it has $k$ points by adding points from sets $S_i(p, T(R))$, where $d(p, q) = d(q, \lambda)$.
15:      return $\mathcal{P}$.
16:   end if
17:   Set $j \leftarrow \max_{a \in R_i} \text{Next}(a, i, T(R))$. (If such $j$ is undefined, we set $j = l_{\min} - 1$)
18:   Set $\eta(j) \leftarrow i$ and $i \leftarrow j$.
19: end while
20: Compute and output $k$-nearest neighbors of query point $q$ from the set $R_{l_{\min}}$.

By Algorithm [G.2], set $\mathcal{P}$ contains all points of $\mathcal{A}$ and rest of the points are filled form $\mathcal{B}$. We will now form $f : \mathcal{P} \rightarrow \mathcal{N}_k$ by mapping every point $p \in \mathcal{A} \cap \mathcal{P}$ into itself and then by extending $f$ to be injective map on whole set $\mathcal{P}$. The claim holds trivially for all points $p \in \mathcal{A} \cap \mathcal{P}$. Let us now consider points $p \in \mathcal{P} \setminus \mathcal{A}$. Let $\gamma \in C_i(R_{\eta(i)})$ be such that $p \in S_i(\gamma, T(R))$. Let $\psi \in C_i(R_{\eta(i)})$ be such that $f(p) \in S_i(\psi, T(R))$. By using triangle inequality, Lemma [B.6] and the fact that $p \in \mathcal{A} \cup \mathcal{B}$ we obtain:

$$d(q, p) \leq d(q, \gamma) + d(\gamma, \lambda) \leq d(q, \lambda) + 2^{i+1}$$

On the other hand since $f(p) \notin \mathcal{A}$ we have

$$(1 + \epsilon) \cdot d(q, f(p)) \geq (1 + \epsilon) \cdot (d(q, \psi) - d(\psi, f(p))) \geq (1 + \epsilon) \cdot (d(q, \lambda) - 2^{i+1})$$

Note that by line 7 we have $\frac{2^{i+2} + 2^{i+1}}{\epsilon} \leq d(q, \lambda)$. It follows that $2^{i+2} \leq \epsilon \cdot d(q, \lambda) - \epsilon \cdot 2^{i+1}$. Therefore we have:

$$d(q, \lambda) + 2^{i+1} \leq d(q, \lambda) + 2^{i+2} - 2^{i+1} \leq (1 + \epsilon) \cdot (d(q, \lambda) - 2^{i+1})$$

By combining Equations (15), (16) we obtain $d(q, p) \leq (1 + \epsilon) \cdot d(q, f(p))$. If the condition on line 7 of Algorithm [G.2] is never satisfied, then the Algorithm finds real $k$-nearest neighbors of point $q$ in the end of the algorithm and therefore the claim holds.

**Theorem G.6** (Time complexity of Algorithm [G.2]). In the notations of Definition [G.7] the complexity of Algorithm [G.2] is:

$$O\left((c_m(R))^{8 + \lceil \log(2 + \frac{1}{\epsilon}) \rceil} \cdot \log_2(k) \cdot \log_2(\Delta(R)) + k\right).$$

**Proof.** Similarly to Lemma [4.5] it can be shown that Algorithm [G.2] is bounded by:

$$O((c_m(R))^4 \cdot \max_i |R_i| \cdot |H(T(R))| + \#\text{Line } 7-16)$$

Note first that in lines 7-16 we loop over set $C_i(R_{\eta(i)})$ and select $k$ points from it. Therefore $\#\text{Line } 7-16 = k + |C_i(R_{\eta(i)})|$. 

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Let us now bound the size of $R_i$. By line 7 of Algorithm G.2, either Algorithm G.2 is launched that terminates the program or $\frac{2^{i+2}}{\epsilon} + 2^{i+1} > d(q, \lambda)$. Let $C_i$ be the $i$th cover set of $T(R)$. To bound $|R_i|$ we can assume the latter. Similarly to Theorem 4.9 we have:

$$R_i = \{ r \in C_i(R_{\eta(i)}) \mid d(p, q) \leq d(q, \lambda) + 2^{i+2} \}$$

$$= \bar{B}(q, d(q, \lambda) + 2^{i+2}) \cap C_i(R_{\eta(i)})$$

$$\subseteq \bar{B}(q, d(q, \lambda) + 2^{i+2}) \cap C_i$$

$$\subseteq \bar{B}(q, 2^{i+2}(\frac{3}{2} + \frac{1}{\epsilon}) \cap C_i$$

Since the cover set $C_i$ is a $2^i$-sparse subset of the ambient metric space $X$, we can apply Lemma 2.2 with $t = 2^{i+2}(\frac{3}{2} + \frac{1}{\epsilon})$ and $\delta = 2^i$. Since $4\frac{3}{8} + 1 = 2^{4}(\frac{3}{2} + \frac{1}{\epsilon}) + 1 \leq 2^{4}(2 + \frac{1}{\epsilon})$, we get $\max |R_i| \leq (c_m(R))^4 + \log d(2^{i+2})$. The final complexity is obtained by plugging the upper bound of $|R_i|$ above into (18).

**Corollary G.7** (complexity for approximate $k$-nearest neighbors set $P$). *In the notations of Definition G.7 an approximate $k$-nearest neighbors set is found for all $q \in Q$ in time $O \left( \left| Q \right| \cdot (c_m(R))^{8 + \log d(2^{i+2})} \cdot \log(k) \cdot \log d(\Delta(R)) \right) + |Q| \cdot k$.*

**Proof.** This corollary follows directly from Theorem G.6.

**H. Discussions: current contributions and future steps**

This paper rigorously proved the time complexity of the exact $k$-nearest neighbor search. The motivations were the past gaps in the proofs of time complexities in Beygelzimer et al. (2006a, Theorem 5), Ram et al. (2009, Theorem 3.1), March et al. (2010, Theorem 5.1). Though Elkin & Kurlin (2022a) provided concrete counterexamples, no corrections were published. Main Theorem 4.9 and Corollary 5.10 have finally filled the above gaps.

To overcome all past obstacles, first Definition 1.2 and Problem 1.3 rigorously dealt with a potential ambiguity of $k$-nearest neighbors at equal distances, which was not discussed in the past work.

A new compressed cover tree in Definition 2.1 substantially simplified the navigating nets Krauthgamer & Lee (2004) and original cover trees Beygelzimer et al. (2006a) by avoiding any repetitions of given data points. This compression has substantially clarified the construction and search Algorithms E.2 and F.2.

Second, section C showed that the new minimized expansion constant $c_m$ of any finite subset $R$ of a normed vector space $\mathbb{R}^n$ has the upper bound $2^m$. In the future, it can be similarly shown that if $R$ is uniformly distributed then classical expansion constant $c(R)$ is $2^m$ as well.

Third, sections E and F corrected the approach of Beygelzimer et al. (2006a) as follows. Assuming that expansion constants and aspect ratio of a reference set $R$ are fixed, Corollaries 5.10 and F.9 rigorously showed that the times are linear in the maximum size of $R, Q$ and near-linear $O(k \log k)$ in the number $k$ of neighbors.

The future problem is to improve the complexity of $k$-nearest neighbor search to a pure linear time $O(c(R)^{O(1)} |R|)$ by using cover trees on both sets $Q, R$. Since a similar approach Ram et al. (2009) was shown to have incorrect proof in Elkin & Kurlin (2022a, Counterexample 6.5) and Curtin et al. (2015); Elkin & Kurlin (2022b) used additional parameters $I, \theta$, this goal will require significantly more effort to understand if $O(c(R)^{O(1)} |R|)$ is achievable by using a compressed cover tree.

Corollary F.9 allowed us to justify the near-linear time of generically complete PDD Widdowson & Kurlin (2021) invariants (Pointwise Distance Distributions), which recently distinguished all (more than 600 thousand) periodic crystals in the world’s largest database of real materials Widdowson et al. (2022). Due to these ultra-fast invariants, more than 200 billion pairwise comparisons were completed over two days on a modest desktop while past tools were estimated to require over 34 thousand years Widdowson & Kurlin (2022). The huge speed of PDD is complemented by slower but provably complete invariant isosets Anosova & Kurlin (2021) with continuous metrics that allow polynomial-time approximations Anosova & Kurlin (2022).
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