Approximation Algorithms for Fair Range Clustering

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Abstract
This paper studies the fair range clustering problem in which the data points are from different demographic groups and the goal is to pick $k$ centers with the minimum clustering cost such that each group is at least minimally represented in the centers set and no group dominates the centers set. More precisely, given a set of $n$ points in a metric space $(P, d)$ where each point belongs to one of the $\ell$ different demographics (i.e., $P = P_1 \cup P_2 \cup \cdots \cup P_\ell$) and a set of $\ell$ intervals $[\alpha_1, \beta_1], \ldots, [\alpha_\ell, \beta_\ell]$ on desired number of centers from each group, the goal is to pick a set of $k$ centers $C$ with minimum $\ell_p$-clustering cost (i.e., $\left( \sum_{v \in P} d(v, C)^p \right)^{1/p}$) such that for each group $i \in \ell$, $|C \cap P_i| \in [\alpha_i, \beta_i]$. In particular, the fair range $\ell_p$-clustering captures fair range $k$-center, $k$-median and $k$-means as its special cases. In this work, we provide an efficient constant factor approximation algorithm for the fair range $\ell_p$-clustering for all values of $p \in [1, \infty)$.

1. Introduction
In recent years, the centroid-based clustering problem has been studied extensively from the fairness point of view. As in the human-centric applications, the input data usually comes from different demographics and the solution has supposedly some (in many cases, long-lasting) effects on the participants (e.g., college admissions, loan applications, and criminal justice), it is crucial to take into account the societal implications of the solution output by large scale automated processes in use. Specifically, since clustering is commonly used as a prepossessing step for more complicated ML pipelines, it is both easier and more effective to handle fairness consideration and bias reduction earlier in the pipeline rather than later.

Fair clustering was first studied by Chierichetti et al. (2017) and since then it has been studied with respect to various notions of fairness (Bera et al., 2019; Jung et al., 2020; Mahabadi & Vakilian, 2020; Ahmadi et al., 2022; Chen et al., 2019; Abbasi et al., 2021; Ghadiri et al., 2021; Brubach et al., 2020; 2021). Motivated by the application of centroid-based clustering as a means of data summarization (e.g., (Moens et al., 1999; Girdhar & Dudek, 2012)) for socioeconomic data, Kleindessner, Awasthi, and Morgenstern (2019) studied a notion of fair clustering in which the points belong to disjoint protected groups $P_1, \ldots, P_\ell$, and the goal is to pick exactly $k_i$ centers from each population $P_i$, s.t. it minimizes the $k$-center clustering cost (i.e., maximum distance of any point to the selected centers), where $k = \sum_{i \in \ell} k_i$.

To give an example, consider an image search system. In practice, when a query is made, the user will only check the first few images output by the system. Those images (say, the first $k$ ones) act as a summary or representative of the relevant images to the searched query. Notably, (Kay et al., 2015) observed that in a few jobs, including CEO, women are significantly underrepresented in Google image search results. An approach to get around this disparity, as proposed by Kleindessner et al. (2019), is to force the solution to contain exactly $k_i$ examples (or representative) from each protected group $i$, where $k_i$s are input parameters. Although this strict center selection requirement seems to be a plausible fix for the unfairness issue, it may incur a significant loss in the quality of the output solution! See Figure 1.

In this paper, we study a relaxed requirement, called fair range, which only requires the number of selected centers from each protected group to be in a given interval specified by a lower bound and an upper bound. As we can easily tolerate slight deviations from the “expected” presence of each protected group, fair range is a more natural requirement and has a better alignment with practice, compared to the strict requirement. On the other hand, the larger the requirement interval becomes, the better the quality of the clustering becomes. While fair clustering with strict

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1More recent studies also show that this phenomenon of unfairness in image search results is not fully resolved yet, e.g., (Feng & Shah, 2022).
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Figure 1. In this example, \( \frac{n}{2} \) points with pairwise distance \( m \) belong to the red group on the right side of the figure. The other \( \frac{n}{2} \) points belong to the blue group on the left side, partitioned into groups \( B_1, \ldots, B_{2k} \), each of size \( \frac{m}{2k} \). For every \( i \neq j \in \{1, 2, \ldots, 2k\} \), the pairwise distance of \( v, u \in B_i \) is \( m \) and the pairwise distance of \( v \in B_i, u \in B_j \) is \( M \gg m \). Moreover, the pairwise distance of any red and blue point is \( M \). Note that the described distance function is metric. In the “strict” fair \( k \)-center, where \( \frac{k}{2} \) centers should be picked from each of the red and blue group, the optimal solution has cost \( M \). However, if we relax the requirement as in the fair range clustering and require \( \lfloor \frac{k}{4}, \frac{3k}{4} \rfloor \) centers from each group, the \( k \)-center cost significantly reduces from \( M \) to \( m \) (the solution corresponds to picking one center from every \( B_i \), for \( i \in \{1, 2, \ldots, 2k\} \) and arbitrary \( \frac{k}{2} \) centers from \( R \)). While the latter solution is admissibly fair too, its quality is significantly better.

(i.e., exactly \( k_i \) from each group \( i \)) requirement—which itself is a special case of clustering under partition matroid constraints—is a well-studied problem in the domain of fair clustering (Hajiaghayi et al., 2010; Krishnaswamy et al., 2011; Charikar & Li, 2012; Swamy, 2016; Chen et al., 2016; Krishnaswamy et al., 2018; Jones et al., 2020; Chiplunkar et al., 2020), this natural generalization of the problem has not been studied in the context of fair clustering. We remark that, very recently, Nguyen et al. (2022) studied fair range \( k \)-center. They studied the problem both in the standard offline and the streaming models and provided constant factor approximations in both regimes. Moreover, Thejaswi et al. (2021) studied the problem when we are provided with lower bounds only and the objective is \( k \)-median cost. They showed fixed parameter algorithms for this problem which is referred to as diversity-aware \( k \)-median. However, the approximability of the general problem of fair range clustering with the \( \ell_p \)-objective which includes the standard \( k \)-median and \( k \)-means still remains open.

Our contributions. In this paper, we study the fair range clustering with the \( \ell_p \)-objective and provide a constant factor approximation algorithm for all values of \( p \in [1, \infty) \). More precisely, our main result is as follows.

Theorem 1.1. For all \( p \in [1, \infty) \), there exists a constant factor approximation algorithm for fair range \( k \)-clustering with the \( \ell_p \)-objective that runs in polynomial time.

More generally, our algorithm works for fair range facility location with zero opening cost facilities which generalizes fair range clustering.

As fair range \( k \)-clustering with \( \ell_p \)-objective has \( k \)-center as its special case (\( k \)-center is equivalent to fair range \( k \)-center where for every \( i \in [k] \), \( \alpha_i = 0 \) and \( \beta_i = k \)), it is NP-hard to approximate it within a factor better than \( 2 \) (Gonzalez, 1985). So, our approximation factor for the problem is tight up to a constant factor.

To design our algorithm, we start with the framework of (Charikar et al., 2002; Krishnaswamy et al., 2011) to sparsify the input instance and find a feasible fractional solution of the standard LP-relaxation of the problem, called FAIRRANGE-LP, on the sparse instance. However, the main difficulty is within the rounding part. In our algorithm, we further exploit the combinatorial structures of an approximately optimal fractional solution of FAIRRANGE-LP. We consider another relaxation of the problem, STRUCTURED-LP, whose optimal solution is within an \( e^{O(p)} \) factor of the optimal solution of FAIRRANGE-LP. Then, using the techniques from Polyhedral Combinatorics, we show that STRUCTURED-LP has a half-integral optimal solution. Next, by a reduction to the network flow problem, we design a combinatorial algorithm that outputs an \( e^{O(p)} \)-approximation integral solution of STRUCTURED-LP, relying on the properties of a half-integral optimal solution of the LP. Finally, we can convert this feasible integral solution of STRUCTURED-LP to a feasible integral solution of FAIRRANGE-LP without losing more than a constant factor in the approximation ratio.

Refer to Section 2 for an overview of our algorithm and a summary of the first part of our algorithm which is standard and also used in some of prior works on (fair) clustering. Then, in Section 3, we describe our efficient rounding approach for fair range clustering.

Other related work. Clustering has been an active area of research in the domain of fairness for algorithms and machine learning. In particular, it has been studied under various settings including group fairness notions such as fair representation (Chierichetti et al., 2017; Bera et al., 2019; Bercea et al., 2019; Backurs et al., 2019; Ahmadian et al., 2019; Dai et al., 2022), social fairness (Abbasi et al., 2021; Ghadiri et al., 2021; Makarychev & Vakilian, 2021; Chlamtac et al., 2022; Ghadiri et al., 2022), proportional fairness (Chen et al., 2019; Micha & Shah, 2020) and in-
individual fairness (Jung et al., 2020; Mahabadi & Vakilian, 2020; Chakrabarty & Negahbani, 2021; Vakilian & Yalcın, 2022; Ahmadi et al., 2022; Brubach et al., 2020; 2021).

A similar notion has been studied for the related problem of nearest neighbor search (Har-Peled & Mahabadi, 2019; Aumüller et al., 2020; 2021) and submodular maximization (El Halabi et al., 2020).

Remark 1.2. While the collection of solutions satisfying the given range constraints does not constitute a matroid (since these constraints do not satisfy the downward-closed property), (El Halabi et al., 2020) showed that the family of all “extendable” subsets is indeed a matroid.\(^3\) In our context, a set of facilities \(F\) is extendable if it is a subset of facilities \(F'\) where \(F'\) has size at most \(k\) and satisfies all given range constraints. More precisely, \(F\) is extendable if and only if for all \(i \in [ℓ]\), \(|F \cap P_i| \leq β_i\) and \(\sum_{i ∈ [∗]} \max\{|F \cap P_i|, α_i\} ≤ k\). Then, one can use an existing algorithm for the \(k\)-clustering under matroid constraint (e.g., (Krishnaswamy et al., 2011; Swamy, 2016; Krishnaswamy et al., 2018)) and find a constant factor approximation for fair range clustering. While the above algorithm only considers the \(k\)-median objective, it is known that this approach can be generalized to any value of \(p \in [1, ∞)\), see (Vakilian & Yalcın, 2022).

While working with the extendable sets reduces our problem to matroid \(k\)-clustering, the known algorithm will require solving LPs with exponentially many constraints. Although these LPs can be solved in polynomial time using the ellipsoid algorithm, the best known algorithm for solving such LPs, which is via cutting plane methods, runs in time \(O(n^4 \log(1/ε))\) and returns a \((1 + ε)\)-approximate solution (Jiang et al., 2020). However, in our approach, we work with small size LPs of the fair range clustering that can be solved in time \(O(n^{1.5}k^{1.5})\) via the interior point method (Van Den Brand et al., 2021). So, arguably our algorithm is more efficient (in particular, for the case \(k = O(1)\), we obtain a quadratic improvement in the run-time).

2. Description of Our Algorithm

For the sake of clarity, similarly to (Swamy, 2016) on clustering under matroid constraints, we describe our algorithm for a more general problem of fair range facility location with the \(ℓ_p\)-objective in which the number of open facilities are required to be (at most) \(k\). For our application of clustering, it suffices to consider the instances of fair range facility locations in which the set of facilities \(F\) is disjoint from the set of clients, where the clients are specified by a pair of set of locations \(D\) and a demand function \(w : D → Z_{≥0}\). Note that while \(F\) and \(D\) are disjoint, they may have points at the same location.

More specifically, let \((P, d)\) be a metric space. In this problem, we are given a set of facilities \(F := F_1 \cup \cdots \cup F_ℓ\) where \(F \subseteq P\), a set of integral range parameters \(\{α_i, β_i\}_{i ∈ [ℓ]}\) where \(α_i, β_i ∈ Z_{≥0}\), a set of locations \(D \subseteq P\), and a demand function \(w : D → Z_{≥0}\) denoting the total number of clients in each location of \(D\).\(^4\) The goal is to open a subset \(C ⊆ F\) of \(k\) facilities with minimum \(ℓ_p\)-cost assignment of clients to their closest facilities (i.e., \(\sum_{v ∈ D} w(v) \cdot d(v, C)^p\))\(^{1/p}\), such that for every group \(i \in [ℓ]\), \(|C \cap F_i| ∈ [α_i, β_i]\).

Remark 2.1. In the description of our algorithm and in particular, in the objective of the LP-relaxations, we consider the \(ℓ_p\)-clustering raised to the power of \(p\). Throughout the paper, we provide all approximation factors according to \(\sum_{v ∈ D} w(v) \cdot d(v, C)^p\) denoted as cost. Then, at the end, to derive the result of the main theorem, we raise the approximation factor to \(1/p\).

Our algorithm has two major components: (1) finding an approximate fractional solution \((x, y)\) for an LP-relaxation of the problem with certain well-separatedness and locality properties, (2) rounding the fractional solution to an integral solution without losing more than a factor of \(e^{O(p)}\) in the LP objective. The first part is essentially a standard approach in approximation algorithm for clustering, introduced in (Charikar et al., 2002), and is similar to the one used in (Krishnaswamy et al., 2011; Swamy, 2016). So, here we only describe the properties of the fractional solution at the end of the first part and for a detailed exposition, we refer to Appendix B.

The main technical contribution of our paper is an efficient rounding algorithm that preserves the clustering with \(ℓ_p\)-objective (for all \(p \in [1, ∞)\)) up to a constant factor, does not violate any of fairness constraints, and opens at most \(k\) centers.

2.1. Reducing the number of locations

Given a set of points \(P\), we first run an existing efficient constant factor approximation algorithm for \(k\)-clustering with \(ℓ_p\)-objective (ignoring all range constraints) to get a set of centers \(C = (c_1, \cdots, c_k)\). Then, we construct the following instance of fair range clustering. We separate the set of clients \((D)\) and facilities \((F)\) as follows: Each point in \(P\) is moved to its closest center in \(C\) and the resulting set of points located at \(k\) locations \(c_1, \cdots, c_k\) constitutes the set of clients. In other words, the set of clients can be described in terms of \(k\) locations plus the total number of clients in each location determined by \(w' : D → Z_{≥0}\). For

\(^3\)The work of (El Halabi et al., 2020) used this idea to study the fairness of an objective different from clustering, namely submodular maximization.

\(^4\)The integrality of \(α_i, β_i\) are without loss of generality as we can always replace them with \([α_i]\) and \([β_i]\) and the solution space remains the same.
the facilities, we set \( F = P \). Then, we solve the minimum cost fair-range clustering (or facility location w.r.t. \( D \) and \( F \)) that picks \( k \) centers (or facilities) from \( F \) and serves all clients.

**Theorem 2.2.** Given an \( O(\alpha) \)-approximation algorithm of \( k \)-clustering with \( \ell_p \)-objective, a \( \beta \)-approximate solution \( S \) of fair range clustering with \( \ell_p \)-objective on the described instance \((D, F)\) is an \( O(\alpha \beta) \)-approximation for fair range clustering with \( \ell_p \)-objective on the original instance \( P \).

The proof of the theorem is in Appendix C.

In the rest of this paper, at the expense of losing a factor of \( O(\alpha) \) in the approximation factor and running an \( \alpha \)-approximation algorithm of \( k \)-clustering with \( \ell_p \)-objective (with no range constraints), we assume that \( |D| = k \) and \( |F| = n \).

### 2.2 Constructing a structured fractional solution

In this section, we describe the sparsification approach of (Charikar et al., 2002) which outputs an instance with a subset of locations where pairwise distances of survived locations are “relatively large”. First, we state a standard LP-relaxation of the problem as follows.

\[
\text{FAIR}\text{RANGE}\text{L}\text{P}(D, F, w, \{\alpha_i, \beta_i\}_{i \in [\ell]})
\]

\[
\text{minimize} \quad \sum_{v \in D, u \in F} w(v) \cdot d(v, u)^p \cdot x_{vu}
\]

\[
s.t. \quad \sum_{u \in F} x_{vu} \geq 1 \quad \forall v \in D \tag{1}
\]

\[
\alpha_i \leq \sum_{u \in F_i} y_{iu} \leq \beta_i \quad \forall i \in [\ell] \tag{2}
\]

\[
\sum_{u \in F} y_{iu} \leq k \tag{3}
\]

\[
0 \leq x_{vu} \leq y_{iu} \quad \forall v \in D, u \in F \tag{4}
\]

In our algorithm, we never modify the set \( F \) and the fairness constraints \( \{\alpha_i, \beta_i\}_{i \in [\ell]} \). So, to specify an instance, we will provide the set of clients \((D, w)\). Consider an optimal fractional solution \((x^*, y^*)\) of \( \text{FAIR}\text{RANGE}\text{L}\text{P}(D, w) \). For any location \( v \in D \), define \( R(v) := (\sum_{u \in F} x_{vu}^* \cdot d(v, u)^p)^{1/p} \) as the fractional distance of a unit of demand at location \( v \) w.r.t. the optimal solution \((x^*, y^*)\). When \( y^* \) is integral, then \( R(v) \) is the distance of \( v \) to its closest open facility, specified by the vector \( y^* \).

The sparsification approach of (Charikar et al., 2002) applied to our setting, described in Appendix B, outputs a sparse instance with the following properties. The proofs of theorems in this section are deferred to Appendix C.

**Theorem 2.3.** Given an instance \((D, w)\) of fair range clustering with \( \ell_p \)-cost and an optimal fractional solution \((x, y)\) of \( \text{FAIR}\text{RANGE}\text{L}\text{P}(D, w) \) with cost \( \text{OPT}_D \), there exists a polynomial time algorithm that returns a set of locations \( D' \subseteq D \) and a demand function \( w' : D' \to \mathbb{R} \) such that

\[\text{(Q1)} \quad \text{For every pair of } v_i, v_j \text{ in } D', \; d(v_i, v_j) \geq 2^{1+1/p} \max\{R(v_i), R(v_j)\} \]

\[\text{(Q2)} \quad (x, y) \text{ is a feasible solution of } \text{FAIR}\text{RANGE}\text{L}\text{P}(D', w') \text{ of cost at most } \text{OPT}_D. \]

\[\text{(Q3)} \quad \text{Any integral solution } C \text{ of } \text{FAIR}\text{RANGE}\text{L}\text{P}(D', w') \text{ of cost } z \text{, can be converted in polynomial time to a feasible solution of } \text{FAIR}\text{RANGE}\text{L}\text{P}(D, w) \text{ of cost at most } 4^p \cdot \text{OPT}_D + 2^{p-1} \cdot z. \]

Next, for every location \( v \in D' \), we define the ball \( B(v) := \{ u \in F : d(v, u) \leq \frac{1}{2} \cdot R(v) \} \) to denote the set of facilities at distance at most \( \frac{1}{2} \cdot R(v) \) from \( v \). Further, for every location \( v \in D' \), we define \( P(v) \) as the super ball of \( v \) which consists of \( B(v) \) and a set of “private facilities” of \( v \).

**Observation 2.4.** For any pair of \( v', v'' \in D' \) and \( u' \in R(v'), \frac{1}{2} d(v', u') \leq d(v, u') \leq \frac{3}{2} d(v, v''). \]

**Proof.** Since \( d \) is a metric distance, by the triangle inequality, \( d(v, u') + d(u', v'') \geq d(v, v'') \). Next, we bound \( d(u', v'') \) in terms of \( d(v, v'') \). Since \( u' \in B(v') \), \( d(v', u') \leq 2^{p-1} R(v') \). On the other hand, by \( (Q1) \), \( d(v, v') \geq 2^{1+1/p} R(v') \). Hence, \( d(v', u') \leq \frac{1}{2} d(v, v') \). By another application of the triangle inequality, \( d(v, u') \leq d(v, v') + d(v', u') \leq \frac{3}{2} d(v, v') \).

Next, in polynomial time, we convert a fractional solution of \( \text{FAIR}\text{RANGE}\text{L}\text{P}(D, w) \) to a fractional solution \((x, y)\) of \( \text{FAIR}\text{RANGE}\text{L}\text{P}(D', w') \) with the following further structural properties.

**Theorem 2.5.** There exists a polynomial time algorithm that outputs a fractional solution \((x, y)\) of \( \text{FAIR}\text{RANGE}\text{L}\text{P}(D', w') \) of cost \( 9^p \cdot \text{OPT}_D \), where \( \text{OPT}_D \) is the cost of an optimal solution of \( \text{FAIR}\text{RANGE}\text{L}\text{P}(D, w) \), and a collection of super balls \( \{P(v)\}_{v \in D'} \) that satisfy the following properties:

\[\text{(P1)} \quad \text{For every } v \in D', \; B(v) \subseteq P(v). \]

\[\text{(P2)} \quad \text{For every } v \in D' \text{ and } u \in P(v) \setminus B(v), \; x_{uv} > 0 \text{ only if } \sum_{u \in B(v)} y_{uv} < 1. \text{ Similarly, for every } v \in D' \text{ and } u \in F \setminus P(v), \; x_{uv} > 0 \text{ only if } \sum_{u \in B(v)} y_{uv} < 1. \]

\[\text{(P3)} \quad \text{For every } v \in D', \; x_{uv} > 0, \text{ then either } u \in P(v) \text{ or } u \in B(v') \text{ where } v' = \text{NN}_{D'}(v) \text{ denotes the nearest location in } D' \text{ (other than } v \text{ itself) to } v. \]

\[\text{(P4)} \quad \text{For every } v \in D', \;
\sum_{u \in P(v)} x_{uv} \geq \sum_{u \in B(v)} x_{uv} \geq 1/2. \]
(P5) For every \( v \in D' \), \( u \in \mathcal{P}(v) \setminus \mathcal{B}(v) \), \( d(v, u) \leq 2 \cdot d(v, v') \), where \( v' = \text{NN}_{D'}(v) \).

(P2) The set of super balls, \( \{\mathcal{P}(v)\}_{v \in D'} \), are disjoint.

3. Rounding Algorithm

To do the rounding, first we write a new LP relaxation, called \( \text{STRUCTUREDLP} \) which is a simplification of \( \text{FAIRRANGE-LP} \) via the further structures of fractional solution \((x, y)\) guaranteed by Theorem 2.5. In particular, \( \text{STRUCTUREDLP} \) is useful for our rounding algorithm because as we show in this section, the polyhedron constructed by the constraints of \( \text{STRUCTUREDLP} \) is half-integral. A solution \( y \) to an LP relaxation is half-integral if every coordinates in \( y \) has value either 0, 1/2 or 1.

In \( \text{STRUCTUREDLP} \), we define \( \Delta(v) := d(v, v')^p + \sum_{u \in \mathcal{P}(v)} (d(v, u)^p - d(v, v')^p) \cdot y_u \) to denote the minimum (fractional) distance of \( v \) to open facilities (i.e., the facilities vector \( y \)).

\[
\text{STRUCTUREDLP}(D', w')
\]

\[
\begin{align*}
\text{minimize} & \quad \sum_{v \in D'} w'(v) \cdot \Delta(v) \\
\text{s.t.} & \quad \alpha_i \leq \sum_{u \in F_i} y_u \leq \beta_i \quad \forall i \in [\ell] \tag{5} \\
& \quad \sum_{u \in F} y_u \leq k \tag{6} \\
& \quad \sum_{u \in \mathcal{B}(v)} y_u \geq 1/2 \quad \forall v \in D' \tag{7} \\
& \quad \sum_{u \in \mathcal{P}(v)} y_u \leq 1 \quad \forall v \in D' \tag{8} \\
& \quad y_u \geq 0 \quad \forall u \in F \tag{9}
\end{align*}
\]

Lemma 3.1. The optimal fractional solution of \( \text{STRUCTUREDLP}(D', w') \) is a valid solution for fair range clustering on \((D', w')\) and has cost at most \( e^{O(p)} \cdot \text{OPT}_{D'} \).

Proof. By the cardinality constraint on the number of open facilities and the range constraints on the number of facilities from each group, a feasible solution of \( \text{STRUCTUREDLP} \) is a feasible solution of fair range clustering on \((D', w')\).

Consider the solution \((x, y)\) guaranteed by Theorem 2.5. It is straightforward to verify that \( y \) satisfies all constraints of \( \text{STRUCTUREDLP}(D', w') \) and is a feasible solution for the LP: \((x, y)\) satisfies the first two sets of constraints in \( \text{STRUCTUREDLP}(D', w') \) because \((x, y)\) is a feasible solution of \( \text{FAIRRANGE-LP}(D', w') \) and constraint 7 follows from (P4) in Theorem 2.5. Note that if \( y \) does not satisfy constraint (8), we can decrease the value of \( y \) accordingly so that \((x, y)\) remains a feasible solution of \( \text{FAIRRANGE-LP}(D', w') \) with a lower cost.

Next, we bound the cost of \( y \) with respect to \( \text{STRUCTUREDLP}, \text{cost}_{STLP}(y) \). We rewrite \( \Delta(v) \) as follows:

\[
\Delta(v) = \left(1 - \sum_{u \in \mathcal{P}(v)} y_u\right) d(v, v')^p + \sum_{u \in \mathcal{P}(v)} d(v, u)^p \cdot y_u
\]

For every \( v \in D' \),

\[
\begin{align*}
\sum_{u \in F} d(v, u)^p x_{vu} \\
= \sum_{u \in \mathcal{P}(v)} d(v, u)^p x_{vu} + \sum_{u \in \mathcal{B}(v)} d(v, u)^p x_{vu} \\
\geq \sum_{u \in \mathcal{P}(v)} d(v, u)^p x_{vu} + \frac{1}{2p} \cdot d(v, v')^p \sum_{u \in \mathcal{B}(v)} x_{vu} \\
= \sum_{u \in \mathcal{P}(v)} d(v, u)^p y_u + \frac{1}{2p} \cdot d(v, v')^p \left(1 - \sum_{u \in \mathcal{P}(v)} y_u\right)
\end{align*}
\]

(10)

where the first equality follows from (P3), the first inequality is by Observation 2.4 and the second equality follows from (P2). Thus,

\[
\text{cost}_{STLP}(y) = \sum_{v \in D'} w'(v) \Delta(v) \\
\leq 2^p \sum_{v \in D'} w'(v) \sum_{u \in F} d(v, u)^p x_{vu} \\
= 2^p \cdot \text{cost}_{FLP}(x, y) \\
= e^{O(p)} \cdot \text{OPT}_{D'}
\]

where \( \text{OPT}_{D} \) denotes the cost of an optimal solution of \( \text{FAIRRANGE-LP}(D, w) \). The last inequality bounding \( \text{cost}_{FLP}(x, y) \) is by Theorem 2.5.

Lemma 3.2. Consider a half-integral solution \( \hat{y} \) of \( \text{STRUCTUREDLP}(D', w') \) of cost \( z \). Then, \( \hat{y} \) is a feasible solution for \( \text{FAIRRANGE-LP}(D', w') \) with cost at most \( \left(\frac{3}{2}\right)^p \cdot z \).

Proof. Let \( \hat{x} \) be the assignment of clients to the opened facilities as follows: For every \( v \in D' \),

- **Step 1.** For every \( u \in \mathcal{P}(v) \), set \( \hat{x}_{vu} = \hat{y}_u \). Let \( \hat{Y}(v) = \sum_{u \in \mathcal{P}(v)} y_u \) denote the total assignment of \( v \) at the end of this step.

- **Step 2.** While \( \hat{Y}(v) < 1 \), iterate over \( u \in \mathcal{B}(v) \) and at each iteration set \( \hat{x}_{vu} = \min\{1 - \hat{Y}(v), \hat{y}_u\} \) and \( \hat{Y}(v) = \hat{Y}(v) + \hat{x}_{vu} \).
Since for every \( v \in D' \), \( \frac{1}{2} \leq \sum_{u \in B(v)} \tilde{y}_u \leq 1 \), this procedure terminates with a feasible fractional assignment of clients to facilities, \( \tilde{x} \); for every \( v \in D' \), \( u \in F \), \( \tilde{x}_{vu} \leq \tilde{y}_u \) and \( \sum_{u \in F} \tilde{x}_{vu} = 1 \). Moreover, if \( \tilde{y} \) is half-integral, clearly \( \tilde{x} \) is half-integral too.

For every \( v \in D' \),

\[
\sum_{u \in F} d(v, u)p \cdot \tilde{x}_{vu} = \sum_{u \in P(v)} d(v, u)p \cdot \tilde{x}_{vu} + \sum_{w' \in B(v')} d(v, u')p \cdot \tilde{x}_{vu'} \leq \sum_{u \in P(v)} d(v, u)p \cdot \tilde{y}_u + \left( \frac{3}{2} \right) p \sum_{u \in P(v)} \tilde{x}_{vu} \quad \text{by Obs. 2.4} \leq \sum_{u \in P(v)} d(v, u)p \cdot \tilde{y}_u + \left( 1 - \sum_{u \in P(v)} \tilde{y}_u \right) d(v, v')p \leq \left( \frac{3}{2} \right) p \cdot \Delta(v) \quad \text{(11)}
\]

Thus,

\[
\text{cost}_{\text{FLP}}(\tilde{x}, \tilde{y}) = \sum_{v \in D'} w'(v) \sum_{u \in F} d(v, u)p \tilde{x}_{vu} \leq \sum_{v \in D'} w'(v) \left( \frac{3}{2} \right) p \cdot \Delta(v) \quad \text{by Eq. (11)} \leq \left( \frac{3}{2} \right) p \cdot \text{cost}_{\text{STLP}}(\tilde{y})
\]

\[
\square
\]

3.1. Constructing a half-integral solution of \text{STRUCTUREDLP}

Next, we show that the constraints of \text{STRUCTUREDLP} forms a totally unimodular (TU) matrix. Then, by scaling the constraint properly, we can find a half-integral optimal solution of \text{STRUCTUREDLP} in polynomial time. Recall that a matrix is TU if every square submatrix has determinant \(-1\), 0 or 1. Totally unimodular matrices are extremely important in polyhedral combinatorics and combinatorial optimization since if \( A \) is TU and \( b \) is integral, then the linear programs of the form \( \{ \min cx \mid Ax \geq b, x \geq 0 \} \) has integral optima, for any cost vector \( c \).

\textbf{Lemma 3.3.} The matrix corresponding to the constraints of \text{STRUCTUREDLP} is a TU matrix.

\textbf{Proof.} We consider \text{STRUCTUREDLP} in the form \( \{ \min cy \mid Ay \geq b, y \geq 0 \} \). Note that values of all entries in \( A \) belong to \( \{0, \pm 1\} \). Moreover, in each column \( A_i \), corresponding to each variable \( y_i \), there are at most five non-zero entries: one with value \(-1\) corresponding to “picking at most \( k \) centers” constraint, one with value \(-1 \) to upper bound the contribution of the facilities of the color class to which facility \( i \) belongs, one with value \( 1 \) to lower bound the contribution of the facilities of the color class to which facility \( i \) belongs, one with value \( 1 \) to open at least \( 1/2 \) facilities from the super ball containing facility \( i \) and finally one with value \(-1 \) to open at most \( 1 \) from the super ball containing facility \( i \).

Now, we show that \( A \) is a TU matrix by the result of Ghouila-Houri (1962); see Theorem A.4. They showed that if for every subset \( R \) of the rows there is an assignment \( s_R : R \to \{-1, 1\} \) of signs to rows so that \( \sum_{r \in R} s_R(r) A_r \) has all entries in \( \{0, \pm 1\} \), then \( A \) is a TU matrix. To show that, for any subset of rows \( R \) we write \( R = R_{\text{center}} \cup R_{\text{upper}} \cup R_{\text{lower}} \cup R_{\text{superball}} \). Next, we consider the following two cases:

- \( R_{\text{center}} \neq \emptyset \). In this case, we assign \(-1 \) to the row corresponding to the total facility budget constraint, (6). Next, we consider the color classes \( C_{\text{both}} \) for which both rows corresponding to the upper and lower bounds constraints on the number of centers from the color class exist in \( R \), \( s_R \) assigns \( 1 \) to both such rows. Note that as the rows corresponding to the lower and upper bound constraints of a specific color class sum up to zero vector, such assignment makes the contribution of such rows in the sum of rows of \( A \) in \( R \) zero. Otherwise, as color classes are disjoint, if exactly one of the rows corresponding to the upper and lower bound constraints belong to \( R \), we can set \( s_R \) so that the contribution of entries in such rows become exactly 1.

Hence, so far (by summing up terms \( \sum_{r \in R_{\text{center}}} s_R(r) A_r \), \( \sum_{r \in R_{\text{upper}}} s_R(r) A_r \) and \( \sum_{r \in R_{\text{lower}}} s_R(r) A_r \)), each entry has value either 0 or \(-1 \). Next, we consider the set of rows in \( R_{\text{superball}} \). If both rows corresponding to the upper and lower bounds constraints on the contributions of facilities in a super ball are in \( R \), then \( s_R \) assigns \( 1 \) to both rows and their total contributions in the final sum will become zero. For the remaining rows in \( R_{\text{superball}} \), we set the sign so that the contribution of each row to the final sum becomes exactly 1 (i.e., if the row corresponds to constraint (7), \( s_R \) assigns 1, and assigns \(-1 \) otherwise). Since the super balls are disjoint, in the weighted sum of rows with \( s_R \), each entry is either 0 or \pm 1.

- \( R_{\text{center}} = \emptyset \). Our approach is similar to the previous case. However, here first we set the assignment of rows corresponding to \( R_{\text{lower}}, R_{\text{upper}} \) and makes sure that by summing up \( \sum_{r \in R_{\text{upper}}} c(r) A_r \), \( \sum_{r \in R_{\text{lower}}} c(r) A_r \), each entry has value either 0 or 1. Then, we follow a similar approach to the previous case and exploiting the fact super balls are disjoint, we show that it is possible to set the sign of rows in \( R_{\text{superball}} \) so that in the overall signed sum of the rows, each entry is either 0 or \pm 1.

\[
\square
\]
**Algorithm 1 Partitioning Facilities.**

1: **Input:** A set of locations $D'$, half-integral vector $y$  
2: **for all** location $v_i \in D'$ do  
3: \ $R_i \leftarrow$ the minimum assignment cost of a unit of demand at $v_i$ w.r.t. $y$: i.e., $R_i = \frac{1}{2} (d(v_i, u_{i_1})^p + d(v_i, u_{i_2})^p)$ where $u_{i_1}, u_{i_2}$ are respectively the primary and secondary facilities serving $v_i$  
4: \ $S_i \leftarrow \{u_{i_1}\} \cup \{u_{i_2}\}$  
5: **end for**  
6: \ $D'' \leftarrow D' \setminus \{D\}$  
7: **while** $D''$ is nonempty do  
8: \ let $v_i \leftarrow \text{argmin}_{v_i \in D''} R_j$  
9: \ add $v_i$ to $D$  
10: **remove** all locations $v_j \in D''$ such that $S_j \cap S_i \neq \emptyset$  
11: **end while**  

Next, we show the following useful property of the $\{S_i\}_{i \mid v_i \in D}$.

**Lemma 3.5.** The clustering cost of any set of $k$ facilities $C$ that opens at least one facility from each $\{S_i\}_{i \mid v_i \in D}$ is at most $\left(\frac{9}{2}\right)^p$ times the cost of an optimal solution of $\text{STRUCTUREDLP}(D', w)$.

**Proof.** We compute the cost of a unit of demand in each location of $v_i \in D'$ when it is assigned to its closest facility in $S := \bigcup_{v_i \in D} S_i$. In particular, we compare this cost to the fractional cost imposed by the half-integral solution $y$ of $\text{STRUCTUREDLP}$. We consider the following two cases:

- $v_i \in D$. It is straightforward to check that in this case the assignment cost of $v_i$ to a facility in $S$ is at most twice the its fractional assignment cost with respect to $y$.
- $v_i \notin D$. Let $v_j \in C$ be the client who removed $v_i$ from $D''$, i.e., $v_j$ is the minimum cost location whose opened facilities, $S_j$, has non-empty intersection with $S_i$. Let $u = S_j \cap C$. Then, we bound $d(v_i, u)^p$ in terms of $R_i$ and $R_j$. Note that in this case $S_i \cap S_j \neq \emptyset$, however, their intersection might be a facility $u'$ different from $u$. So, by the approximate triangle inequality (see Corollary A.2)

$$d(v_i, u)^p \leq 3^{p-1} (d(v_i, u')^p + d(u', v_j)^p + d(v_j, u)^p) \leq 3^{p-1} (R_i + 2R_j) \leq 3^p \cdot R_i$$

Hence, the total cost of such clustering is at most

$$\sum_{v_i \in D'} w'(v) \cdot d(v_i, C)^p \leq 3^p \cdot \sum_{v_i \in D'} R_i \leq \left(\frac{9}{2}\right)^p \cdot \text{cost}_{\text{STLP}}(y),$$

where the last inequality follows from Lemma 3.2. 

**Step 2. Constructing an integral solution.** Finally, we show that we can always find an integral solution of $\text{STRUCTUREDLP}$ that picks at least one center from every set $S_1, \ldots, S_L$. Note that by showing the existence of such a solution, automatically (via Lemma 3.5) we have the guarantee that it is a $3^p$-approximation solution of $\text{STRUCTUREDLP}$. We show the existence of such an integral solution via an application of max-flow problem.

**Lemma 3.6.** Given a collection of disjoint sets $S_1, \ldots, S_L$ where $L \leq k$, there exists an integral solution that picks a set of $k$ centers $C$ with the following extra properties:

- For every $j \in [L]$, $C \cap S_j \neq \emptyset$, and
- For every group $i \in [\ell]$, $\alpha_i \leq |C \cap P_i| \leq \beta_i$.

**Proof.** To show the existence of such a solution, we construct the following instance of network-flow. As in Figure 2, we create a network with 6 layers. Layer 0 consists of a single source vertex $s$ and layer 1 consists of $L + 1$ vertices corresponding to sets $S_1, \ldots, S_L$ and a dummy set $\bar{S}$. Moreover, for every $i \in [L]$, the source vertex $s$ is connected to $S_i$ with an edge of capacity 1. There is also an edge from $s$ to $\bar{S}$ with capacity $k - L$. In layer 2, there are $|F|$ vertices corresponding to each facility in $F$. Moreover, for every
Theorem 2.5. Next, we find a half-integral optimal solution \( y \) of \( \text{STLP} \) with the specified set of fairness constraints, first we compute the sparse instance \((D', w')\) guaranteed in Theorem 2.5. Next, we find a half-integral optimal solution \( y \) of \( \text{STLP}(D', w') \) (Theorem 3.4). Then, by Lemma 3.5 and 3.6, we can find a set of centers \( C \) such that the clustering \((D', w')\) using the set of centers \( C \) has cost at most \((\frac{9}{2})^p \cdot \text{cost}_{\text{STLP}}(y)\). Next, by the optimality of \( y \) for \( \text{STLP} \) and by Lemma 3.1, we show that the clustering cost of the instance with the set of centers \( C \) is at most \((\frac{9}{2})^p \cdot \text{cost}_{\text{STLP}}(y) = e^{O(p)} \cdot \text{OPT}_D \) where \( \text{OPT}_D \) is the cost of an optimal solution of \( \text{FAIRRANGLP}(D, w) \).

Finally, by Theorem 2.3 and raising the approximation factor to the power of \( 1/p \), the clustering using the center set \( C \) is an \( O(1) \)-approximate solution on instance \((D, w)\) with the given fairness constraints \( \{\alpha_i, \beta_i\}_{i \in [\ell]} \).

Conclusion

In this paper, we study the fair range clustering problem which is a generalization of several well-studied problems including fair \( k \)-center (Kleindessner et al., 2019) and clustering under partition matroid. We designed efficient constant-factor approximation algorithms. Our result is the first pure multiplicative approximation algorithm for fair range clustering with general \( \ell_p \)-objective.

References


Approximation Algorithms for Fair Range Clustering


A. Preliminaries

A.1. Approximate Triangle Inequality

**Lemma A.1** (Lemma A.1 (Makarychev et al., 2019)). Let $x, y_1, \cdots, y_n$ be non-negative real numbers and $\lambda > 0, p \geq 1$. Then,

$$
(x + \sum_{i=1}^{n} y_i)^p \leq (1 + \lambda)^{p-1} x^p + \left(\frac{(1+\lambda)n}{\lambda}\right)^{p-1} \sum_{i=1}^{n} y_i^p.
$$

The following approximate variants of triangle inequalities are direct corollaries of the above lemma.

**Corollary A.2.** Let $(P, d)$ be a metric space. Consider distance function $d(u, v)^p$. Then, $\forall u_0, \cdots, u_r \in P, d(u_0, u_r)^p \leq r^{p-1} \cdot \sum_{i=0}^{r-1} d(u_i, u_{i+1})^p$.

**Proof.** Follows from the triangle inequality for the distance function $d$ and application of Lemma A.1 with $\lambda = r - 1$. □

**Corollary A.3.** Let $(P, d)$ be a metric space. Consider distance function $d(u, v)^p$. Then, $\forall u, v, w \in P, d(u, w)^p \leq 2^{p-1} \cdot (d(u, v)^p + d(v, w)^p)$.

A.2. Totally Unimodular Matrices

A matrix is called totally unimodular if every square submatrix of the given matrix has a determinant that is either 0, 1, or $-1$. This property has significant implications in linear programming (LP). One of the key benefits of totally unimodular matrices is their close relationship with integer programming and LP. When a LP has a constraint matrix that is totally unimodular, it guarantees that the LP has an integral solution. Moreover, the integral solution can be found in polynomial time using various algorithms such as the ellipsoid method or the interior point method.

**Theorem A.4** (Ghouila-Houri (1962)). A matrix $A \in \mathbb{R}^{m \times n}$ is totally unimodular if and only if for every subset of the rows $R \subseteq [m]$, there is a partition of $R = R_+ \sqcup R_-$ such that for every $j \in [n]$,

$$
\sum_{i \in R_+} A_{ij} - \sum_{i \in R_-} A_{ij} \in \{-1, 0, 1\}.
$$

B. Constructing well-separated locations

In this section, we follow the location consolidation approach of (Charikar et al., 2002) and output an instance with a sparsified set of locations where pairwise distance of survived locations are “relatively large”. To describe this step, we work with the relaxation FAIRRANGLP. As in our algorithm we never modify the set $F$ and the range constraints $\{\alpha_i, \beta_i\}_{i \in \ell}$, to specify an instance, from now on we will only specify the set of clients $(D, w)$. Consider an optimal fractional solution $(x, y)$ of FAIRRANGLP($D, w$)—throughout the paper we refer to the cost of an optimal solution of this LP as OPT$_D$. For any location $v \in D$, we define $R(v) := \left(\sum_{u \in F} x_{vu} \cdot d(v, u)^p\right)^{1/p}$ as the fractional distance of a unit of demand at location $v$ w.r.t. the optimal solution $(x, y)$. Note that if $(x, y)$ is an integral solution, then $R(v)$ is simply the distance from $v$ to its closest open facility, which is specified by $y$. Next, we process the locations as follows. We sort locations in a non-decreasing order of their fractional distance; we index the locations in $D$ as $v_1, \cdots, v_n$ such that $R(v_1) \leq R(v_2) \leq \cdots \leq R(v_n)$. We iterate over the locations in this order and for every $i \in [n]$, when we are processing $v_i$, we check for all locations $v_j$ s.t. $j > i$ and $d(v_i, v_j) \leq 2^{1+1/p} \cdot R(v_j)$. For each such location, we move the demand at location $v_j$ to $v_i$ and set the demand at location $v_j$ to zero. At the end of this step, the algorithm returns a new demand function $w'$ supported on $D' \subseteq D$.

For every location $v \in D'$, we define the ball $B(v) := \{u \in F | d(v, u) \leq 2^{1/p} R(v)\}$ to denote the set of facilities at distance at most $2^{1/p} R(v)$ from $v$. The above claim implies the following lemma.

**Lemma B.1.** For every $v \in D'$, $\sum_{u \in B(v)} x_{vu} \geq 1/2$.

**Proof.** Note that

$$
R(v)^p = \sum_{u \in B(v)} x_{vu} d(u, v)^p + \sum_{u \notin B(v)} x_{vu} d(u, v)^p \geq \sum_{u \notin B(v)} x_{vu} d(u, v)^p + (1 - \sum_{u \notin B(v)} x_{vu}) \cdot 2R(v)^p
$$

which proves that $\sum_{u \in B(v)} x_{vu} \geq 1/2$. □
Algorithm 2 Consolidating Locations.

1: **Input:** \((x, y)\) is an optimal solution of FAIRRANGE\(LP(D, w)\)
2: \(\mathcal{R}(v) \leftarrow (\sum_{u \in F} d(v, u)^p \cdot d(vu))^{1/p}\) for all \(v \in D\)
3: \(w'(v) \leftarrow w(v)\) for all \(v \in D\)
4: sort locations in \(D\) so that \(\mathcal{R}(v_1) \leq \cdots \leq \mathcal{R}(v_n)\)
5: for \(i = 1\) to \(n - 1\) do
6:  if \(w'(v_i) > 0\) then
7:      for \(j = i + 1\) to \(n\) do
8:         if \(w'(v_j) > 0\) and \(d(v_i, v_j) \leq 2^{1+1/p} \cdot \mathcal{R}(v_j)\) then
9:            \(w'(v_i) \leftarrow w'(v_i) + w(v_j)\) and \(w'(v_j) \leftarrow 0\)
10:       end if
11:  end for
12: end if
13: end for

Next, we follow the reduction steps in (Krishnaswamy et al., 2011; Swamy, 2016) to construct a fractional solution of FAIRRANGE\(LP\) with further structures. We construct a fractional solution \((x', y)\) in which for each location \(v\), its demands are served by the facilities in a super ball \(P(v)\) with fraction \(\gamma \geq 1/2\) and by the facilities in \(\mathcal{P}(v')\) with fraction \(1 - \gamma_u\) where \(v' := \text{NN}_{D'}(v)\) denotes the nearest location in \(D'\) (other than \(v\) itself) to \(v\) and the collection of super balls \(\{P(v)\} v \in D'\) are disjoint.

We start with the feasible solution \((x^1, y)\) of FAIRRANGE\(LP(D', w')\) where \(x^1_{vu} = x_{vu}\) for all \(v \in D'\) and \(u \in F\) and \((x, y)\) is the optimal solution of FAIRRANGE\(LP(D, w)\) from which the sparsified set of locations \((D', w')\) is constructed. Note that by Theorem 2.3-(Q2), \((x^1, y)\) is a feasible solution of FAIRRANGE\(LP(D', w')\) with cost at most \(\text{OPT}_D\).

**Private facilities.** In Theorem 2.3-(Q1), we showed the collection of balls \(\{B(v)\} v \in D'\) are disjoint. Here, we construct a solution \((x^2, y)\) of FAIRRANGE\(LP(D', w')\) with cost \(e^{O(p)} \cdot \text{OPT}_D\) such that each facility \(u \in F \setminus \bigcup_{v \in D'} \mathcal{B}(v)\), serves the clients of at most one location. In other words, each (fractionally) opened facility outside the balls only serves one location. For each \(v \in D'\), the super ball \(P(v)\) consists of \(B(v)\) and the private facilities of \(v\).

Consider a facility \(u\) that does not belong to any ball in \(\{B(v)\} v \in D'\) and serves clients from more than one locations \(v_1, \cdots, v_r\). Let us assume locations are ordered according their distance to \(u\); \(d(v_1, u) \leq \cdots \leq d(v_r, u)\).

**Claim B.2.** For every \(j \in \{2, \cdots, r\}\) and every facility \(u' \in B(v_1)\), \(d(v_j, u') \leq 3d(v_j, u)\).

**Proof.** Since for every \(j \in \{2, \cdots, r\}\), \(d(v_j, u) \geq d(v_1, u)\),
\[
d(v_j, v_1) \leq d(v_j, u) + d(v_1, u) \leq 2d(v_j, u). \tag{12}
\]
Moreover, for a facility \(u' \in B(v_1)\),
\[
d(v_j, u') \leq d(v_j, v_1) + d(v_1, u') \leq \frac{3}{2} d(v_j, v_1) \leq 3d(v_j, u) \quad \triangleright \text{Theorem 2.3-(Q1)} \quad \triangleright \text{Eq. (12)}
\]
\[\square\]

Now, in \((x^2, y)\), for every \(j \in \{2, \cdots, r\}\), we set the assignment of \(v_j\) to \(u\) to zero; \(x^2_{vu} = 0\) and instead increase the assignment of \(v_j\) to facilities \(u \in B(v_1)\) with total increase of \(x^1_{vu}\). A formal procedure of this reassignment step is described in Algorithm 3.

**Lemma B.3.** \((x^2, y)\) is a feasible solution of FAIRRANGE\(LP(D', w')\) with cost at most \(3^p \cdot \text{OPT}_D\).
Algorithm 3 Assignments to Private Facilities.

1: **Input:** \((x^1, y)\) is a feasible solution of \(\text{FAIRRANGELP}(D', w')\)
2: \(x^2_{vu} \leftarrow x^1_{vu}\) for all \(v \in D', u \in F\)
3: for all \(u \in F \setminus \bigcup_{v \in D'} B(v)\) with \(y(u) > 0\) do
4: sort locations \(\{v \in D' \mid x^1_{vu} > 0\}\) according to their distance to \(u\) so that \(d(v_1, u) \leq \cdots \leq d(v_r, u)\)
5: for \(j = 2\) to \(r\) do
6: \(x^2_{vj, u} \leftarrow 0\) and \(b = x^1_{vj, u}\)
7: for all \(u' \in B(v_1)\) do
8: \(x^2_{v'j, u} \leftarrow \min(y_{u'}, b + x^1_{v'j, u'})\) and \(b \leftarrow b - \min(y_{u'}, x^1_{v'j, u'})\)
9: end for
10: end for
11: end for

**Proof.** First we prove the feasibility of \((x^2, y)\).

\[
\sum_{u_1 \in B(v_1)} y_{u_1} \geq \sum_{u_1 \in B(v_1)} x^1_{v_1 u_1} \quad \triangleright \text{by feasibility of } (x^1, y) \\
\geq 1/2 \quad \triangleright \text{Lemma B.1}
\]

By another application of Lemma B.1, for every \(j \in \{2, \cdots, r\}\),

\[
x^1_{v_1 u_1} + \sum_{u'_j \in F \setminus B(v_j)} x^1_{v_j u'_j} \leq 1 - \sum_{u'_j \in B(v_j)} x^1_{v_j u'_j} \leq 1/2
\]

Hence, the facilities in \(B(v_1)\) have enough slack to accommodate extra \(x^1_{v_1 u}\) in their assignment from \(v_1\). This implies that \((x^2, y)\) as constructed in Algorithm 3 is a feasible solution of \(\text{FAIRRANGELP}(D', w')\).

Moreover, since by Claim B.2 for every \(j \in \{2, \cdots, r\}\) and \(u' \in B(v_1), d(v_j, u') \leq 3d(v_j, u), \quad \text{cost}(x^2, y) \leq 3^p \cdot \text{cost}(x^1, y) = 3^p \cdot \text{OPT}_D\).

\(\square\)

C. Omitted Proofs of Section 2

**Proof of Theorem 2.2.** As the set of facilities \(F = P\), the set \(S\) is a feasible solution for fair range clustering on the original instance \(P\) too.

Let \(D = \{c_1, \cdots, c_k\}\) denote an \(\alpha\)-approximate solution of clustering with \(\ell_p\)-objective on \(P\). Let \(\text{OPT}_{\text{org}}\) and \(\text{OPT}_{\text{rdc}}\) respectively denote optimal solutions of fair range clustering with \(\ell_p\)-objective on the original instance \(P\) and the reduced instance \((D, F)\). First we bound the cost of \(\text{OPT}_{\text{org}}\) on the reduced instance \((D, F)\):

\[
\text{cost}_{\text{rdc}}(\text{OPT}_{\text{org}}) = \sum_{v \in D} w'(v) \cdot d(v, \text{OPT}_{\text{org}})^P
\]

\[
\leq \sum_{v \in P} w(v) \cdot 2^{p-1} \cdot (d(v, D)^P + d(v, \text{OPT}_{\text{org}})^P) \quad \triangleright \text{approximate triangle inequality}
\]

\[
\leq 2^{p-1} \left( \sum_{v \in P} w(v)d(v, D)^P + \sum_{v \in P} w(v)d(v, \text{OPT}_{\text{org}})^P \right)
\]

\[
\leq 2^{p-1} \cdot (\alpha^p \cdot \text{cost}(\text{OPT}_{\text{org}}) + \text{cost}(\text{OPT}_{\text{org}})) \quad \triangleright D \text{ is an } \alpha\text{-approximate solution}
\]

\[
= 2^{p-1}(\alpha^p + 1) \cdot \text{cost}(\text{OPT}_{\text{org}})
\] \hspace{1cm} (13)
Next, we bound the cost of the solution $S$ on the original instance $P$,

$$
cost(S) = \sum_{v \in P} w(v) \cdot d(v, S)^p
= \sum_{v \in P} w(v) \cdot 2^{p-1} \cdot (d(v, D)^p + d(ND(v), S)^p) \quad \triangleright \text{approximate triangle inequity}
\leq 2^{p-1} \left( \sum_{v \in P} w(v) d(v, D)^p + \sum_{v \in D} w(v) d(v, S)^p \right)
\leq 2^{p-1} \cdot (\alpha^p \cdot \text{cost(OPT)} + \beta^p \cdot \text{cost}_{\text{rdc}}(\text{OPT}_{\text{rdc}}))
\leq 2^{p-1} \cdot (\alpha^p + \beta^p \cdot 2^{p-1}(\alpha^p + 1)) \cdot \text{cost(OPT)} \quad \triangleright \text{by Eq (13)}
$$

Thus, $S$ is an $O(\alpha \beta)$-approximation for fair range clustering with $\ell_p$-objective on the original instance $P$. \hfill \Box

**Proof of Theorem 2.3.** (Q1): Wlog, let assume that $v_1$ located before $v_2$ in the ordering considered by Algorithm 2; hence, $R(v_2) \geq R(v_1)$. However, if $d(v_1, v_2) \geq 2^{1+1/p} R(v_2)$, then Algorithm 2 moves the demand of $v_2$ to $v_1$ and $v_2$ will not survive the process which is a contradiction. Thus, $d(v_1, v_2) \leq 2^{1+1/p} R(v_2) = 2^{1+1/p} \cdot \max\{R(v_2), R(v_1)\}$.

(Q2): In FAIRRANGLP, no constraint depends on demands and thus the constraints of FAIRRANGLP$(D', w')$ are subsets of the ones in FAIRRANGLP$(D, w)$. Thus, $(x, y)$ is a feasible solution of FAIRRANGLP$(D', w')$ too.

Next, we show that the cost of $(x, y)$ w.r.t. FAIRRANGLP$(D', w')$ is not more than the cost $(x, y)$ w.r.t. FAIRRANGLP$(D, w)$. For each location $v$, let $\pi$ denote the location in $D'$ that receives the demand of $v$ by the end of Algorithm 2.

$$
cost(x, y; D', w') := \sum_{q \in D'} w'(q) \sum_{u \in F} x_{q,u} d(q, u)^p = \sum_{q \in D'} w'(q) R(q)^p
= \sum_{q \in D'} \sum_{v \in D' : \pi = q} w(v) R(q)^p
= \sum_{v \in D} w(v) R(\pi)^p
\leq \sum_{v \in D} w(v) R(v)^p = \text{OPT}_D
$$

(Q3): Algorithm 2 may either move the demand of a location $v$ to some other location $\pi$ or keep it at $v$. Let $v' = \pi$ in the former case and $v' = v$ in the latter case. In either case $d(v, v') \leq 2^{1+1/p} \cdot R(v)$. Therefore, by the approximate triangle inequality (Corollary A.3),

$$
d(v, C)^p \leq 2^{p-1} (d(v, v')^p + d(v', C)^p) \leq 2^{2p} R(v)^p + 2^{p-1} d(v', C)^p
$$

Summing over all locations,

$$
\sum_{v \in D} w(v) d(v, C)^p \leq \sum_{v \in D} w(v) (4^p R(v)^p + 2^{p-1} d(v', C)^p) \leq 4^p \cdot \text{OPT}_D + 2^{p-1} \cdot z
$$

**Proof of Theorem 2.5.** We initialize $(\pi, y)$ to the solution $(x^2, y)$ as constructed in Algorithm 3; $\pi = x^2$ and $y = y$. We keep $y = y$ throughout the process but modify $\pi$ to satisfy the desired properties.

By Claim B.2, we can partition open facilities, $\{u \in F \mid y_u > 0\}$ into disjoint super balls $\{P(v)\}_{v \in D'}$. So, (P1) will be satisfied by $y$.

Next, we modify the assignment vector $x_2$ so that for every $v \in D'$ and $u \in P(v) \setminus B(v)$, $\pi_{vu} > 0$ only if $\sum_{u' \in B(v)} y_{u'} < 1$. Similarly, for every $v \in D'$ and $u \in F \setminus P(v)$, $\pi_{vu} > 0$ iff $\sum_{u \in P(v)} y_u < 1$. So, by the construction of $(\pi, y)$, (P2) is satisfied.
Consider a location \( v \in D' \). Since the total \( \overline{y} \) contributions in each of \( B(v) \) and \( B(v') \) is at least half, in \( \overline{x} \) we assign the remaining demand of each location \( v \), which are not satisfied by the facilities in \( P(v) \), to the facilities in \( B(v') \). So, \((\overline{x}, \overline{y})\) satisfies (P3) and (P4).

Without loss of generality, we can assume that (P5) holds, otherwise, we could transfer the \( \overline{y}_{vu} \) fraction of assignment of \( v \) from \( u \) to a facility in \( B(v') \), set \( \overline{y}_{u} = 0 \), and reduce the total cost. Again, this reassignment is always possible because for every \( v \in D' \),

\[
\overline{y}_{vu} + \sum_{u' \in F \setminus P(v)} \overline{y}_{vu'} \leq 1/2 \leq \sum_{u' \in B(v')} \overline{y}_{u'}.
\]

Finally, the last property follows from Claim B.2 and the following bound. For every \( u'' \in B(v'') \) and \( u' \in B(v') \) where \( w \notin \{v, v'\} \),

\[
d(v, u') \leq d(v, v') + d(v', u') \quad \triangleright \text{triangle inequality}
\]
\[
\leq (1 + \frac{1}{2}) \cdot d(v, v')
\]
\[
\leq (1 + \frac{1}{2}) \cdot d(v, v'') \quad \triangleright d(v, v') \leq d(v, v'')
\]
\[
\leq 3 \cdot d(v, v''),
\]

where the second inequality holds since \( d(v, v') \geq 2^{1+\frac{1}{p}} R(v') \) and \( d(v', u') \leq 2^\frac{1}{p} R(v') \). Similarly, the fourth inequality follows from \( d(v, v'') \geq 2^{1+\frac{1}{p}} R(v'') \) and \( d(v'', u'') \leq 2^\frac{1}{p} R(v'') \) (which implies that \( d(v, u'') \geq \frac{1}{2} d(v, v'') \)). Thus, \( \text{cost}(\overline{x}, \overline{y}) \leq 3^p \cdot \text{cost}(x^2, y) \leq 9^p \cdot \text{OPT}_D \). \( \square \)