Describing and Quantifying Contradiction between Pieces of Evidence via Belnap Dunn Logic and Dempster-Shafer Theory

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Abstract

Belnap Dunn logic is a four-valued logic introduced to model reasoning with incomplete or contradictory information. In this article, we show how Dempster-Shafer theory can be used over Belnap Dunn logic in order to formalise reasoning with incomplete and/or contradictory pieces of evidence. First, we discuss how to encode different kinds of evidence, and how to interpret the resulting belief and plausibility functions. Then, we discuss the behaviour of Dempster's rule in this framework and present a variation of the rule. Finally, we show how to construct credal sets of classical probability measures based on this kind of evidence.

Keywords: Dempster-Shafer theory, Belnap Dunn logic, contradictory evidence

1. Introduction

Combination of evidence and conflict in Dempster-Shafer theory. Combination of conflicting or contradictory sources has been a major topic of study in Dempster-Shafer (DS) theory [9, 14, 13, 4]. In Dempster's original combination rule (DS-rule) [9], it was assumed that the sources are completely reliable, and hence any conflict between them is considered impossible. In addition, it was assumed that the frame of discernment Ω is composed of a list of mutually incompatible and exhaustive events. That is, every possible outcome is listed in Ω and no two outcomes in Ω can take place at the same time. Zadeh [14] gives an example to show that DS-rule can lead to counterintuitive results when it is used to aggregate pieces of evidence that are not fully reliable and with a significant degree of conflict between them. Several modifications of DS-rule have been proposed and studied in the literature to aggregate pieces of evidence both from not fully reliable sources and from sources strongly contradicting each other. Shafer [9] describes *discounting* or *tradeoff* method to deal with conflict. In this method, when the sources have a conflict between them, the analyst discounts sources based on their reliability before using DS-rule. Yager [13] proposes a combination rule which is similar to DS-rule but the mass attached to conflicting evidence is assigned to the whole frame of discernment. That is, having conflicting evidence is considered equivalent to having no information. Dubois and Prade [4] propose instead that if two sources attache mass to sets A and B, with $A \cap B = \varnothing$, then in the combination the mass $m(A) \cdot m(B)$ is attached to the set $A \cup B$. Intuitively this corresponds to the idea that if sources are contradictory, then the analyst concludes that one and only one of them is correct without knowing which one. Another alternative is to consider the non-normalized version of DS-rule and to assign a non-zero mass to the empty set whenever there is conflicting information. As discussed by Smets [11, 12], in the open-world context, this mass on the empty set can be interpreted as evidence supporting the fact that there is a possible outcome not listed in the frame of discernment.

In this work, we use an expansion of Belnap Dunn logic (BD) to represent and combine conflicting evidence. BD was proposed to represent and reason about incomplete and contradictory information.

Belnap Dunn logic. BD was introduced to reason about information rather than about truth [1]. In classical logic, a statement p is either *true* or *false*, meaning that p is *true* (resp. *false*) iff the statement p is true (resp. false) in the world. In BD, a statement p is either "supported by the available information", "contradicted by the available information", "neither supported nor contradicted by the available information", or "both supported and contradicted by the available information". These four truth values are

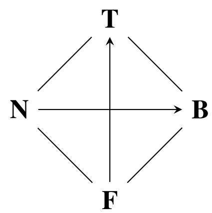

Figure 1: Belnap–Dunn square

respectively denoted **T** (*true*), **F** (*false*), **N** (*neither*), **B** (*both*) and are interpreted over the **4**-valued De Morgan algebra (Figure 1).

The four elements ordered from bottom to top define the so-called *truth lattice*. The truth tables of the conjunction $\wedge$ and disjunction $\vee$ of this lattice are given in Definition 1. By going from **F** to **T**, one goes from a situation where the information available fully supports the falsity of the statement, to a situation where the information available fully supports the truth of the statement. The four elements ordered from left to right define the so-called *information lattice*. By going from **N** to **B**, one goes from a situation where there is no information about the statement, to a situation where there is contradictory information about the statement. The truth tables of the disjunction $\sqcup$ and the conjunction $\sqcap$ of the information lattice are given in Definition 10.

A logic such that $p \vee \neg p$ (resp. $p \wedge \neg p$) is not an axiom is called *paracomplete* (resp. *paraconsistent*). BD is a weakening of classical propositional logic that is both paracomplete and paraconsistent. For instance, in BD, the conjunction of all variables and their negations $\bigwedge_{p \in \mathtt{Prop}} p \wedge \neg p$ is not equivalent to $\perp$ and is not equivalent to $q \wedge \neg q$ for some variable q. The first formula is true when there is contradictory information on every topic p, while the second one is true when there is contradictory information on the specific topic q.

Paraconsistent probabilities. In the classical case, $\mathrm{p}(\phi)$ (resp. $\mathrm{p}(\neg\phi)$) encodes the probability that ϕ is true (resp. false). Dunn [5] introduces paraconsistent probabilities that describe the information available about ϕ via four numbers (b, d, u, c). They encode the degree of belief b, disbelief d, uncertainty u (ignorance), and conflict c (contradiction) about ϕ. Klein et al. [6] presents a probabilistic extension of BD with a sound and complete axiomatization.

Our project. In Bílková et al. [2], we introduce belief functions over BD models and present logics to reason both with probabilities and belief functions over BD. This work is a first step towards understanding (imprecise) probabilities within a paracomplete and paraconsistent framework. In this article, we show how situations, where highly contradictory information is available, can be formalised using BD and DS-theory. BD is used to reason about information (that can very well be wrong) and not about facts of the real world. Therefore, when we combine BD and DS-theory, BD takes care of formalising contradictions and DS-theory handles uncertainty and incompleteness of the information. We consider sources that are fallible and we do not assume we are able to estimate their reliability. We work within the closed-world context because BD allows us to describe the pieces of information available and the conflict between them, but that conflict is not a contradiction that needs to be resolved like in classical logic.

Structure of the paper. First, in Section 2, we provide preliminaries on the semantics of BD, paraconsistent probabilities, and in Section 3, we discuss how to encode evidence via mass functions over BD-models and how to interpret the resulting belief and plausibility functions. Then, in Section 4, we discuss a variation of DS-rule over BD-models and its interpretation in BD. Finally, in Section 5 and Section 6, we introduce different notions of support of a statement that induce different belief functions over BD formulas, and we show that some of them allow to deduce credal sets of classical probabilities based on mass functions over BD-models.

2. An Expansion of Belnap Dunn Logic and Non-Standard Probabilities

Semantics of Belnap Dunn logic. In this paragraph, we present an expansion of BD with an additional connective $\triangle$ from Sano and Omori [8]. Throughout the paper, $\mathtt{Prop}$ will denote a *finite* set of propositional variables and $\mathtt{Lit} = \mathtt{Prop} \cup \{\neg p \mid p \in \mathtt{Prop}\}$ the associated set of literals.

Definition 1 (Semantics of BD$\triangle$ and BD) *Let $\mathcal{V} = \{\mathbf{T}, \mathbf{B}, \mathbf{N}, \mathbf{F}\}$ and define the following grammar in the Backus–Naur form.*

$\mathscr{L}_{\mathsf{BD}\triangle} \ni \phi := \perp \mid \top \mid p \in \mathtt{Prop} \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \triangle\phi.$ *Let $\mathscr{L}_{\mathsf{BD}}$ denote the $\triangle$-free fragment of $\mathscr{L}_{\mathsf{BD}\triangle}$.*

A **4***-valuation is a map $v_{\mathbf{4}} : \mathtt{Prop} \to \mathcal{V}$ that is extended to complex formulas using the following definitions.*

	$\neg$
T	**F**
B	**B**
N	**N**
F	**T**

$\wedge$	**T**	**B**	**N**	**F**
T	**T**	**B**	**N**	**F**
B	**B**	**B**	**F**	**F**
N	**N**	**F**	**N**	**F**
F	**F**	**F**	**F**	**F**

$\top$	**T**
$\perp$	**F**

	$\triangle$
T	**T**
B	**T**
N	**F**
F	**F**

$\vee$	**T**	**B**	**N**	**F**
T	**T**	**T**	**T**	**T**
B	**T**	**B**	**T**	**B**
N	**T**	**T**	**N**	**N**
F	**T**	**B**	**N**	**F**

We say that ϕ BD-entails χ ($\phi \models_{\mathsf{BD}} \chi$) iff it holds that if $v_{\mathbf{4}}(\phi) \in \{\mathbf{T}, \mathbf{B}\}$, then $v_{\mathbf{4}}(\chi) \in \{\mathbf{T}, \mathbf{B}\}$ for every **4***-valuation. ϕ BD$\triangle$-entails χ ($\phi \models_{\mathsf{BD}\triangle} \chi$) iff it holds that if $v_{\mathbf{4}}(\phi) \in$*

$\{\mathbf{T},\mathbf{B}\}$, *then* $v_4(\chi) \in \{\mathbf{T},\mathbf{B}\}$ *and if* $v_4(\chi) \in \{\mathbf{F},\mathbf{B}\}$, *then* $v_4(\phi) \in \{\mathbf{F},\mathbf{B}\}$ *for every* **4**-*valuation.*

We will use the fact that BD and classical logic (CL) share the same language and that $\models_{\mathsf{BD}}\subseteq\models_{\mathsf{CL}}$ (see e.g. [10]).

Observe that $\neg$, $\wedge$, and $\vee$ preserve **B** and **N**, thus there is no $\{\triangle,\bot,\top\}$-free formula that is always true or always non-false. For instance, saying that the formula $p \vee \neg p$ is always true would be interpreted as *"it is always the case that we have either the information that p is true or the information that ¬p is true"*. On the other hand, for every v_4, $v_4(\triangle p \vee \neg\triangle p) = \mathbf{T}$ and $v_4(\triangle p \wedge \neg\triangle p) = \mathbf{F}$. Indeed, the formula $\triangle p \vee \neg\triangle p$ is interpreted as *"either we have information stating that p is true or we do not have information stating that p is true"*.

Definition 2 (Lindenbaum algebras) *Let $\cong$ be the congruence relation on $\mathscr{L}_{\mathsf{BD}}$ defined as $\phi \cong \phi'$ iff $\phi \models_{\mathsf{BD}} \phi'$ and $\phi' \models_{\mathsf{BD}} \phi$. The* BD Lindenbaum algebra *over $\mathscr{L}_{\mathsf{BD}}$ is the De Morgan algebra $\langle\{[\phi]_{\phi\in\mathscr{L}_{\mathsf{BD}}}\},\wedge,\vee,\neg\rangle$, where $[\phi]$ is the equivalence class of the formula ϕ, $\neg[\phi] = [\neg\phi]$ and $[\phi] \odot [\psi] = [\phi \odot \psi]$ for $\odot \in \{\wedge,\vee\}$. The* CL Lindenbaum algebra *over $\mathscr{L}_{\mathsf{BD}}$ is the Boolean algebra defined similarly using $\models_{\mathsf{CL}}$.*

BD Lindenbaum algebra[1], the free de-Morgan algebra generated by the set Prop and carried by (equivalence classes of) formulas of $\mathscr{L}_{\mathsf{BD}}$, serves as an essential algebraic model of the BD entailment.

The previous definitions will be used to handle some technicalities, however, the intuitive understanding of the proposed framework relies on the interpretation of the formulas on the canonical model.

Definition 3 (BD-model) *A* BD-model *is a tuple $\mathfrak{M} = \langle W, v_4\rangle$ with W a finite nonempty set and $v_4 : \mathtt{Prop}\times W \to \mathcal{V}$. v_4 is extended to complex formulas using tables from Definition 1. For every nonempty $\mathsf{V} \subseteq \mathcal{V}$, the* V-extension of a formula ϕ on $\mathfrak{M}$ *is* $|\phi|^{\mathsf{V}} = \{w \in W : v_4(\phi, w) \in \mathsf{V}\}$.

For instance, $v_4(p, w) = \mathbf{B}$ means that *"the state w has information supporting both the truth and the falsity of p"* and $|\phi|^{\mathbf{T},\mathbf{F}}$ is the set of states that either provide only information supporting the truth of ϕ or only information supporting the falsity of ϕ. Note also that if all variables of ϕ are in the scope of $\triangle$, then $|\phi|^{\mathbf{B},\mathbf{N}} = \varnothing$.

Definition 4 (Canonical model) *The* canonical BD-model *over* Prop *is a tuple $\mathfrak{M}_c = \langle W_c, v_4\rangle$, where $W_c = \mathcal{P}(\mathtt{Lit})$ and the valuation $v_4 : \mathtt{Prop}\times W_c \to \mathcal{V}$ is defined as follows:*

$$
\begin{aligned}
v_4(p,w) &= \mathbf{N} \text{ if } p, \neg p \notin w,\\
v_4(p,w) &= \mathbf{T} \text{ if } p \in w \text{ and } \neg p \notin w,\\
v_4(p,w) &= \mathbf{F} \text{ if } p \notin w \text{ and } \neg p \in w,\\
v_4(p,w) &= \mathbf{B} \text{ if } p, \neg p \in w.
\end{aligned}
$$

[1] We refer the readers to [3] for an introduction into algebraic study of logics.

Intuitively, W_c is the set of all possible valuations v_4 over Prop. Similarly, $\mathcal{P}(\mathtt{Prop})$ can be seen as the set of all classical valuations: i.e. p is true at $S \in \mathcal{P}(\mathtt{Prop})$ iff $p \in S$.

Consider the canonical model over $\mathtt{Prop} = \{p, q, r\}$. The set W_c corresponds to the set of **4**-valuations over Prop. Since, each variable can be assigned 4 different values in $\{\mathbf{T},\mathbf{B},\mathbf{N},\mathbf{F}\}$, there are $4^{|\mathsf{Prop}|} = 64$ possible valuations. Take, for instance, $w = \{p, \neg p, q\}$. Then, $v_4(p, w) = \mathbf{B}$, $v_4(q, w) = \mathbf{T}$, and $v_4(r, w) = \mathbf{N}$. That is, the state w both supports and rejects p, it supports q, and it says nothing about r. One can look at each state w as a different source of information.

A state $w \in W_c$ is said *incomplete* (resp. *classical*) if $v_4(p, w) \in \{\mathbf{T},\mathbf{N},\mathbf{F}\}$ (resp. $v_4(p, w) \in \{\mathbf{T},\mathbf{F}\}$) for every $p \in \mathtt{Prop}$.

Definition 5 *We consider formulas in $\mathscr{L}_{\mathsf{BD}}$. A* conjunctive clause *is a conjunction of literals. A conjunctive clause is* irredundant *if it contains each literal at most once, e.g. $\bigwedge_{l\in L} l$ with $L \subseteq \mathtt{Lit}$. A formula φ is in* irredundant disjunctive normal form *(iDNF) if it is a disjunction of irredundant conjunctive clauses, and moreover, if $\varphi = \bigvee_{i\in I}\varphi_i$ implies $\varphi_i \not\models_{\mathsf{BD}} \varphi_j$ for every $i \neq j$.*

Every formula in $\mathscr{L}_{\mathsf{BD}}$ is BD-equivalent to a *unique* (up to permutation of clauses and literals) formula in iDNF.

Definition 6 *We consider formulas in $\mathscr{L}_{\mathsf{BD}\triangle}$. A $\mathcal{V}$*-atom *is one of the following formulas.*

$$
\begin{array}{ll}
\mathbf{T}p \coloneqq \triangle p \wedge \neg\triangle\neg p & \mathbf{B}p \coloneqq \triangle p \wedge \triangle\neg p\\
\mathbf{N}p \coloneqq \neg\triangle p \wedge \neg\triangle\neg p & \mathbf{F}p \coloneqq \neg\triangle p \wedge \triangle\neg p
\end{array}
$$

A V-literal *is a formula of the form $\mathsf{V}p \coloneqq \bigvee_{\mathbf{X}\in\mathsf{V}}\mathbf{X}p$ with $\mathsf{V} \subseteq \mathcal{V}$. A* witnessed clause *is a conjunction of* V*-literals. A witnessed clause is* reduced *iff no* variable *occurs twice. A witnessed clause is* complete *iff it is reduced, every variable occurs once, and it contains only $\mathcal{V}$-atoms. A formula is in* witnessed disjunctive normal form *(wDNF) if it is a disjunction of witnessed clauses. A formula in* wDNF *$\bigvee_{i\in I} c_i$ is* maximal *iff it is composed of complete witnessed clauses and $|c_i|^{\mathbf{T}} = |c_j|^{\mathbf{T}}$ implies $i = j$.*

Note that a $\mathcal{V}$-atom $\mathbf{X}p$ means 'p has value $\mathbf{X}$'. A V-literal over p designates the set of values p can have. A wDNF specifies the possible values of variables, e.g.,
$\phi = (\{\mathbf{T},\mathbf{N}\}p\wedge\{\mathbf{B},\mathbf{N}\}q\wedge\{\mathbf{T},\mathbf{F}\}r)\vee(\{\mathbf{N}\}p\wedge\{\mathbf{N}\}q\wedge\{\mathbf{N}\}r)$
means that 'either p is non-false, q is non-classical, and r is classical, or p, q, and r have value $\mathbf{N}$.' For sake of readability, we write $\mathbf{TN}p$ instead of $\{\mathbf{T},\mathbf{N}\}p$. Using this convention for ϕ, we get $\phi = (\mathbf{TN}p\wedge\mathbf{BN}q\wedge\mathbf{TF}r)\vee(\mathbf{N}p\wedge\mathbf{N}q\wedge\mathbf{N}r)$.

For any formula $\phi \in \mathscr{L}_{\mathsf{BD}\triangle}$ and for any nonempty set of truth values $\mathsf{V} \subseteq \mathcal{V}$, one can write a formula $\mathsf{V}\phi \in \mathscr{L}_{\mathsf{BD}\triangle}$ such that $|\mathsf{V}\phi|^{\mathbf{T}} = |\phi|^{\mathsf{V}}$. To reduce the use of superscripts, whenever there is no ambiguity, we will write $|\phi|$ instead of $|\phi|^{\mathbf{T}}$ and $|\mathsf{V}\phi|$ instead of $|\phi|^{\mathsf{V}}$.

Notice that the $\mathbf{T}$-extension on the canonical model of a complete clause is a singleton, that is, any state w of the canonical model can be uniquely (up to permutation of the $\mathcal{V}$-atoms) represented by a complete witnessed clause c_w and every set X of states can be uniquely (up to permutation of the clauses) represented by the corresponding maximal wDNF formula $\phi_X = \bigvee_{w \in X} c_w$.

In this paper, we will mostly consider reasoning within classical logic CL and Belnap Dunn logic BD. The logic BD$\triangle$ is used as a tool to name and manipulate the sets of the states of the canonical model.

Paraconsistent probabilities Several equivalent representations of paraconsistent probabilities have been introduced in the literature. In this article, we focus our attention on probabilities over BD-models by Klein et al. [6].

Definition 7 (Probability assignments) *A* (paraconsistent) probability assignment *is a function* $\mathrm{p} : \mathscr{L}_{\mathsf{BD}} \to [0,1]$ *satisfying, for all* $\phi, \psi \in \mathscr{L}_{\mathsf{BD}}$,
1. $\mathrm{p}(\bot) = 0$ *and* $\mathrm{p}(\top) = 1$,
2. monotonicity. *if* $\phi \models_{\mathsf{BD}} \psi$, *then* $\mathrm{p}(\phi) \leq \mathrm{p}(\psi)$,
3. import-export rule. $\mathrm{p}(\phi \vee \psi) + \mathrm{p}(\phi \wedge \psi) = \mathrm{p}(\phi) + \mathrm{p}(\psi)$.

A classical probability assignment *is a function* $\mathsf{p} : \mathscr{L}_{CL} \to [0,1]$ *satisfying, for all* $\phi, \psi \in \mathscr{L}_{CL}$,
1. normalization. $\mathsf{p}(\top) = 1$,
2. monotonicity. *if* $\phi \models_{\mathsf{CL}} \psi$, *then* $\mathsf{p}(\phi) \leq \mathsf{p}(\psi)$,
3. finite additivity. $\mathsf{p}(\phi \vee \psi) = \mathsf{p}(\phi) + \mathsf{p}(\psi)$, *for* $(\phi \wedge \psi) \models_{\mathsf{CL}} \bot$.

Monotonicity ensures that equivalent formulas have the same probability. For classical probabilities, if we define probability measures over the CL Lindenbaum algebra, monotonicity follows from additivity. In addition, one can easily show that any classical probability assignment is also a probability assignment.

Definition 8 (Probabilistic BD-models) *A* probabilistic BD-model *is a tuple* $\mathfrak{M} = \langle W, v_{\mathbf{4}}, \mu \rangle$ *such that* $\langle W, v_{\mathbf{4}} \rangle$ *is a* BD*-model and* $\mu : \mathcal{P}(W) \to [0,1]$ *is a probability measure on* $\mathcal{P}(W)$.

Then, the *induced probability assignment* is defined as follows: for any formula $\phi \in \mathscr{L}_{\mathsf{BD}}$,

$$\mathrm{p}(\phi) = \mu(|\mathbf{TB}\phi|).$$

Klein et al. [6] present the results over the two-valuation semantics of BD. Here, we choose to work with an equivalent semantics using a $\mathbf{4}$-valued valuation. We straightforwardly rephrase their results on that semantics.

While working with BD, some subsets of states of the canonical model cannot be represented by a formula, however using the axioms of probabilities, one can compute the measure of each subset of states. We use BD$\triangle$ as a tool to designate each subset of states via a formula. This will be useful to describe the focal elements of mass functions and to present aggregation rules in a reader-friendly manner. The algebraic semantic of BD$\triangle$ is presented in [7], but probabilities over the logic BD$\triangle$ are not known in the literature. Therefore, the study of (imprecise) probabilities over BD$\triangle$ is an open problem we do not discuss in this article. We now define the notion of belief and plausibility over BD formulas.

Definition 9 *A* belief *and a* plausibility *assignment over* BD *formulas are functions* $\mathtt{Bel}, \mathtt{Pl} : \mathcal{L}_{\mathsf{BD}} \to [0,1]$ *s.t.*
(1) they are monotone w.r.t. BD *entailment,*
(2) they satisfy respectively: for every $n \in \mathbb{N}$, *for all* $\phi_1 \ldots \phi_n \in \mathscr{L}_{\mathsf{BD}}$,

$$\mathtt{Bel}(\phi_1 \vee \cdots \vee \phi_n) \geq \sum_{\varnothing \neq I \subseteq [1,n]} (-1)^{|I|+1} \mathtt{Bel}\left(\bigwedge_{i \in I} \phi_i\right), \tag{1}$$

and

$$\mathtt{Pl}(\phi_1 \wedge \cdots \wedge \phi_n) \leq \sum_{\varnothing \neq I \subseteq [1,n]} (-1)^{|I|+1} \mathtt{Pl}\left(\bigvee_{i \in I} \phi_i\right). \tag{2}$$

A belief (resp. plausibility) assignment is said to be normal *if* $\mathtt{Bel}(\bot) = 0$ *and* $\mathtt{Bel}(\top) = 1$ *(resp.* $\mathtt{Pl}(\bot) = 0$ *and* $\mathtt{Pl}(\top) = 1$*).*

3. Dempster-Shafer Theory and BD.

In this section, we discuss how to encode information via a mass function over the canonical model. Then we discuss the behaviour of DS-rule on the canonical model via examples. The *frame of discernment* Ω is the set of possibilities under consideration. Here, we work within the closed-world context, that is, we assume that the possibilities listed in Ω are exhaustive (no option has been forgotten). However, we do not apriori assume that a piece of evidence supporting $p \in \Omega$ contradicts any other possibility $q \in \Omega \setminus \{p\}$, even though such a piece of evidence can perfectly be encoded in our framework (see Example 1). Therefore, whether or not one assumes that the frame of discernment consists of mutually exclusive possibilities will not affect the reasoning within BD. But it can affect how the analyst encodes a piece of information into a mass function.

Let S be a set and $\mathcal{P}(S)$ its powerset. $\mathtt{m} : \mathcal{P}(S) \to [0,1]$ is a *mass function over* S if $\sum_{X \subseteq S} \mathtt{m}(X) = 1$. $\mathtt{m}$ is *normal* if $\mathtt{m}(\varnothing) = 0$. Let $\mathtt{m}$ be a mass function over S. We note $\mathtt{bel}_{\mathtt{m}}$ and $\mathtt{pl}_{\mathtt{m}}$ the maps from $\mathcal{P}(S)$ to $[0,1]$ defined as follows:

$$\mathtt{bel}_{\mathtt{m}}(X) = \sum_{Y \subseteq X} \mathtt{m}(Y), \quad \text{and} \quad \mathtt{pl}_{\mathtt{m}}(X) = 1 - \mathtt{bel}_{\mathtt{m}}(X^c)$$

where X^c is the set complement of X. $\mathtt{bel}_{\mathtt{m}}$ (resp. $\mathtt{pl}_{\mathtt{m}}$) is a belief (resp. plausibility) function.

In this section, we consider the canonical model $\mathfrak{M}_c = \langle W_c, v_4 \rangle$ over the finite set of propositional variables $\mathtt{Prop}$ and a normal mass function $\mathtt{m}$ over W_c.

Encoding of evidence on the canonical model. Within $\mathsf{BD}\triangle$, the statement *"there is information supporting p"* is encoded by the formula p and equivalently by the V-literal $\mathbf{T}p \vee \mathbf{B}p$ noted $\mathbf{TB}p$. Indeed, the formula p is satisfied by every state w such that $v_4(w)(p) \in \{\mathbf{T}, \mathbf{B}\}$ which is equivalent to $v_4(w)(\mathbf{TB}p) = \mathbf{T}$. Therefore, one encodes the statement *"the information 100% supports p"* via the mass function $\mathtt{m}_p : \mathcal{P}(W_c) \to [0,1]$ such that $\mathtt{m}(|\mathbf{TB}p|) = 1$. The mass function $\mathtt{m}_{\mathbf{T}p}$ such that $\mathtt{m}_{\mathbf{T}p}(|\mathbf{T}p|) = 1$ encodes classical evidence supporting p, i.e. *"the information 100% supports p and there is no information available supporting ¬p"*. We say that *"the information supports exactly p."*

Encoding of contradiction. In this framework, we can encode a source stating *"I have information supporting p with certainty* 0.3*, ¬p with certainty* 0.3 *and the remaining of my information supports exactly q, and is contradictory about p"* via the mass function $\mathtt{m}$ such that $\mathtt{m}(|\mathbf{TB}p|) = \mathtt{m}(|\mathbf{TB}\neg p|) = 0.3$ and $\mathtt{m}(|\mathbf{T}q \wedge \mathbf{B}p|) = 0.4$.

How to interpret a non-zero mass on *falsum*? In the classical framework, $\mathtt{m}(\bot) \neq 0$ can be used to represent contradictory pieces of information. When using BD, this will not happen, because there is a formula (different from $\bot$) that can describe the information available and exactly on which topics ($p \in \mathtt{Prop}$) it is contradictory. However, one could still imagine situations where it makes sense to have $\mathtt{m}(\bot) \neq 0$. For instance, if one works in the open-world context, and one gets a piece of information about an outcome not taken into consideration in the frame of discernment, then the mass corresponding to this information could be assigned to $\bot$. Here, we provide a first analysis of formalising conflicting pieces of evidence within BD and DS-theory. We work within the closed-world context, but further studies on the formalisation of the open-world context are necessary.

Encoding absence of evidence. One can differentiate between the statement *"I have no information about p"* and transmitting no information by either assigning the mass to the set $|\mathbf{N}p|$ or to the set $W_c = |\top|$.

Encoding incomplete sources. A source is said to be *incomplete* if, for every $p \in \mathtt{Prop}$, it says *"either there is information supporting exactly p or there is information supporting exactly ¬p or there is no information about p"*. For instance, a coherent database, that contains no contradictory information but has no information about some entries.

Encoding classical sources. A source is said *classical*, if it affirms that, for every $p \in \mathtt{Prop}$, "the information supports exactly p or supports exactly $\neg p$."

How to encode lack of information? When the source announces *"p with certainty* 0.8*"*, where do we assign the remaining 0.2? The formulas $\psi_{\mathsf{CL}} = \bigwedge_{p \in \mathtt{Prop}} \mathbf{TF}p$ and $\psi_{\mathsf{PCL}} = \bigwedge_{p \in \mathtt{Prop}} \mathbf{TNF}p$ correspond to the set of states assigning values in the set $\{\mathbf{T}, \mathbf{F}\}$ or $\{\mathbf{T}, \mathbf{N}, \mathbf{F}\}$ to every variable, that is the so-called *classical* and *incomplete* states. The formula $\psi_{W_c} = \top$ corresponds to the set of all states. The choice between these alternatives depends on the nature of the source (e.g. is the source contradictory?) and the reasoning framework of the analyst (e.g. does the analyst accepts contradictory evidence?).

Interpreting belief and plausibility over BD-models. As the mass function is defined on the powerset algebra of the canonical model, $\mathtt{bel}_\mathtt{m}$ and $\mathtt{pl}_\mathtt{m}$ are respectively belief and plausibility functions and every combination rule using only the fact that the underlying algebra is a powerset algebra can still be used.

The BD-negation $\neg$ does not correspond to the set theoretic complement, however the map $\mathtt{pl}^{\neg}_\mathtt{m} : \mathcal{P}(W_c) \to [0,1]$ such that $\mathtt{pl}^{\neg}_\mathtt{m}(X_\phi) = 1 - \mathtt{bel}_\mathtt{m}(X_{\neg\phi})$ with $X_\phi = |\mathbf{TB}\phi|$ and $X_{\neg\phi} = |\mathbf{TB}\neg\phi|$ is still a plausibility function. $\mathtt{pl}_\mathtt{m}(|\mathbf{TB}\phi|)$ and $\mathtt{pl}^{\neg}_\mathtt{m}(|\mathbf{TB}\phi|)$ represent respectively the amount of evidence *coherent with the truth of* ϕ and *not supporting the falsity of* ϕ. But one can have models such that $\mathtt{pl}_\mathtt{m}(|\mathbf{TB}\phi|) \neq \mathtt{pl}^{\neg}_\mathtt{m}(|\mathbf{TB}\phi|)$ and $\mathtt{bel}_\mathtt{m}(|\mathbf{TB}\phi|) > \mathtt{pl}^{\neg}_\mathtt{m}(|\mathbf{TB}\phi|)$. See Bílková et al. [2] for more details.

Belief and plausibility as lower and upper paraconsistent probabilities. Recall that a probabilistic BD-model induces a probability assignment on BD-formulas s.t. $\mathrm{p}(\phi) = \mu(|\phi|^{\mathbf{TB}})$. That is, the measure of the set of states that support the truth of ϕ, but can be contradictory about ϕ. A mass function $\mathtt{m}$ over W_c induces the following lower (resp. upper) bound $\mathtt{Bel}_{\mathbf{TB}}(\phi)$ (resp. $\mathtt{Pl}_{\mathbf{TB}}(\phi)$) on $\mathrm{p}(\phi)$:

$$\mathtt{Bel}_{\mathbf{TB}}(\phi) := \mathtt{bel}_\mathtt{m}(|\phi|^{\mathbf{TB}}), \tag{3}$$

$$\mathtt{Pl}_{\mathbf{TB}}(\phi) := \mathtt{pl}_\mathtt{m}(|\phi|^{\mathbf{TB}}) = 1 - \mathtt{bel}_\mathtt{m}((|\phi|^{\mathbf{TB}})^c). \tag{4}$$

Since $|\phi \vee \psi|^{\mathbf{TB}} = |\phi|^{\mathbf{TB}} \cup |\psi|^{\mathbf{TB}}$, $|\phi \wedge \psi|^{\mathbf{TB}} = |\phi|^{\mathbf{TB}} \cap |\psi|^{\mathbf{TB}}$, and $\mathtt{bel}_\mathtt{m}$ and $\mathtt{pl}_\mathtt{m}$ are belief and plausibility function, $\mathtt{Bel}_{\mathbf{TB}}$ and $\mathtt{Pl}_{\mathbf{TB}}$ are respectively belief and plausibility assignments over BD formulas. Moreover, if $\mathtt{m}$ is normal, $\mathtt{Bel}_{\mathbf{TB}}(\top) = 1$ and $\mathtt{Bel}_{\mathbf{TB}}(\bot) = 0$. Therefore, $\mathtt{Bel}_{\mathbf{TB}}$ and $\mathtt{Pl}_{\mathbf{TB}}$ are normal.

Let $\mathcal{F}_\mathtt{m}$ be the set of probability measures on $\mathcal{P}(W_c)$ such that $\mathtt{bel}_\mathtt{m}$ and $\mathtt{pl}_\mathtt{m}$ are the lower and upper probabilities given by $\mathcal{F}_\mathtt{m}$. Every probability measure $\mu : \mathcal{P}(W) \to [0,1]$ induces a probability assignment p_μ on BD formulas such that, for any formula ϕ, $\mathrm{p}_\mu(\phi) := \mu(|\phi|^{\mathbf{TB}})$ as described in

Section 2. Then $\mathtt{Bel}_{\mathbf{TB}}$ and $\mathtt{Pl}_{\mathbf{TB}}$ are the lower and upper probabilities given by the set $\mathcal{F}_{\mathbf{TB}} = \{\mathrm{p}_\mu \mid \mu \in \mathcal{F}_{\mathtt{m}}\}$.

Behaviour of DS-rule. Since we are considering mass functions over powerset algebras, one may still apply existing combination rules. However, the results for non-classical sources may be unintuitive, as to be expected. Let $\mathtt{m}_1, \mathtt{m}_2$ be two normal mass functions over S. Their aggregation $\mathtt{m}_{1\oplus 2}$ via DS-rule is defined as follows: $\mathtt{m}_{1\oplus 2}(\varnothing) = 0$, otherwise

$$\mathtt{m}_{1\oplus 2}(X) = \frac{\sum\{\mathtt{m}_1(X_1)\cdot\mathtt{m}_2(X_2) \mid X_1 \cap X_2 = X\}}{1 - \sum\{\mathtt{m}_1(X_1)\cdot\mathtt{m}_2(X_2) \mid X_1 \cap X_2 \neq \varnothing\}}.$$

Example 1 (Two disagreeing doctors) *It is assumed that a patient can have one and only one of the diseases in $S = \{a, b, c\}$ (the events a, b and c are exhaustive and incompatible).*

Doctor 1 thinks that the patient has disease a with certainty 0.9 and that it is very unlikely the patient has disease b, therefore assigning certainty 0.1 to that option. Doctor 2 thinks that the patient has disease c with certainty 0.9 and that it is very unlikely the patient has disease b, therefore assigning certainty 0.1 to that option.

An analyst can interpret the claim of Doctor 1 as "I have information that $a\wedge\neg b\wedge\neg c$ is true and that $\neg(a\wedge\neg b\wedge\neg c)$ is false with certainty 0.9", which corresponds to assigning mass 0.9 to the set $|\mathbf{T}a \wedge \mathbf{F}b \wedge \mathbf{F}c|^{\mathbf{T}}$. Following the same kind of reasoning for Doctor 2, the analyst gets the following mass functions:

$\mathtt{m}_1(|\mathbf{T}a \wedge \mathbf{F}b \wedge \mathbf{F}c|) = 0.9, \quad \mathtt{m}_1(|\mathbf{F}a \wedge \mathbf{T}b \wedge \mathbf{F}c|) = 0.1,$
$\mathtt{m}_2(|\mathbf{F}a \wedge \mathbf{F}b \wedge \mathbf{T}c|) = 0.9, \quad \mathtt{m}_2(|\mathbf{F}a \wedge \mathbf{T}b \wedge \mathbf{F}c|) = 0.1.$

Since $\mathbf{T}a \wedge \mathbf{F}a$ is a contradictory statement in $\mathsf{BD}\triangle$, one will get exactly the same conclusion as in the classical case by applying DS-rule, that is: $\mathtt{m}_{1\oplus 2}(|\mathbf{F}a \wedge \mathbf{T}b \wedge \mathbf{F}c|) = 1$.

This conclusion may be perfectly justified if Doctor 1 (resp. 2) is an expert on disease c (resp. a), therefore when they say it cannot be disease c (resp. a), they are necessarily correct. However, one may want to be able to consider doctors who might be wrong (without having to evaluate how reliable they are) or to describe situations in which the doctors refuse to give an opinion about the diseases they are not knowledgeable about.

Example 2 (Reasoning with BD) *We still consider our two doctors, but we do not assume that they are fully reliable experts. Therefore, when they announce p we assign the corresponding mass to $|\mathbf{TB}p|$ rather than $|\mathbf{T}p|$. In this framework, this means that two contradictory pieces of information $|\mathbf{TB}p|$ and $|\mathbf{FB}p|$ are not inconsistent. When doctor 1 (resp. 2) announces the evidence supports disease x, they still imply that they reject the other diseases. We get the following mass functions*

$$\mathtt{m}_1(|\mathbf{TB}a \wedge \mathbf{FB}b \wedge \mathbf{FB}c|) = 0.9$$
$$\mathtt{m}_1(|\mathbf{FB}a \wedge \mathbf{TB}b \wedge \mathbf{FB}c|) = 0.1$$
$$\mathtt{m}_2(|\mathbf{FB}a \wedge \mathbf{FB}b \wedge \mathbf{TB}c|) = 0.9$$
$$\mathtt{m}_2(|\mathbf{FB}a \wedge \mathbf{TB}b \wedge \mathbf{FB}c|) = 0.1$$

and the following aggregated mass function

$$\mathtt{m}_{1\oplus 2}(X) = \begin{cases} 0.81 & \textit{if } X = |\mathbf{B}a \wedge \mathbf{FB}b \wedge \mathbf{B}c| \\ 0.09 & \textit{if } X = |\mathbf{B}a \wedge \mathbf{B}b \wedge \mathbf{FB}c| \\ & \textit{or } X = |\mathbf{FB}a \wedge \mathbf{B}b \wedge \mathbf{B}c| \\ 0.01 & \textit{if } X = |\mathbf{FB}a \wedge \mathbf{TB}b \wedge \mathbf{FB}c| \\ 0 & \textit{otherwise.} \end{cases}$$

$\mathtt{bel}_{1\oplus 2}(|\mathbf{TB}\phi|)$ *and* $\mathtt{pl}_{1\oplus 2}(|\mathbf{TB}\phi|) = 1 - \mathtt{bel}_{1\oplus 2}(|\mathbf{NF}\phi|)$ *provide the following lower and upper probabilities:*

ϕ	a	$\neg a$	b	$\neg b$	$a \wedge \neg a$	$b \wedge \neg b$
$\mathtt{bel}_{1\oplus 2}$	0.9	1	0.19	0.99	0.9	0.18
$\mathtt{pl}_{1\oplus 2}$	1	1	1	1	1	1

Notice that $v_{\mathbf{4}}(p \wedge \neg p, w) = \mathbf{B}$ if $v_{\mathbf{4}}(p, w) = \mathbf{B}$ and $v_{\mathbf{4}}(p \wedge \neg p, w) = \mathbf{N}$ otherwise. The values for a (resp. $\neg a$, $a \wedge \neg a$) are the same as those for c (resp. $\neg c$, $c \wedge \neg c$).

We have that $\mathtt{bel}_{1\oplus 2}(|\mathbf{TB}a|) = \mathtt{bel}_{1\oplus 2}(|\mathbf{TB}c|) = 0.9$ is 4.7 times larger than $\mathtt{bel}_{1\oplus 2}(|\mathbf{TB}b|) = 0.19$ where $\mathtt{bel}_{1\oplus 2}(|\mathbf{TB}a|)$ encodes how much evidence supports the truth of a. In addition, $\mathtt{bel}_{1\oplus 2}(|\mathbf{B}a|) = \mathtt{bel}_{1\oplus 2}(|\mathbf{B}c|) = 0.9$ is 5 times larger than $\mathtt{bel}_{1\oplus 2}(|\mathbf{B}b|) = 0.18$ which means that the evidence is highly contradictory regarding a and c and 5 times less contradictory regarding b. This is confirmed by the fact that $\mathtt{bel}_{1\oplus 2}(|\mathbf{FB}a|) = \mathtt{bel}_{1\oplus 2}(|\mathbf{FB}c|) = 1$ - that is all the evidence supports the fact that it is not disease a or c - and $\mathtt{bel}_{1\oplus 2}(|\mathbf{TB}a|) = \mathtt{bel}_{1\oplus 2}(|\mathbf{B}a|) = 0.9$ - that is, all the evidence supporting the truth of a is contradictory.

The mass function highlights that (1) doctors agree the patient most likely does not have disease b, (2) evidence strongly supports the hypothesis that they have disease a or c and (3) the evidence is highly contradictory which may suggest that further investigation is necessary.

Example 3 (Reasoning with incomplete information) *Now, Doctor 1 considers only diseases a and b and states that it is disease a with certainty 0.9 and disease b with certainty 0.1. They do not provide any information about disease c. Doctor 2 states that it is disease b with certainty 0.1, disease c with certainty 0.9 and that they could not find any conclusive information about disease a and cannot estimate the induced uncertainty. If one assumes that the sources reason classically, one can encode them as follows:*

$\mathtt{m}_1(|\mathbf{T}a \wedge \mathbf{F}b|) = 0.9 \quad \mathtt{m}_2(|\mathbf{N}a \wedge \mathbf{F}b \wedge \mathbf{T}c|) = 0.9$
$\mathtt{m}_1(|\mathbf{F}a \wedge \mathbf{T}b|) = 0.1 \quad \mathtt{m}_2(|\mathbf{N}a \wedge \mathbf{T}b \wedge \mathbf{F}c|) = 0.1.$

Here, it is impossible to aggregate using DS-rule because $\mathbf{T}a \wedge \mathbf{N}a = \mathbf{F}a \wedge \mathbf{N}a = \bot$. That is, DS-rule behaves as if the information about a was contradictory. However, if one source says 'I have information supporting a' and one source says 'I have no information about a', one would expect to conclude 'the sources have information supporting a'. This problem will be resolved by the variation of DS-rule proposed in Section 4.

Example 1 shows that one can still encode classical reasoning and computation in the proposed BD framework, by assigning non-zero mass only to sets of classical states. Example 2 shows that if one assigns non-zero mass only to sets representable as the **TB**-extensions of BD formulas, then DS-rule allows to describe the nature (contradictory or not) of the available information. Example 3 shows that DS-rule may not be suitable for the combination of more general types of sources. In the next section, we propose a variation of DS-rule more suitable for our framework.

4. Aggregation Rule Based on the Information Lattice

In the previous section, we discussed the behaviour of DS-rule within the framework of BD. If one works only with set of states that are the **TB** extension of BD-formulas, then DS-rule provides a natural way to aggregate information: if two sources agree about the truth of ϕ, the corresponding mass is assigned to $|\mathbf{TB}\phi|$, otherwise it is assigned to $|\mathbf{B}\phi|$. However, if one wants to work with mass functions whose focal elements are arbitrary sets of states, the behaviour of DS-rule is less intuitive (see Example 3).

As shown in Example 1, classical pieces of information can be encoded in BD framework by assigning mass to the set $|\mathbf{T}\phi|$. In this case, DS-rule will behave as in the classical framework. However, here we get more options to assign the mass $\mathtt{m}_1(X)\cdot\mathtt{m}_2(Y)$ when $X\cap Y=\varnothing$ (i.e. mass of contradictory evidence). As discussed previously (page 38), the algebra of truth values of BD (Figure 1) contains two lattices: the *truth lattice* and the *information lattice*. In this section, we use the information lattice to propose a variation of DS-rule.

First, we present the behaviour of the logical connectives of the information lattice. Then, we define a fusion operation $\odot$ on maximal wDNF's using $\sqcup$ and use this operation to define an aggregation rule on mass functions over the canonical model. Finally, we discuss the mathematical properties and the behaviour of this aggregation rule.

Definition 10 *We denote $\sqcup$ (resp. $\sqcap$) the disjunction (resp. conjunction) of the information lattice of Figure 1. It is called* information join *(resp.* information meet*). These operators on $\mathcal{V}$ are defined as follows:*

$\sqcup$	**T**	**B**	**N**	**F**
T	**T**	**B**	**T**	**B**
B	**B**	**B**	**B**	**B**
N	**T**	**B**	**N**	**F**
F	**B**	**B**	**F**	**F**

$\sqcap$	**T**	**B**	**N**	**F**
T	**T**	**T**	**N**	**N**
B	**T**	**B**	**N**	**F**
N	**N**	**N**	**N**	**N**
F	**N**	**F**	**N**	**F**

We propose the following fusion operator on formulas. If two formulas convey non-contradictory information, then the fusion returns the common ground. For instance, $\mathbf{TB}p\odot\mathbf{TF}p=\mathbf{T}p$. Indeed, the first formula states *'the evidence supports the truth of p or it is contradictory about p'*, and the second states *'the evidence supports the truth of p or the falsity of p'*. Both agree on the fact that *'the evidence supports the truth of p'*. This behavior is exactly the same as DS-rule on non-contradictory pieces of evidence.

If two formulas convey contradictory information about p, then the fusion returns the information join on the information about p. For instance, $\mathbf{T}p\odot\mathbf{F}p=\mathbf{B}p$ and $(\mathbf{T}p\wedge\mathbf{T}q)\odot(\mathbf{F}p\wedge\mathbf{T}q)=\mathbf{B}p\wedge\mathbf{T}q$. One can also consider more complex formulas such as $\mathbf{TB}p\odot\mathbf{FN}p=\mathbf{TB}p$. The first formula states *'the information supports the truth of p or it is contradictory about p'* and the second formula states *'the information supports the falsity of p or is inconclusive about p'*. The fusion conveys the fact that *'the aggregated information either supports the truth of p or is contradictory about p'*.

In the following, we define the fusion on maximal wDNF. This is enough because every set of states is uniquely represented by a maximal wDNF and vice versa.

Definition 11 (Fusion) *Let $\odot$ be a binary operation defined on BD$\triangle$ formulas in maximal wDNF as follows. Let $\phi=\bigvee_{i\in I}c_i$ and $\psi=\bigvee_{j\in J}d_j$ be BD$\triangle$ formulas in maximal wDNF. If $\phi\wedge\psi\not\models_{\mathsf{BD}}\bot$, then $\phi\odot\psi=\phi\wedge\psi$, else*

$$\phi\odot\psi=\bigvee_{i\in I,j\in J}c_i\odot d_j\quad\text{with}\quad c_i\odot d_j=\bigwedge_{p\in\mathtt{Prop}}(X_i\sqcup X_j)p.$$

c_i and d_j are complete clauses, therefore each variable appears exactly once in a $\mathcal{V}$-atom.

Proposition 12 *$\odot$ is idempotent and commutative. The neutral element is the mass function assigning 1 to W_c.*

Notice that the empty set is represented by the empty disjunction and that the fusion between a formula and the empty disjunction is the empty disjunction. In addition, since $\wedge$ is idempotent, $\odot$ is idempotent. Since, $\wedge$ and $\sqcup$ are commutative, $\odot$ is commutative. However, it is not associative. For instance, consider the formulas: $\phi=(\mathbf{T}\wedge\mathbf{N}q)\vee(\mathbf{N}p\wedge\mathbf{F}q)$, $\chi=(\mathbf{N}p\wedge\mathbf{T}q)\vee(\mathbf{N}p\wedge\mathbf{N}q)$ and $\psi=(\mathbf{T}p\wedge\mathbf{N}q)\vee(\mathbf{N}p\wedge\mathbf{T}q)\vee(\mathbf{T}p\wedge\mathbf{F}q)$.

Now, we can define the aggregation rule $\odot$ over mass functions on canonical models.

Definition 13 *Let $\mathtt{m}_1,\mathtt{m}_2:\mathcal{P}(W_c)\to[0,1]$ be two mass functions on the canonical model. Let ϕ_X denote the maximal wDNF such that $|\phi_X|=X$ for any $X\subseteq W_c$. Let $X\odot Y$ denote the set $|\phi_X\odot\phi_Y|$. The aggregation $\mathtt{m}_{1\odot2}:\mathcal{P}(W_c)\to[0,1]$ is defined as follows.*

$$\mathtt{m}_{1\odot2}(Z)=\sum_{X,Y\in\mathcal{P}(W_c)\text{ s.t. }X\odot Y=Z}\mathtt{m}_1(X)\cdot\mathtt{m}_2(Y).\tag{5}$$

Example 1. Two disagreeing doctors. One gets the following aggregated mass function

$$\mathtt{m}_{1\odot 2}(X) = \begin{cases} 0.81 & \text{if } X = |\mathbf{B}a \wedge \mathbf{F}b \wedge \mathbf{B}c| \\ 0.09 & \text{if } X = |\mathbf{B}a \wedge \mathbf{B}b \wedge \mathbf{F}c| \\ & \text{or } X = |\mathbf{F}a \wedge \mathbf{B}b \wedge \mathbf{B}c| \\ 0.01 & \text{if } X = |\mathbf{F}a \wedge \mathbf{T}b \wedge \mathbf{F}c| \\ 0 & \text{otherwise.} \end{cases}$$

The focal elements of $\mathtt{m}_{1\odot 2}$ are singletons, therefore it induces a probability assignment p such that

ϕ	a	$\neg a$	b	$\neg b$	$a \wedge \neg a$	$b \wedge \neg b$
p	0.9	1	0.19	0.99	0.9	0.18

and $\mathrm{p}(\phi) = \mathtt{bel}_{1\odot 2}(|\mathbf{TB}\phi|) = \mathtt{pl}_{1\odot 2}(|\mathbf{TB}\phi|)$. Notice that the conclusions are very similar to Example 2. The mass function encodes the fact that the information is highly contradictory about a and c with $\mathrm{p}(|\mathbf{B}a|) = \mathrm{p}(\mathbf{B}c) = 0.9$ and that the evidence supporting b is rather weak and mostly non-contradictory: $\mathrm{p}(|\mathbf{T}b|) = 0.01$, $\mathrm{p}(|\mathbf{F}b|) = 0.81$, $\mathrm{p}(|\mathbf{B}b|) = 0.18$. Thus, unlike in the case of DS-rule where all the contradictory evidence is discarded, our combination rule describes the extent of contradictions between doctors about different diseases.

Example 2. Two disagreeing doctors. Here, the aggregation will be identical to DS-rule, since the mass functions do not contradict each other in the sense of BD.

Example 3. Reasoning with incomplete information. One gets the following aggregated mass function.

$$\mathtt{m}_{1\odot 2}(X) = \begin{cases} 0.81 & \text{if } X = |\mathbf{T}a \wedge \mathbf{F}b \wedge \mathbf{T}c| \\ 0.09 & \text{if } X = |\mathbf{T}a \wedge \mathbf{B}b \wedge \mathbf{F}c| \\ & \text{or } X = |\mathbf{F}a \wedge \mathbf{B}b \wedge \mathbf{T}c| \\ 0.01 & \text{if } X = |\mathbf{F}a \wedge \mathbf{T}b \wedge \mathbf{F}c| \\ 0 & \text{otherwise.} \end{cases}$$

The focal elements of $\mathtt{m}_{1\odot 2}$ are singletons, therefore it induces a probability assignment p such that

ϕ	a	$\neg a$	b	$\neg b$	$a \wedge \neg a$	$b \wedge \neg b$
p	0.9	0.1	0.19	0.99	0	0.18

the values for a (resp. $\neg a$, $a \wedge \neg a$) are the same as those for c (resp. $\neg c$, $c \wedge \neg c$). Here, the mass function encodes the fact that the available information strongly supports the truth of a and c. This result comes from the fact that $\sqcup$ tells us the nature of the available information: incomplete, contradictory, supporting the statement.

Thus, within our framework, the proposed combination rule $\odot$ has the following advantages over DS-rule. (1) It behaves more in accordance with intuition than DS-rule when dealing with pieces of evidence which cannot be encoded using BD-formulas as shown in Example 3. (2) It provides a more nuanced framework to pinpoint topics leading to conflict between classical sources and quantify the conflict on different topics rather than overall conflict as shown in Example 1.

5. What Would Be a 'Good Classical' Piece of Evidence?

In this section, we discuss what can be considered as a *good classical* piece of evidence for a statement ϕ in the paraconsistent framework, and therefore which notions - based on the evidence- of "belief" and "plausibility" would be the most pertinent to estimate a lower and upper bound on an unknown classical probability assignment p on the formulas.

Let $\mathfrak{M}_c = \langle W_c, v_4\rangle$ be the canonical model over $\mathtt{Prop}$. Let $\mathtt{m} : \mathcal{P}(W_c) \to [0,1]$ be a mass function and $\mathtt{bel}_\mathtt{m}$ (resp. $\mathtt{pl}_\mathtt{m}$) its associated belief (resp. plausibility) function (cf. Section 3). Consider a formula $\phi \in \mathscr{L}_{\mathsf{BD}}$.

T-support. A state $w \in W_c$ provides $\mathbf{T}$-*support* to ϕ, if $w \in |\phi|^{\mathbf{T}}$. That is, w supports the truth of ϕ, and does not support the falsity of ϕ. Notice that if $v_4(p,w) = \mathbf{N}$ and $v_4(q,w) = \mathbf{B}$, then $v_4(p \vee q, w) = \mathbf{T}$. Thus, in general $|\phi \vee \psi|^{\mathbf{T}} \neq |\phi|^{\mathbf{T}} \cup |\psi|^{\mathbf{T}}$. We define the following lower (resp. upper) bound $\mathtt{Bel}_{\mathbf{T}}(\phi)$ (resp. $\mathtt{Pl}_{\mathbf{T}}(\phi)$) on $\mathrm{p}(\phi)$:

$$\mathtt{Bel}_{\mathbf{T}}(\phi) := \mathtt{bel}_\mathtt{m}(|\phi|^{\mathbf{T}}) \text{ and } \mathtt{Pl}_{\mathbf{T}}(\phi) := 1 - \mathtt{bel}_\mathtt{m}(|\phi|^{\mathbf{F}}).$$

Since $\phi \models_{\mathsf{BD}} \psi$ implies that $|\phi|^{\mathbf{T}} \subseteq |\psi|^{\mathbf{T}}$, $\mathtt{Bel}_{\mathbf{T}}$ and $\mathtt{Pl}_{\mathbf{T}}$ are monotone w.r.t. $\models_{\mathsf{BD}}$. In addition, we have $|\phi|^{\mathbf{T}} \cup |\psi|^{\mathbf{T}} \subseteq |\phi \vee \psi|^{\mathbf{T}}$ and $|\phi|^{\mathbf{T}} \cap |\psi|^{\mathbf{T}} = |\phi \wedge \psi|^{\mathbf{T}}$. Thus, for every $n \in \mathbb{N}$, for every $\phi_1, \ldots, \phi_n \in \mathscr{L}_{\mathsf{BD}}$,

$$\mathtt{Bel}_{\mathbf{T}}(\bigvee_{1\leq i\leq n} \phi_i) = \mathtt{bel}_\mathtt{m}(|\bigvee_{1\leq i\leq n} \phi_i|^{\mathbf{T}}) \geq \mathtt{bel}_\mathtt{m}(\bigcup_{1\leq i\leq n} |\phi_i|^{\mathbf{T}}),$$
$$\mathtt{Bel}_{\mathbf{T}}(\bigwedge_{1\leq i\leq n} \phi_i) = \mathtt{bel}_\mathtt{m}(|\bigwedge_{1\leq i\leq n} \phi_i|^{\mathbf{T}}) = \mathtt{bel}_\mathtt{m}(\bigcap_{i\in I} |\phi_i|^{\mathbf{T}}).$$

Moreover, $\mathtt{Bel}_{\mathbf{T}}(\perp) = 0$ and $\mathtt{Bel}_{\mathbf{T}}(\top) = 1$. Hence, $\mathtt{Bel}_{\mathbf{T}}$ is a belief assignment over BD. In addition, we have $1 - \mathtt{bel}_\mathtt{m}(|\phi|^{\mathbf{F}}) = 1 - \mathtt{bel}_\mathtt{m}(|\neg\phi|^{\mathbf{T}}) = 1 - \mathtt{Bel}_{\mathbf{T}}(\neg\phi)$. As, $\neg$ is a De Morgan negation, $\mathtt{Pl}_{\mathbf{T}}$ is a normal plausibility function [2, Lemma 2.14]. Thus, $\mathtt{Bel}_{\mathbf{T}}$ and $\mathtt{Pl}_{\mathbf{T}}$ are respectively normal belief and plausibility assignments. Note that since $\mathtt{Bel}_{\mathbf{T}} \leq \mathtt{Bel}_{\mathbf{TB}} \leq \mathtt{Pl}_{\mathbf{TB}} \leq \mathtt{Pl}_{\mathbf{T}}$, every probability assignment $\mathrm{p} \in \mathcal{F}_{\mathbf{TB}}$ satisfies $\mathtt{Bel}_{\mathbf{T}} \leq \mathrm{p} \leq \mathtt{Pl}_{\mathbf{T}}$. Thus, $\mathtt{Bel}_{\mathbf{T}}$ and $\mathtt{Pl}_{\mathbf{T}}$ define a non-empty credal set of probabilities.

This notion of support in a formula ϕ does not require a piece of evidence to be consistent about every subformula of ϕ, it simply requires for the evidence to support ϕ without supporting its negation.

Support based on classical 'proofs'. Let $Q_w := \{l \in \mathtt{Lit} \mid v_4(l,w) = \mathbf{T}\}$. A state $w \in W$ supports ϕ if it provides a classical 'proof' of ϕ, i.e. if $\left(\bigwedge_{l\in Q_w} l\right) \models_{\mathsf{CL}} \phi$. For any formula ϕ, let $|\phi|^{\mathbf{CP}}$ be the set of states which provide a classical 'proof' of ϕ. We obtain the following lower bound $\mathtt{Bel}_{\mathbf{CP}}(\phi)$ on $\mathrm{p}(\phi)$:

$$\mathtt{Bel}_{\mathbf{CP}}(\phi) = \mathtt{bel}_\mathtt{m}(|\phi|^{\mathbf{CP}}).$$

The associated plausibility function is defined by $\mathtt{Pl}_{\mathbf{CP}}(\phi) = 1-\mathtt{Bel}_{\mathbf{CP}}(|\neg\phi|^{\mathbf{CP}})$. By the properties of classical entailment we have $|\phi|^{\mathbf{CP}} \cup |\psi|^{\mathbf{CP}} \subseteq |\phi \vee \psi|^{\mathbf{CP}}$ and $|\phi \wedge \psi|^{\mathbf{CP}} = |\phi|^{\mathbf{CP}} \cap |\psi|^{\mathbf{CP}}$. This implies that $\mathtt{Bel}_{\mathbf{CP}}$ satisfies equation (1). As $\mathtt{Pl}_{\mathbf{CP}}(\phi) = 1 - \mathtt{bel}_{\mathtt{m}}(|\neg\phi|^{\mathbf{CP}})$, $\mathtt{pl}_{\mathbf{CP}}$ satisfies equation (2). Moreover, since $\bot$ and $\top$ are falsum and tautology, we have $\mathtt{Bel}_{\mathbf{CP}}(\bot) = 0$ and $\mathtt{Bel}_{\mathbf{CP}}(\top) = 1$. Thus, $\mathtt{Bel}_{\mathbf{CP}}$ and $\mathtt{Pl}_{\mathbf{CP}}$ are belief and plausibility assignments.

This notion of support is natural when the analyst only considers evidence as support for ϕ if the part of the evidence which is classical entails ϕ.

Remark 14 *Note that for any propositional variable p, we have $|p|^{\mathbf{CP}} = |p|^{\mathbf{T}}$. Thus, we have $\mathtt{Bel}_{\mathbf{CP}}(p) = \mathtt{Bel}_{\mathbf{T}}(p)$. Moreover, for any formula ϕ, if a state w provides a classical 'proof' of ϕ, then ϕ must be exactly true at w. Therefore, $\mathtt{Bel}_{\mathbf{CP}} \leq \mathtt{Bel}_{\mathbf{T}}$.*

Support from incomplete (resp. classical) states. Here, for the support of ϕ, we consider only incomplete (resp. classical) states. Let $\mathbf{IC} \subseteq W_c$ (resp. $\mathbf{C} \subseteq W_c$), be the set of incomplete (resp. classical) states. These notions of support induce respectively the following lower bounds on $\mathrm{p}(\phi)$,

$$\mathtt{Bel}_{\mathbf{IC}}(\phi) := \mathtt{bel}_{\mathtt{m}}(|\phi|^{\mathbf{T}} \cap \mathbf{IC}),$$

and

$$\mathtt{Bel}_{\mathbf{C}}(\phi) := \mathtt{bel}_{\mathtt{m}}(|\phi|^{\mathbf{T}} \cap \mathbf{C}).$$

The associated upper bounds are given by $\mathtt{Pl}_{\mathbf{IC}}(\phi) = 1 - \mathtt{bel}_{\mathtt{m}}(|\phi|^{\mathbf{F}} \cap \mathbf{IC})$ and $\mathtt{Pl}_{\mathbf{C}}(\phi) = 1 - \mathtt{bel}_{\mathtt{m}}(|\phi|^{\mathbf{F}} \cap \mathbf{C})$. As $\mathtt{Bel}_{\mathbf{T}}$ and $\mathtt{Pl}_{\mathbf{T}}$ satisfy equations (1) and (2) respectively, $\mathtt{Bel}_{\mathbf{IC}}$ (resp. $\mathtt{Bel}_{\mathbf{C}}$) and $\mathtt{Pl}_{\mathbf{IC}}$ (resp. $\mathtt{Pl}_{\mathbf{C}}$) must also satisfy these equations. Therefore, $\mathtt{Bel}_{\mathbf{IC}}$ (resp. $\mathtt{Bel}_{\mathbf{C}}$) and $\mathtt{Pl}_{\mathbf{IC}}$ (resp. $\mathtt{Pl}_{\mathbf{C}}$) are belief and plausibility assignments.

However, since we only consider incomplete (resp. classical) states when we calculate $\mathtt{Bel}_{\mathbf{IC}}$ (resp. $\mathtt{Bel}_{\mathbf{C}}$), $\mathtt{Bel}_{\mathbf{IC}}(\top)$ (resp. $\mathtt{Bel}_{\mathbf{C}}(\top)$) need not be 1. Thus, in general, $\mathtt{Bel}_{\mathbf{IC}}$ and $\mathtt{Bel}_{\mathbf{C}}$ are non-normal belief functions.

This notion of support is natural when the analyst wants to restrict themselves to the incomplete or classical states. That is, they ignore completely any evidence which is contradictory in evaluating belief and plausibility.

Comparison of the different supports.

Proposition 15 *For any mass function $\mathtt{m}$ and any $\phi \in \mathscr{L}_{\mathsf{BD}}$, we have the following chain of inclusions.*

$$\begin{aligned}&[\mathtt{Bel}_{\mathsf{BD}}(\phi), \mathtt{Pl}_{\mathsf{BD}}(\phi)] \subseteq [\mathtt{Bel}_{\mathbf{T}}(\phi), \mathtt{Pl}_{\mathbf{T}}(\phi)]\\ &\subseteq [\mathtt{Bel}_{\mathbf{CP}}(\phi), \mathtt{Pl}_{\mathbf{CP}}(\phi)] \subseteq [\mathtt{Bel}_{\mathbf{IC}}(\phi), \mathtt{Pl}_{\mathbf{IC}}(\phi)]\\ &\subseteq [\mathtt{Bel}_{\mathbf{C}}(\phi), \mathtt{Pl}_{\mathbf{C}}(\phi)].\end{aligned}$$

For any formula ϕ, the interval $[\mathtt{Bel}(\phi), \mathtt{Pl}(\phi)]$ describes the uncertainty in probability of ϕ. Thus, intuitively, this proposition corresponds to the fact that the uncertainty in the formula increases as the requirement for admittance of evidence as support increases. That is, if more information is ignored by the analyst (due to stronger requirements) the uncertainty in the probability of any formula ϕ increases.

6. Lower and Upper Bounds for Classical Probabilities

In this section, we show that $\mathtt{Bel}_{\mathbf{CP}}$, $\mathtt{Bel}_{\mathbf{IC}}$ and $\mathtt{Bel}_{\mathbf{C}}$ induce credal sets of classical probabilities on CL Lindembaum algebra, and that, in general, it is not possible to find classical probabilities coherent with $\mathtt{Bel}_{\mathbf{T}}$.

Let $\mathbb{B}$ be the CL Lindenbaum algebra over $\mathscr{L}_{\mathsf{BD}}$ (Definition 2). Since, $\mathtt{Prop}$ is finite, $\mathbb{B}$ is finite too. Let $\mathtt{m}$ be a normal mass function on the canonical model, $\mathbf{X} \in \{\mathbf{CP}, \mathbf{C}, \mathbf{IC}\}$, and $\mathtt{Bel}_{\mathbf{X}}$ and $\mathtt{Pl}_{\mathbf{X}}$ be the belief and plausibility functions on $\mathscr{L}_{\mathsf{BD}}$ corresponding to $\mathbf{X}$-support. Let $\mathtt{bel}_{\mathbf{X}}, \mathtt{pl}_{\mathbf{X}} : \mathbb{B} \to [0,1]$ be as follows: for any $[\phi] \in \mathbb{B}$,

$$\mathtt{bel}_{\mathbf{X}}([\phi]) = \mathtt{Bel}_{\mathbf{X}}(\hat{\bigvee}[\phi]), \text{ and } \mathtt{pl}([\phi]) = \mathtt{Pl}_{\mathbf{X}}(\hat{\bigwedge}[\phi]),$$

with $\hat{\bigvee}[\phi] := \bigvee\{\psi \in \mathscr{L}_{\mathsf{BD}} \mid \psi \in [\phi] \text{ and } \psi \text{ is in iDNF}\}$, and $\hat{\bigwedge}[\phi] := \bigwedge\{\psi \in \mathscr{L}_{\mathsf{BD}} \mid \psi \in [\phi] \text{ and } \psi \text{ is in iDNF}\}$. $[\phi] \subseteq \mathscr{L}_{\mathsf{BD}}$ is an infinite set of formulas, but each formula in BD is equivalent to a formula in iDNF and there are finitely many formulas in iDNF. Therefore, $\hat{\bigvee}[\phi]$ is well-defined.

Proposition 16 *The following equalities hold:*

$$\begin{aligned}&[\hat{\bigvee}[\phi]] = [\phi] \qquad\qquad [\hat{\bigwedge}[\phi]] = [\phi]\\ &[\hat{\bigvee}[\phi \vee \psi]] = [\hat{\bigvee}([\phi] \vee [\psi])] = [\hat{\bigvee}[\phi] \vee \hat{\bigvee}[\psi]]\\ &[\hat{\bigvee}[\phi \wedge \psi]] = [\hat{\bigvee}([\phi] \wedge [\psi])] = [\hat{\bigvee}[\phi] \wedge \hat{\bigvee}[\psi]]\\ &[\hat{\bigwedge}[\phi \vee \psi]] = [\hat{\bigwedge}([\phi] \vee [\psi])] = [\hat{\bigwedge}[\phi] \vee \hat{\bigwedge}[\psi]]\\ &[\hat{\bigwedge}[\phi \wedge \psi]] = [\hat{\bigwedge}([\phi] \wedge [\psi])] = [\hat{\bigwedge}[\phi] \wedge \hat{\bigwedge}[\psi]]\\ &[\neg\hat{\bigwedge}[\phi]] = [\hat{\bigvee}\neg[\phi]] = [\hat{\bigvee}[\neg\phi]]\\ &[\neg\hat{\bigvee}[\phi]] = [\hat{\bigwedge}\neg[\phi]] = [\hat{\bigwedge}[\neg\phi]]\end{aligned}$$

Theorem 17 *Let* CM *be the set of classical probability measures on the finite Boolean algebra $\mathbb{B}$, let $\mathbb{B}^* = \mathbb{B} \smallsetminus \{\top_{\mathbb{B}}\}$, and for $\mathbf{X} \in \{\mathbf{IC}, \mathbf{C}, \mathbf{CP}\}$,*

$$\mathcal{F}_{\mathbf{X}} := \{\mu \in \mathrm{CM} \mid \forall a \in \mathbb{B},\ \mathtt{bel}_{\mathbf{X}}(a) \leq \mu(a)\}.$$

For $\mathbf{X} \in \{\mathbf{IC}, \mathbf{C}\}$ (resp. $\mathbf{X} = \mathbf{CP}$), $\mathtt{bel}_{\mathbf{X}}$ and $\mathtt{pl}_{\mathbf{X}}$ are non-normal (resp. normal) belief and plausibility functions on $\mathbb{B}$, and, for $a \in \mathbb{B}^*$ (resp. $a \in \mathbb{B}$), $\mathtt{bel}_{\mathbf{X}}(a)$ and $\mathtt{pl}_{\mathbf{X}}(a)$ provide optimal lower and upper bounds on $\mathcal{F}_{\mathbf{X}}$. In addition,

$$\varnothing \subsetneq \mathcal{F}_{\mathbf{CP}} \subseteq \mathcal{F}_{\mathbf{IC}} \subseteq \mathcal{F}_{\mathbf{C}}$$

Proof Let $\phi, \phi' \in \mathscr{L}_{\mathsf{BD}}$. Note that for every $\phi' \in [\phi]$, we have $\mathtt{Bel}_{\mathbf{X}}(\phi') \leq \mathtt{bel}_{\mathbf{X}}([\phi])$ and $\mathtt{pl}_{\mathbf{X}}([\phi]) \leq \mathtt{Pl}_{\mathbf{X}}(\phi')$. Since, $\mathtt{Bel}_{\mathbf{X}}$ and $\mathtt{Pl}_{\mathbf{X}}$ are belief and plausibility assignments, Proposition 16 implies that $\mathtt{bel}_{\mathbf{X}}$ and $\mathtt{pl}_{\mathbf{X}}$ are monotone and satisfy equations (1) and (2) respectively. In addition, for $\phi \in \mathscr{L}_{\mathsf{BD}}$, we have

$$\mathtt{pl}_{\mathbf{X}}([\phi]) = \mathtt{Pl}_{\mathbf{X}}(\hat{\bigwedge}[\phi]) = 1 - \mathtt{Bel}_{\mathbf{X}}(\neg\hat{\bigwedge}[\phi]) = 1 - \mathtt{Bel}_{\mathbf{X}}(\hat{\bigvee}[\neg\phi]) = 1 - \mathtt{bel}_{\mathbf{X}}(\neg[\phi]).$$

Moreover, $\phi \in [\bot]$ implies $\mathtt{Bel}_{\mathbf{X}}(\phi) = 0$, because $|\phi|^{\mathbf{T}} \cap \mathbf{IC} = |\phi|^{\mathbf{T}} \cap \mathbf{C} = \varnothing$ and $\left(\bigwedge_{l \in Q_w} l\right) \not\models_{\mathsf{CL}} \bot$ for every $w \in W_c$. In addition, $\mathtt{bel}_{\mathbf{CP}}([\top]) = 1$, because $\varnothing \models_{\mathsf{CL}} \top$. Therefore, $\mathtt{bel}_{\mathbf{CP}}$ and $\mathtt{pl}_{\mathbf{CP}}$ are normal belief and plausibility functions. However, for $\mathbf{X} \in \{\mathbf{IC}, \mathbf{C}\}$, $\mathtt{bel}_{\mathbf{X}}$ and $\mathtt{pl}_{\mathbf{X}}$ are non-normal belief and plausibility functions. Indeed, $\mathtt{bel}_{\mathbf{X}}([\top])$ and $\mathtt{pl}_{\mathbf{X}}([\bot])$ can be different than 1 and 0 respectively, because $\mathtt{bel}_{\mathbf{X}}([\top]) = \mathtt{bel}_{\mathtt{m}}(\mathbf{X}) \neq \mathtt{bel}_{\mathtt{m}}(W_c)$.

For $\mathbf{X} = \mathbf{CP}$, since $\mathtt{bel}_{\mathbf{CP}}$ is a normal belief function on a Boolean algebra, and $\mathtt{pl}_{\mathbf{X}}$ is its associated plausibility function, $\mathcal{F}_{\mathbf{X}}$ is a non-empty credal set and $\mathtt{bel}_{\mathbf{X}}$ and $\mathtt{pl}_{\mathbf{X}}$ are its lower and upper probabilities. Now, let us consider the case $\mathbf{X} \in \{\mathbf{IC}, \mathbf{C}\}$. Let $\mu \in \mathcal{F}_{\mathbf{X}}$. If $\mathtt{bel}_{\mathbf{X}}(a) \leq \mu(a)$ for every $a \in \mathbb{B}$, then, for every $a \in \mathbb{B}$, we have

$$\mu(\neg a) = 1 - \mu(a) \leq 1 - \mathtt{bel}_{\mathbf{X}}(a) = \mathtt{pl}_{\mathbf{X}}(\neg a).$$

Thus, $\mathcal{F}_{\mathbf{X}}$ is the set of classical probability measures lying between $\mathtt{bel}_{\mathbf{X}}$ and $\mathtt{pl}_{\mathbf{X}}$ for every $a \in \mathbb{B}$.

We can describe $\mathcal{F}_{\mathbf{X}}$ constructively as follows. Let $\mathtt{m}_{\mathtt{bel}_{\mathbf{X}}} : \mathbb{B} \to [0,1]$ be defined as follows: for $a \in \mathbb{B}$,

$$\mathtt{m}_{\mathtt{bel}_{\mathbf{X}}}(a) = \mathtt{bel}_{\mathbf{X}}(a) - \sum_{b<a} \mathtt{m}_{\mathtt{bel}_{\mathbf{X}}}(b).$$

We have $\sum_{a \in \mathbb{B}} \mathtt{m}_{\mathtt{bel}_{\mathbf{X}}}(a) = \mathtt{bel}_{\mathbf{X}}(\top_{\mathbb{B}}) = \mathtt{bel}_{\mathbf{X}}([\top]) \leq 1$. Let At be the set of atoms of $\mathbb{B}$, $k = |\mathrm{At}|$, and, for every $a \in \mathbb{B}$, let $\mathrm{At}(a)$ be the set of atoms below a and $k_a = |\mathrm{At}(a)|$. Let $\mathcal{M}_{\mathbf{X}}$ be the set of mass functions $\mathtt{m} : \mathrm{At} \to [0,1]$ that can be constructed as follows.

1. Set $\mathtt{m}(a) = 0$ for all $a \in \mathbb{B}$.

2. Let $\{\alpha_i\}_{i \in [1,k]}$ be a family of real numbers in $[0,1]$ such that $\sum_{1 \leq i \leq k} \alpha_i = 1$. For $a_i \in \mathrm{At}$,

$$\mathtt{m}(a_i) \leftarrow \mathtt{m}(a_i) + \alpha_i(1 - \mathtt{bel}_{\mathbf{X}}(\top_{\mathbb{B}})).$$

3. Let $\{\beta_{i,a}\}_{i \in [1,k_a]}$ be a family of real numbers in $[0,1]$ such that $\sum_{1 \leq i \leq k_a} \beta_{i,a} = 1$. For $a_i \in \mathrm{At}(a)$,

$$\mathtt{m}(a_i) \leftarrow \mathtt{m}(a_i) + \beta_{i,a}\mathtt{m}_{\mathtt{bel}_{\mathbf{X}}}(a).$$

Notice that $\sum_{b \leq a} \mathtt{m}_{\mathtt{bel}_{\mathbf{X}}}(b) \leq \sum_{b \in \mathrm{At}(a)} \mathtt{m}(b)$. Thus, the probability measure $\mu_{\mathtt{m}}$ induced by $\mathtt{m}$ satisfies $\mathtt{bel}_{\mathbf{X}} \leq \mu_{\mathtt{m}}$ and $\mu_{\mathtt{m}} \in \mathcal{F}_{\mathbf{X}}$. On the other hand, if $\mu \in \mathcal{F}_{\mathbf{X}}$, its associated mass function $\mathtt{m}_\mu$ satisfies: for any $a \in \mathbb{B}$, $\sum_{b \leq a} \mathtt{m}_{\mathtt{bel}}(b) \leq \sum_{b \in \mathrm{At}(a)} \mathtt{m}_\mu(b)$ and $\sum_{a \in \mathrm{At}} \mathtt{m}_\mu(a) = 1$. Thus, $\mathtt{m}_\mu$ can be constructed by the above method with the right choice of weights (αs and βs). Therefore, $\mathcal{F}_{\mathbf{X}}$ is the set of probability measures induced by the mass functions in $\mathcal{M}_{\mathbf{X}}$. ■

In the case of $\mathtt{bel}_{\mathbf{T}}$, Theorem 17 does not hold in general. Consider a mass function $\mathtt{m}$ such that $\mathtt{bel}_{\mathtt{m}}(\{w \mid w \in |p|^{\mathbf{B}} \cap |q|^{\mathbf{N}}\}) = 1$. Then the formula $(p \wedge \neg p) \vee (q \wedge \neg q)$ is classically equivalent to a contradiction, but we have $\mathtt{bel}_{\mathbf{T}}((p \wedge \neg p) \vee (q \wedge \neg q)) = 1$. As $(p \wedge \neg p) \vee (q \wedge \neg q) \in [\bot]$, if some probability measure μ is above it, then we have $\mathtt{bel}_{\mathbf{T}}((p \wedge \neg p) \vee (q \wedge \neg q)) = 1 \leq \mu[\bot]$ which is not possible for a classical probability measure. Therefore $\mathbf{T}$-support is not a notion of supporting evidence adequate to deduce classical probabilities from evidence formalised over a BD-model.

7. Conclusion

In this paper, we provide a framework for applying DS-theory to sources that might be (self) contradictory. We describe how to encode different kinds of evidence and show via examples that DS-rule may behave unintuitively. We propose a variation of DS-rule that uses the algebraic structure of the truth values to describe more finely the available information and its nature (incomplete, consistent, contradictory). Finally, we show how one can extract a credal set of classical probability measures consistent with the evidence.

Note that the Dempster-Shafer rule in Belnap Dunn framework and aggregation rule based on information lattice defined by us involve asymptotically same number of operations as the classical Dempster-Shafer rule. Also, note that other common operations like obtaining mass function corresponding to a belief function and vice versa, calculating plausibility transform, etc. are of same computational complexity as the classical Dempster-Shafer theory w.r.t to the size of the underlying algebra. However, the underlying algebra is much bigger in the case of BD. Indeed the size of the complex algebra of the canonical model of Belnap Dunn logic with n propositions is 2^{4^n}, while in classical logic the complex algebra of the canonical model is of size 2^{2^n}.

In the future, we intend to study the behaviour of more existing aggregation rules in the BD framework in order to understand exactly what kind of situations they can be applied to and how it compares to the classical framework. In addition, the semantics of BD offers more possibilities to define aggregation rules via new logical connectives. We wish to understand their meaning and mathematical properties. In addition, we wish to investigate what can be formalised within the open-world context.

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