# A Generalized Notion of Conjunction for Two Conditional Events

**Lydia Castronovo** LYDIA.CASTRONOVO@UNIPA.IT
**Giuseppe Sanfilippo** GIUSEPPE.SANFILIPPO@UNIPA.IT
*Department of Mathematics and Computer Science, University of Palermo, Palermo, Italy*

## Abstract

Traditionally the conjunction of conditional events has been defined as a three-valued object. However, in this way classical logical and probabilistic properties are not preserved. In recent literature, a notion of conjunction of two conditional events as a suitable conditional random quantity, which satisfies classical probabilistic properties, has been deepened in the setting of coherence. In this framework the conjunction $(A|H) \wedge (B|K)$ of two conditional events $A|H$ and $B|K$ is defined as a five-valued object with set of possible values $\{1, 0, x, y, z\}$, where $x = P(A|H)$, $y = P(B|K)$, and $z = \mathbb{P}[(A|H) \wedge (B|K)]$. In this paper we propose a generalization of this object, denoted by $(A|H) \wedge_{a,b} (B|K)$, where the values $x$ and $y$ are replaced by two arbitrary values $a, b \in [0, 1]$. Then, by means of a geometrical approach, we compute the set of all coherent assessments on the family $\{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$, by also showing that in the general case the Fréchet-Hoeffding bounds for the conjunction are not satisfied. We also analyze some particular cases. Finally, we study coherence in the imprecise case of an interval-valued probability assessment and we consider further aspects on $(A|H) \wedge_{a,b} (B|K)$.

**Keywords:** coherence, conjunction, conditional events, conditional random quantity, prevision, imprecise probability, Fréchet-Hoeffding bounds, quasi conjunction

## 1. Introduction

Conditionals have been largely studied in many fields (see, e.g., [1, 9, 10, 11, 13, 14, 15, 17, 20, 36, 37, 39, 40, 41, 42, 43, 46, 51, 53]). Bruno de Finetti introduced conditional events as tri-events and proposed a three-valued logic ([12]). Traditionally logical operations among conditionals or conditional events have been defined, and deeply analyzed, in the context of three-valued logic (see, e.g., [1, 3, 4, 7, 12, 16, 34]). However, these approaches lead to some problems in preserving classical logical and probabilistic properties, for example the Fréchet-Hoeffding bounds for the conjunction are not satisfied ([6, 23, 50]). In recent literature, the study of conjunction and disjunction of conditionals, seen as a suitable conditional random quantities which satisfy classical probabilistic properties, has been deepened in the setting of coherence ([26, 27, 28, 29, 31]). We recall that the notions of conjoined and disjoined conditionals in terms of conditional random quantities involve the conditional probabilities of the basic conditional events. The three-valued logic view of conditional events separates the conditional from its probability. More in general, the Boolean approach of conditionals given in ([17]) is a super-structure that contains the three-valued algebra as a substructure. In the setting of Boolean algebras of conditionals ([17]) a general theory of compound conditionals as suitable conditional random quantities has been developed in [18]. The probabilistic properties on conjunction and disjunction of conditionals in the conditional Boolean algebras are consistent with the results obtained in the field of conditional random quantities under coherence ([19]). Based on the notion of conjunction, a suitable notion of iterated conditionals has been defined ([24]). For some applications of iterated conditionals see e.g. [30, 47, 48, 51, 52]. Given two conditional events $A|H$ and $B|K$, their conjunction $(A|H) \wedge (B|K)$ is defined as the five-valued random quantity $(A|H) \wedge (B|K) = [AHBK + x\overline{H}BK + yAH\overline{K}]|(H \vee K) \in [0, 1]$, where $x = P(A|H)$, $y = P(B|K)$. In this paper we propose a generalization of this object, denoted by $(A|H) \wedge_{a,b} (B|K)$, that, instead of being dependent on the probabilities $x$ and $y$ of the two conditional events, is defined by means of two arbitrary values $a, b \in [0, 1]$, which may be related to $x$ and $y$. The values $a$ and $b$ represent the numerical counterpart of the value 'partially' true of the conjunction, which is obtained when a conjunct is true and the other is void ([5]). Moreover, by exploiting a geometrical approach, we compute the set of all coherent assessments on the family $\{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$, by also showing that in the general case the Fréchet-Hoeffding bounds for the conjunction are not satisfied. We analyze some particular cases and we consider interval-valued prevision assessments.

The paper is organized as follows. After briefly recalling the basics on coherence and on the conjunction of conditional events (Section 2), in Section 3 we give the definition of generalized conjunction of conditional events, $(A|H) \wedge_{a,b} (B|K)$ and we present the main result which characterizes the set of all coherent prevision assessments on the family $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$. In Section 4 we analyze the particular case where $HK = \emptyset$ and the case where the generalized conjunction reduces to some conditional events. Then, in Section 5 we study the coherence of some interval-valued prevision assessments. We deepen further aspects on the generalized conjunction in Section 6. Finally, in Section 7, we conclude and we point out future work.

## 2. Preliminary Notions and Results

We use the same symbol to refer to an event, that is a two-valued logical entity which can be *true*, or *false*, and its indicator, which can be 1, or 0. We denote by $\Omega$ and $\emptyset$, the sure event and the impossible one, respectively. The negation of an event $A$ is denoted by $\overline{A}$. We denote by $A \wedge B$ (resp., $A \vee B$), or simply by $AB$, the conjunction (resp., disjunction) of $A$ and $B$. When, an event $A$ logically implies an event $B$, i.e., $A\overline{B} = \emptyset$, we write $A \subseteq B$. We say that $n$ events $E_1, \dots, E_n$ are logically independent when there are no logical relations among them. Given two events $A$ and $H$, with $H \neq \emptyset$, the conditional event $A|H$ is a three-valued logical entity which is *true*, or *false*, or *void*, according to whether $AH$ is true, or $\overline{A}H$ is true, or $\overline{H}$ is true, respectively. The negation $\overline{A|H}$ of a conditional event $A|H$ is defined as $\overline{A|H} = \overline{A}|H$. We recall the relation of logical implication (also called Goodman-Nguyen inclusion relation) between two conditional events ([33], see also [44]).

**Definition 1** *Given two conditional events $A|H$ and $B|K$, we say that $A|H$ logically implies $B|K$, denoted by $A|H \subseteq B|K$ if and only if $AH \subseteq BK$ and $\overline{B}K \subseteq \overline{A}H$.*

**Conditional Prevision and Coherence.** We denote by $X$ *random quantity* and by $\mathbb{P}(X)$ its prevision. In the betting scheme, given any event $H \neq \emptyset$, agreeing to the betting metaphor, if you assess that $\mathbb{P}(X|H)$, is equal to $\mu$, this means that for any given real number $s$ you are willing to pay an amount $s\mu$ and to receive $sX$, or $s\mu$, according to whether $H$ is true, or false (bet called off), respectively. In particular, when $X$ is (the indicator of) an event $A$, then $\mathbb{P}(X|H) = P(A|H)$, where $P(A|H)$ is the conditional probability on $A|H$. As we will see the conjunction of two conditional events is a conditional random quantity with a finite number of possible (numerical) values. Then, in what follows, for any given conditional random quantity $X|H$, we assume that, when $H$ is true, the set of possible values of $X$ is a finite set of real numbers. In this case we say that $X|H$ is a finite conditional random quantity. Given a prevision function $\mathbb{P}$ defined on an arbitrary family $\mathcal{K}$ of finite conditional random quantities, consider a finite subfamily $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\} \subseteq \mathcal{K}$ and the vector $\mathcal{M} = (\mu_1, \dots, \mu_n)$, where $\mu_i = \mathbb{P}(X_i|H_i)$ is the assessed prevision for the conditional random quantity $X_i|H_i$, $i \in \{1, \dots, n\}$. With the pair $(\mathcal{F}, \mathcal{M})$ we associate the random gain $G = \sum_{i=1}^{n} s_i H_i(X_i - \mu_i) = \sum_{i=1}^{n} s_i(X_i|H_i - \mu_i)$. We denote by $\mathcal{G}_{\mathcal{H}_n}$ the set of values of $G$ restricted to $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$.

**Definition 2** The function $\mathbb{P}$ defined on $\mathcal{K}$ is coherent if and only if, $\forall n \geq 1, \forall s_1, \dots, s_n, \forall \mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\} \subseteq \mathcal{K}$, it holds that: $\min\ \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max\ \mathcal{G}_{\mathcal{H}_n}$.

In other words, $\mathbb{P}$ on $\mathcal{K}$ is incoherent if and only if there exists a finite combination of $n$ bets such that, after discarding the case where all the bets are called off, the values of the random gain are all positive or all negative. In the particular case where $\mathcal{K}$ is a family of conditional events, then Definition 2 becomes the well known definition of coherence for a conditional probability function, denoted as $P$. Notice that, in the general case where the conditional random quantities in $\mathcal{K}$ are bounded but possibly infinitely valued, the condition $\min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n}$ in Definition 2 becomes $\inf \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \sup \mathcal{G}_{\mathcal{H}_n}$.

**Remark 3** *By Definition 2, given any (finite) conditional random quantity $X|H$ and denoting by $x_1, \dots, x_r$, the possible values of $X$ when $H$ is true, and let $\mu$ be a prevision assessment on $X|H$. Then, coherence requires that $\mu \in [\min\{x_1, \dots, x_r\}, \max\{x_1, \dots, x_r\}]$. Indeed, if you pay $\mu < \min\{x_1, \dots, x_r\}$ in a bet on $X|H$, it holds that $X - \mu > 0$ and hence $\min \mathcal{G}_H > 0$. Likewise, if you pay $\mu > \max\{x_1, \dots, x_r\}$ in a bet on $X|H$, it holds that $X - \mu < 0$ and hence $\max \mathcal{G}_H < 0$. Thus, $X|H = XH + \mu\overline{H} \in [\min\{x_1, \dots, x_r\}, \max\{x_1, \dots, x_r\}]$.*

**Geometrical Characterization of Coherence.** Given a family $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\}$, for each $i \in \{1, \dots, n\}$ we denote by $\{x_{i1}, \dots, x_{ir_i}\}$ the set of possible values of $X_i$ when $H_i$ is true; then, we set $A_{ij} = (X_i = x_{ij})$, $i = 1, \dots, n$, $j = 1, \dots, r_i$. We set $C_0 = \overline{H}_1 \cdots \overline{H}_n$ (it may be $C_0 = \emptyset$) and we denote by $C_1, \dots, C_m$ the constituents contained in $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$. Hence $\bigwedge_{i=1}^{n}(A_{i1} \vee \cdots \vee A_{ir_i} \vee \overline{H}_i) = \bigvee_{h=0}^{m} C_h$. With each $C_h$, $h \in \{1, \dots, m\}$, we associate a vector $Q_h = (q_{h1}, \dots, q_{hn})$, where $q_{hi} = x_{ij}$ if $C_h \subseteq A_{ij}$, $j = 1, \dots, r_i$, while $q_{hi} = \mu_i$ if $C_h \subseteq \overline{H}_i$; with $C_0$ we associate $Q_0 = \mathcal{M} = (\mu_1, \dots, \mu_n)$. Denoting by $\mathcal{I}$ the convex hull of $Q_1, \dots, Q_m$, the condition $\mathcal{M} \in \mathcal{I}$ amounts to the existence of a vector $(\lambda_1, \dots, \lambda_m)$ such that: $\sum_{h=1}^{m} \lambda_h Q_h = \mathcal{M}$, $\sum_{h=1}^{m} \lambda_h = 1$, $\lambda_h \geq 0$, $\forall h$; in other words, $\mathcal{M} \in \mathcal{I}$ is equivalent to the solvability of the system $(\Sigma)$, associated with $(\mathcal{F}, \mathcal{M})$,

$$(\Sigma) \quad \begin{cases} \sum_{h=1}^{m} \lambda_h q_{hi} = \mu_i \,, \ i \in \{1, \dots, n\} \,, \\ \sum_{h=1}^{m} \lambda_h = 1, \ \ \lambda_h \geq 0 \,, \ \ h \in \{1, \dots, m\} \,. \end{cases} \tag{1}$$

Given the assessment $\mathcal{M} = (\mu_1, \dots, \mu_n)$ on $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\}$, let $S$ be the set of solutions $\Lambda = (\lambda_1, \dots, \lambda_m)$ of system $(\Sigma)$. We point out that the solvability of system $(\Sigma)$ is a necessary (but not sufficient) condition for coherence of $\mathcal{M}$ on $\mathcal{F}$. When $(\Sigma)$ is solvable, that is $S \neq \emptyset$, we define:

$$\begin{array}{l} \Phi_i(\Lambda) = \Phi_j(\lambda_1, \ldots, \lambda_m) = \sum_{r:C_r \subseteq H_i} \lambda_r, \; ; \; \Lambda \in S, \\ M_i = \max_{\Lambda \in S}, \Phi_i(\Lambda), \; i \in \{1, \ldots, n\}; \\ I_0 = \{i \, : \, M_i = 0\}, \mathcal{F}_0 = \{X_i|H_i, i \in I_0\}, \\ \mathcal{M}_0 = (\mu_i, \, i \in I_0). \end{array} \tag{2}$$

Then, the following theorem can be proved ([2, Thm 3]):

**Theorem 4** A conditional prevision assessment $\mathcal{M} = (\mu_1, \ldots, \mu_n)$ on the family $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$ is coherent if and only if the following conditions are satisfied: (i) the system $(\Sigma)$ defined in (1) is solvable; (ii) if $I_0 \neq \emptyset$, then $\mathcal{M}_0$ is coherent.

**Remark 5** *We observe that if $\Lambda$ is a solution of System $(\Sigma)$, associated with the pair $(\mathcal{F}, \mathcal{M})$, such that $\Phi_j(\Lambda) > 0$, $j = 1, \ldots, n$, then it holds that $I_0 = \emptyset$ and hence by Theorem 4 the assessment $\mathcal{M}$ on $\mathcal{F}$ is coherent.*

We recall the following extension theorem for conditional previsions, which is a generalization of de Finetti's fundamental theorem of probability to conditional random quantities (see, e.g.,[35, 49, 55])

**Theorem 6** *Let $\mathcal{M} = (\mu_1, \ldots, \mu_n)$ be a coherent prevision assessment on a family of bounded conditional random quantities $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$. Moreover, let $X|H$ be a further bounded conditional random quantity. Then, there exists a suitable closed interval $[\mu', \mu'']$ such that the extension $\mu = \mathbb{P}(X|H)$ is coherent if and only if $\mu \in [\mu', \mu'']$.*

**Imprecise Assessments.** We recall below the notions of coherence, for imprecise, or set-valued, prevision assessments. As, in this paper we only consider conditional random quantities with possible values in the unit interval, in our case the imprecise assessments are subsets of the unitary hypercube.

**Definition 7** *Let be given a family of n conditional random quantities $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$. An* imprecise, or set-valued, assessment *$\mathcal{A}$ on $\mathcal{F}$ is a set of precise assessments $\mathcal{M}$ on $\mathcal{F}$.*

Let $\mathcal{A}$ be an imprecise assessment of $\mathcal{F}$. For each $j \in \{1, 2, \ldots, n\}$, the projection $\rho_j(\mathcal{A})$ be an imprecise assessment of $\mathcal{F}$. For each $j \in \{1, 2, \ldots, n\}$, the projection $\rho_j(\mathcal{A})$ of $\mathcal{A}$ onto the $j$-th coordinate, is defined as ) of $\mathcal{A}$ onto the $j$-th coordinate, is defined as

$$\rho_j(\mathcal{A}) = \{x_j : \mu_j = x_j, \text{ for some } (\mu_1, \ldots, \mu_n) \in \mathcal{I}\}.$$

**Definition 8** *Let be given a family of n conditional random quantities $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$. An imprecise assessment $\mathcal{A}$ on $\mathcal{F}$ is* coherent *if and only if, for every $j \in \{1, \ldots, n\}$ and for every $x_j \in \rho_j(\mathcal{A})$, there exists a coherent precise assessment $\mathcal{M} = (p_1, \ldots, p_n)$ on $\mathcal{F}$, such that $\mathcal{M} \in \mathcal{A}$ and $p_j = x_j$.*

In the context of imprecise assessments the notions of g-coherence and total coherence have been also introduced ([22, 32]).

**Remark 9** *We observe that an interval-valued prevision assessment $\mathcal{A} = [l_1, u_1] \times \cdots \times [l_n, u_n]$ is an imprecise prevision assessment. In this case it holds that $\rho_j(\mathcal{A}) = [l_j, u_j]$. Then, based on Definition 8, the imprecise prevision assessment $\mathcal{A}$ on $\mathcal{F}$ is coherent if and only if, given any $j \in \{1, \ldots, n\}$ and any $x_j \in [l_j, u_j]$, there exists a coherent precise prevision assessment $\mathcal{M} = (\mu_1, \ldots, \mu_n)$ on $\mathcal{F}$, with $l_i \leq \mu_i \leq u_i$, $i = 1, \ldots, n$, and such that $p_j = x_j$.*

**Indicator of a Conditional Event.** Given a conditional event $A|H$ and a probability assessment $P(A|H) = x$, the indicator of $A|H$, denoted by the same symbol, is

$$A|H = AH + x\overline{H} = AH + x(1-H) = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ 0, & \text{if } \overline{A}H \text{ is true,} \\ x, & \text{if } \overline{H} \text{ is true.} \end{cases} \tag{3}$$

Notice that it holds that $\mathbb{P}(AH + x\overline{H}) = xP(H) + xP(\overline{H}) = x$. For the indicator of the negation of $A|H$ it holds that $\overline{A}|H = 1 - A|H$. Given two conditional events $A|H$ and $B|K$, for every coherent assessment $(x, y)$ on $\{A|H, B|K\}$, it holds that ([29, formula (15)])

$$\begin{array}{l} AH + x\overline{H} \leq BK + y\overline{K} \\ \qquad \Longleftrightarrow A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } K \subseteq B, \end{array}$$

that is, between the numerical values of $A|H$ and $B|K$, under coherence it holds that

$$A|H \leq B|K \iff A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } K \subseteq B. \tag{4}$$

By following the approach given in [8, 26, 38], once a coherent assessment $\mu = \mathbb{P}(X|H)$ is specified, the conditional random quantity $X|H$ is not looked at as the restriction of $X$ to $H$, but is defined as $X$, or $\mu$, according to whether $H$ is true, or $\overline{H}$ is true; that is,

$$X|H = XH + \mu\overline{H}. \tag{5}$$

Notice that the representation (5) is not circular. Once the value $\mu = \mathbb{P}(X|H)$ is (coherently) specified by the betting scheme, the object $X|H$ in (5) is (subjectively) determined. We observe that $\mathbb{P}(XH + \mu\overline{H}) = \mathbb{P}(X|H)$. Indeed,

$$\mathbb{P}(XH + \mu\overline{H}) = \mathbb{P}(X|H)P(H) + \mu P(\overline{H}) = \mu.$$

**Conjunction of Two Conditional Events.** We recall below the notion of conjunction of two conditional events in the framework of conditional random quantities ([26], see also [36, 39]).

**Definition 10** *Given two conditional events $A|H$, $B|K$ and a (coherent) probability assessment $P(A|H) = x$, $P(B|K) = y$, the conjunction $(A|H) \wedge (B|K)$ is defined as the following conditional random quantity*

$$(A|H) \wedge (B|K) = (AHBK + x\overline{H}BK + yAH\overline{K})|(H \vee K). \quad (6)$$

**Remark 11** *Notice that the conjunction in (6) and can be represented as $X|H$ in (5) and, once the (coherent) assessment $(x, y, z)$, where $z = \mathbb{P}[(A|H) \wedge (B|K)]$, is given, the conjunction is (subjectively) determined by*

$$(A|H) \wedge (B|K) = AHBK + x\overline{H}BK + yAH\overline{K} + z\overline{H}\,\overline{K}.$$

*Then, the set of possible values of $(A|H) \wedge (B|K)$, i.e. $\{1, 0, x, y, z\}$, is associated to a given (coherent) subjective assessment $(x, y, z)$.*

Within the betting scheme, by starting with a coherent assessment $(x, y)$ on $\{A|H, B|K\}$, if you extend $(x, y)$ (in a coherent way) by adding the assessment $\mathbb{P}[(A|H) \wedge (B|K)] = z$, then you agree for instance to pay $z$, by receiving the random amount
1, if both conditional events $A|H$ and $B|K$ are true;
0, if $A|H$ or $B|K$ is false;
$x = P(A|H)$, if $A|H$ is void and $B|K$ is true ;
$y = P(B|K)$, if $A|H$ is true and $B|K$ is void
$z$, that is the paid amount, if both conditional events $A|H$ and $B|K$ are void. Notice that, in some particular case, the conjunction $(A|H) \wedge (B|K)$, which is a five-valued object, reduces to a conditional event, that is a three-valued object.

We recall that, the Fréchet-Hoeffding bounds, i.e., the lower and upper bounds

$$z' = \max\{x + y - 1, 0\}, z'' = \min\{x, y\} \quad (7)$$

obtained under logical independence in the unconditional case for the coherent extensions $z = P(A \wedge B)$ of $P(A) = x$ and $P(B) = y$, are still valid when $P(A)$, $P(B)$, and $P(A \wedge B)$ are replaced by $P(A|H)$, $P(B|K)$, and $\mathbb{P}[(A|H) \wedge (B|K)]$ ([26]). We recall that the Fréchet-Hoeffding bounds are not satisfied by other notions of conjunction, e.g., the quasi conjunction, defined in suitable trivalent logics ([50]).

**Quasi Conjunction.** The Sobocinski conjunction ($\wedge_S$), or quasi conjunction ([1]), of two conditional events $A|H$ and $B|K$ is defined, in a trivalent logic, as the following conditional event

$$(A|H) \wedge_S (B|K) = [(AH \vee \overline{H}) \wedge (BK \vee \overline{K})]|(H \vee K).$$

In terms of conditional random quantity, it holds that

$$(A|H) \wedge_S (B|K) = (AHBK + \overline{H}BK + AH\overline{K})|(H \vee K). \quad (8)$$

We recall that, by setting $x = P(A|H)$, $y = P(B|K)$, under logical independence, the lower and upper bounds $z'_S, z''_S$ for $(A|H) \wedge_S (B|K)$ are ([21])

$$z'_S = \max\{x+y-1, 0\}, \; z''_S = \begin{cases} \frac{x+y-2xy}{1-xy}, & \text{if } (x, y) \neq (1, 1) \\ 1, & \text{if } (x, y) = (1, 1). \end{cases} \quad (9)$$

## 3. A Generalized Notion of Conjunction

We generalize the notion of conjunction between two conditional events given in Definition 10 by replacing $x$ and $y$ with two arbitrary values $a, b$ in $[0, 1]$.

**Definition 12** *Given four events $A, B, H, K$, with $H \neq \emptyset$ and $K \neq \emptyset$, and two values $a, b \in [0, 1]$, we define the* the generalized conjunction w.r.t. $a$ and $b$ *of the conditional events $A|H$ and $B|K$ as the following conditional random quantity*

$$(A|H) \wedge_{a,b} (B|K) = (AHBK + a\overline{H}BK + bAH\overline{K})|(H \vee K). \quad (10)$$

In the betting framework, if you assess $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, then you agree for instance to pay $z$, by receiving the random amount

$$(A|H) \wedge_{a,b} (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true}, \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true}, \\ a, & \text{if } \overline{H}BK \text{ is true}, \\ b, & \text{if } AH\overline{K} \text{ is true}, \\ z, & \text{if } \overline{H}\,\overline{K} \text{ is true}. \end{cases}$$

In other words, you agree to pay $z$ in order to receive:
1, if both conditional events $A|H$ and $B|K$ are true;
0, if $A|H$ or $B|K$ is false;
$a$, if $A|H$ is void and $B|K$ is true ;
$b$, if $A|H$ is true and $B|K$ is void;
$z$, that is the paid amount, if both conditional events $A|H$ and $B|K$ are void. Therefore

$$\begin{aligned} &\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] = P(AHBK|(H \vee K)) + \\ &+aP(\overline{H}BK|(H \vee K)) + bP(AH\overline{K}|(H \vee K)). \end{aligned} \quad (11)$$

We observe that $(A|H) \wedge_{a,b} (B|K)$, when $H \vee K$ is true, assumes values in $[0, 1]$. Then, by coherence (see Remark 3) it must be $z \in [0, 1]$ and hence $(A|H) \wedge_{a,b} (B|K) \in [0, 1]$. Of course, $(A|H) \wedge_{a,b} (B|K) = (B|K) \wedge_{b,a} (A|H)$.

**Remark 13** *When we assess $P(A|H) = x$ and $P(B|K) = y$, from definitions 6 and 10 it holds that*

$$(A|H) \wedge_{x,y} (B|K) = (A|H) \wedge (B|K),$$

*that is $(A|H) \wedge_{a,b} (B|K)$ reduces to $(A|H) \wedge (B|K)$, when $a = x$ and $b = y$. Moreover,*

$$\mathbb{P}[(A|H) \wedge_{x,y} (B|K)] = P(AHBK|(H \vee K))+ \\ +P(A|H)P(\overline{H}BK|(H\vee K))+P(B|K)P(AH\overline{K}|(H\vee K)). \tag{12}$$

*We also observe that, if $H = K$, then $\overline{H}BK = AH\overline{K} = \emptyset$, and hence $(A|H) \wedge_{a,b} (B|H) = AB|H = ABH|H = AB|H$ and $z = P(ABH|H) = P(AB|H)$.*

The next result allows to compute the set of all coherent assessments $(x, y, z)$ on the family $\{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$, where $a, b$ are arbitrary numbers in $[0, 1]$.

**Theorem 14** *Let $A$, $B$, $H$, $K$ be any logically independent events. A prevision assessment $\mathcal{M} = (x, y, z)$ on the family of conditional random quantities $\mathcal{F} = \{A|H, (B|K), (A|H) \wedge_{a,b} (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where*

$$z' = \begin{cases} (x+y-1)\cdot \min\{\frac{a}{x}, \frac{b}{y}, 1\}, & \text{if } x+y-1>0, \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

*and*

$$z'' = \max\{z''_1, z''_2, \min\{z''_3, z''_4\}\}, \tag{14}$$

*where*

$$z''_1 = \min\{x, y\},$$
$$z''_2 = \begin{cases} \dfrac{x(b-ay)+y(a-bx)}{1-xy}, & \text{if } (x,y) \neq (1,1), \\ 1, & \text{if } (x,y) = (1,1), \end{cases}$$

$$z''_3 = \begin{cases} \dfrac{x(1-a)+y(a-x)}{1-x}, & \text{if } x \neq 1, \\ 1, & \text{if } x = 1, \end{cases}$$

$$z''_4 = \begin{cases} \dfrac{x(b-y)+y(1-b)}{1-y}, & \text{if } y \neq 1, \\ 1, & \text{if } y = 1. \end{cases}$$

**Proof** First of all we observe that, by logical independence of $A, H, B, K$, the assessment $(x, y)$ is coherent for every $(x, y) \in [0, 1]^2$. The constituents $C_h$'s contained in $H \vee K$ and the points $Q_h$'s associated with the assessment $\mathcal{M} = (x, y, z)$ on $\mathcal{F}$ are $C_1 = AHBK, C_2 = AH\overline{B}K, C_3 = \overline{A}HBK, C_4 = \overline{A}H\overline{B}K, C_5 = AH\overline{K}, C_6 = \overline{H}BK, C_7 = \overline{A}H\overline{K}, C_8 = \overline{H}\,\overline{B}K$ and $Q_1 = (1, 1, 1), Q_2 = (1, 0, 0), Q_3 = (0, 1, 0), Q_4 = (0, 0, 0), Q_5 = (1, y, b), Q_6 = (x, 1, a), Q_7 = (0, y, 0), Q_8 = (x, 0, 0)$. Considering the convex hull $\mathcal{I}$ of $Q_1, \dots, Q_8$, the coherence of $\mathcal{M}$ requires that the condition $\mathcal{M} \in \mathcal{I}$ be satisfied, that is

$$\mathcal{M} = \textstyle\sum_{h=1}^{8} \lambda_h Q_h,\ \sum_{h=1}^{8} \lambda_h = 1,\ \lambda_h \geq 0,\ h = 1, \dots, 8\,.$$

We observe that $Q_7 = yQ_3(1-y)Q_4$ and $Q_8 = xQ_2 + (1-x)Q_4$. Then, $\mathcal{I}$ is the convex hull of $Q_1, \dots, Q_6$. Thus, the condition $\mathcal{M} \in \mathcal{I}$ amounts to the solvability of the following system in the unknowns $\lambda_1, \dots, \lambda_6$

$$\mathcal{M} = \textstyle\sum_{h=1}^{6} \lambda_h Q_h,\ \sum_{h=1}^{6} \lambda_h = 1,\ \lambda_h \geq 0,\ h = 1, \dots, 6 \tag{15}$$

that is

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_5 + x\lambda_6 = x, \\ \lambda_1 + \lambda_3 + y\lambda_5 + \lambda_6 = y, \\ \lambda_1 + b\lambda_5 + a\lambda_6 = z, \\ \lambda_1 + \cdots + \lambda_6 = 1,\ \lambda_i \geq 0,\ \forall i = 1, \dots, 6. \end{cases} \tag{16}$$

For each solution $\Lambda = (\lambda_1, \dots, \lambda_6)$ of system (16) we have that the functions $\Phi_j$ defined in (2) are

$$\begin{array}{l} \Phi_1(\Lambda) = \sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, \\ \Phi_2(\Lambda) = \sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_6, \\ \Phi_3(\Lambda) = \sum_{h:C_h \subseteq H \vee K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6. \end{array} \tag{17}$$

For each given $(x, y) \in [0, 1]$, based on Theorem 6 we determine the lower and upper bounds $z', z''$ for the coherent extension $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.

**Lower Bound.** We distinguish two cases: (**A**) $x+y-1 \leq 0$; (**B**) $x + y - 1 > 0$.
Case (**A**). We show that $z' = 0$ is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ by proving that the assessment $(x, y, 0)$ on $\mathcal{F}$ is coherent. Let a pair $(x, y) \in [0, 1]^2$ be given. We first observe that $\mathcal{M} = (x, y, 0) = xQ_2 + yQ_3 + (1 - x - y)Q_4$. Then, $\mathcal{M} \in \mathcal{I}$, where $\mathcal{I}$ is the convex hull of $Q_1, \dots, Q_6$, with a solution of (16) given by $\Lambda = (0, x, y, 1 - x - y, 0, 0)$. For the functions $\Phi_j$ given in (17) it holds that $\Phi_1(\Lambda) = \Phi_2(\Lambda) = \Phi_3(\Lambda) = 1 > 0$. Then, from Remark 5, the assessment $(x, y, 0)$ on $\mathcal{F}$ is coherent. Thus, for every $(x, y) \in [0, 1]^2$ the assessment $(x, y, 0)$ is coherent and hence $z' = 0$.
Case (**B**). As $x + y - 1 > 0$, it holds that $x \neq 0$ and $y \neq 0$. We consider three subcases: *(B.1)* $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = 1$; *(B.2)* $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{a}{x}$; *(B.3)* $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{b}{y}$.

*Case (B.1).* We observe that $a \geq x$ and $b \geq y$. We first prove that the assessment $(x, y, x + y - 1)$ is coherent. Then, we show that $z' = x + y - 1$ is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$. As $\mathcal{M} = (x, y, x + y - 1) = (x + y - 1)Q_1 + (1 - y)Q_2 + (1 - x)Q_3$, it follows that $\mathcal{M} \in \mathcal{I}$. Then, a solution of system (16) is given by $\Lambda = (x + y - 1, 1 - y, 1 - x, 0, 0, 0)$. From (17) it holds that $\Phi_1(\Lambda) = \Phi_2(\Lambda) = \Phi_3(\Lambda) = 1 > 0$. Then, from Remark 5, the assessment $(x, y, x + y - 1)$ on $\mathcal{F}$ is coherent.

In order to prove that $z' = x+y-1$ is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, we verify that the assessment $\mathcal{M} = (x, y, z)$, with $(x, y) \in [0, 1]^2$ and $z < z' = x + y - 1$, is not coherent because $(x, y, z) \notin \mathcal{I}$. We observe that the points $Q_1, Q_2, Q_3$ belong to the plane $\pi : X + Y - Z = 1$. We set $f(X, Y, Z) = X + Y - Z - 1$ and we obtain $f(Q_1) = f(Q_2) = f(Q_3) = 0,\ \ f(Q_4) = -1 < 0,\ f(Q_5) = y - b \leq 0,\ f(Q_6) = x - a \leq 0$. Then, by considering $\mathcal{M} = (x, y, z)$, with $z < x + y - 1$, it holds that

$$f(\mathcal{M}) = x + y - 1 - z > 0 \geq f(Q_h),\ \ h = 1, \ldots, 6,$$

and hence $\mathcal{M} = (x, y, z) \notin \mathcal{I}$. Indeed, if it were $\mathcal{M} \in \mathcal{I}$, that is $\mathcal{M}$ linear convex combination of $Q_1, \ldots, Q_6$, it would follow that $f(\mathcal{M}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leq 0$. Thus, the lower bound for $z$ is $z' = x + y - 1$, for every $(x, y) \in [0, 1]^2$ such that $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = 1$.

*Case (B.2)*. We notice that $a \leq x$ and $\frac{a}{x} \leq \frac{b}{y}$.
We show that in this case

$$z' = (x + y - 1) \min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{a}{x}(x + y - 1).$$

We first prove that that $(x, y, \frac{a}{x}(x + y - 1))$ is coherent. Indeed, we observe that $\mathcal{M} = (x, y, \frac{a}{x}(x + y - 1)) = (1 - y)Q_2 + \frac{(1-x)(1-y)}{x}Q_3 + \frac{x+y-1}{x}Q_6$. Then, $\mathcal{M} \in \mathcal{I}$, where $\mathcal{I}$ is the convex hull of $Q_1, \ldots, Q_6$, with a solution of (16) given by $\Lambda = (0, 1 - y, \frac{(1-x)(1-y)}{x}, 0, 0, \frac{x+y-1}{x})$. By recalling (17), it holds that $\Phi_1(\Lambda) = \frac{1-y}{x}$, $\Phi_2(\Lambda) = \Phi_3(\Lambda) = 1 > 0$. We distinguish two cases: $(i)$ $y \neq 1$, $(ii)$ $y = 1$. In the case $(i)$ we get $\Phi_j(\Lambda) > 0$, $j = 1, 2, 3$ and hence by Remark 5 it follows that the assessment $(x, y, \frac{a}{x}(x + y - 1))$ is coherent. In case $(ii)$, as $\Phi_1(\Lambda) = 0$, it holds that $\mathcal{I}_0 \subseteq \{1\}$, with the sub-assessment $\mathcal{M}_0 = x$ on $\mathcal{F}_0 = \{A|H\}$ coherent because $x \in [0, 1]$. Then, by Theorem 4, the assessment $(x, y, \frac{a}{x}(x+y-1)) = (x, 1, a)$ on $\mathcal{F}$ is coherent. Thus, in this sub-case the assessment $(x, y, \frac{a}{x}(x+y-1))$ on $\mathcal{F}$ is coherent for every $(x, y) \in [0, 1]^2$. In order to prove that $\frac{a}{x}(x+y-1)$ is the lower bound $z'$ for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, we verify that $(x, y, z)$, with $(x, y) \in [0, 1]^2$ and $z < \frac{a}{x}(x+y-1)$, is not coherent because $(x, y, z) \notin \mathcal{I}$. We observe that the points $Q_2, Q_3, Q_6$ belong to the plane $\pi : aX + aY - xZ = a$. We set $f(X, Y, Z) = a(X + Y - 1) - xZ$ and we obtain $f(Q_2) = f(Q_3) = f(Q_6) = 0$, $f(Q_1) = a - x \leq 0$, $f(Q_4) = -a \leq 0$, $f(Q_5) = ay - bx \leq 0$. Then, by considering $\mathcal{M} = (x, y, z)$, with $z < \frac{a}{x}(x + y - 1)$, it holds that $f(\mathcal{M}) = f(x, y, z) = a(x+y-1) - xz > 0 \geq f(Q_h), h = 1, \ldots, 6$, and hence $\mathcal{M} = (x, y, z) \notin \mathcal{I}$. Indeed, if it were $\mathcal{M} \in \mathcal{I}$, that is $\mathcal{M}$ linear convex combination of $Q_1, \ldots, Q_6$, it would follow that $f(\mathcal{M}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leq 0$. Thus, in this sub-case the lower bound for $z = \mathbb{P}((A|H) \wedge_{a,b} (B|K))$ is $z' = \frac{a}{x}(x + y - 1)$, for every $(x, y) \in [0, 1]^2$ such that $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{a}{x}$.

*Case (B.3)*. We notice that $b \leq y$ and $\frac{a}{x} \geq \frac{b}{y}$. We prove that $(x, y, \frac{b}{y}(x + y - 1))$ is coherent and that $z' = \frac{b}{y}(x + y - 1)$ is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.

We observe that $\mathcal{M} = (x, y, \frac{b}{y}(x + y - 1)) = \frac{(1-x)(1-y)}{y}Q_2 + (1 - x)Q_3 + \frac{x+y-1}{y}Q_5$. Then, $\mathcal{M} \in \mathcal{I}$, with a solution of (16) given by $\Lambda = (0, \frac{(1-x)(1-y)}{y}, 1 - x, 0, \frac{x+y-1}{y}, 0)$.It holds that $\Phi_1(\Lambda) = \Phi_3(\Lambda) = 1$, $\Phi_2(\Lambda) = \frac{1-x}{y}$. We distinguish two cases: $(i)$ $x \neq 1$, $(ii)$ $x = 1$. In the case $(i)$ we get $\Phi_j(\Lambda) > 0$, $j = 1, 2, 3$, and hence by Remark 5, the assessment $(x, y, \frac{b}{y}(x + y - 1))$ is coherent. In the case $(ii)$ we get $\mathcal{I}_0 \subseteq \{2\}$, with the sub-assessment $\mathcal{M}_0 = y$ on $\mathcal{F}_0 = \{B|K\}$ coherent because $y \in [0, 1]$. Then, by Theorem 4, the assessment $(x, y, \frac{b}{y}(x + y - 1)) = (1, y, b)$ on $\mathcal{F}$ is coherent. Thus, the assessment $(x, y, \frac{b}{y}(x + y - 1))$ on $\mathcal{F}$ is coherent for every $(x, y) \in [0, 1]^2$ such that $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{b}{y}$. In order to prove that $\frac{b}{y}(x + y - 1)$ is the lower bound $z'$ for $z = \mathbb{P}((A|H) \wedge_{a,b} (B|K))$, we verify that $(x, y, z)$, with $(x, y) \in [0, 1]^2$ and $z < \frac{b}{y}(x+y-1)$, is not coherent because $(x, y, z) \notin \mathcal{I}$. We observe that the points $Q_2, Q_3, Q_5$ belong to the plane $\pi : bX + bY - yZ = b$. We set $f(X, Y, Z) = b(X + Y - 1) - yZ$ and we obtain $f(Q_2) = f(Q_3) = f(Q_5) = 0$, $\ f(Q_1) = b - y < 0$, $f(Q_4) = -b \leq 0$, $f(Q_5) = bx - ay \leq y$. Then, by considering $\mathcal{M} = (x, y, z)$, with $z < \frac{b}{y}(x + y - 1)$, it holds that $f(\mathcal{M}) = f(x, y, z) = b(x + y - 1) - yz > 0 \geq f(Q_h)$, $h = 1, \ldots, 6$, and hence $\mathcal{M} = (x, y, z) \notin \mathcal{I}$. Indeed, if it were $\mathcal{M} \in \mathcal{I}$, that is $\mathcal{M}$ linear convex combination of $Q_1, \ldots, Q_6$, it would follow that $f(\mathcal{M}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leq 0$. Thus, the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ is $z' = \frac{b}{y}(x + y - 1)$.

Therefore, for every $(x, y) \in [0, 1]^2$ the value $z'$ given in formula (13) is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.

**Upper Bound.** Due to lack of space this part of the proof is omitted. ■

A summary of the different values of the lower and upper bounds $z'$ and $z''$ for the prevision of $(A|H) \wedge_{a,b} (B|K)$ are given in Table1 and in Table 2, respectively. Moreover,

Table 1: Values of the lower bound $z'$ of $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ for different values of $x = P(A|H)$, $y = P(B|K)$, $a, b \in [0, 1]$.

| Case | $z'$ |
|---|---|
| **(A):** $x + y - 1 \leq 0$ | 0 |
| **(B):** $x + y - 1 > 0$ | |
| Sub-cases: | |
| *(B.1)*:$a \geq x$ and $b \geq y$ | $x + y - 1$ |
| *(B.2)*:$a < x$ and $\frac{a}{x} \leq \frac{b}{y}$ | $\frac{a}{x}(x + y - 1)$ |
| *(B.3)*:$b < y$ and $\frac{b}{y} \leq \frac{a}{x}$ | $\frac{b}{y}(x + y - 1)$ |

Table 2: Values of the upper bound $z''$ of $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ for different values of $x = P(A|H), y = P(B|K), a, b \in [0, 1]$.

| Case | $z''$ |
|---|---|
| **(C)** $a(1-y)+b(1-x) > 1-xy$ | $z''_2 = \frac{x(b-ay)+y(a-bx)}{1-xy}$ |
| **(D)** $a(1-y)+b(1-x) \leq 1-xy$ Sub-cases: | |
| $x \leq y$ and $a \leq x$ | $z''_1 = x$ |
| $x \leq y$ and $a > x$ | $z''_3 = \frac{x(1-a)+y(a-x)}{1-x}$ |
| $y < x$ and $b \leq y$ | $z''_1 = y$ |
| $y < x$ and $b > y$ | $z''_4 = \frac{x(b-y)+y(1-b)}{1-y}$ |

from Theorem 14 it follows that

**Corollary 15** *Let $A, B, H, K$ be any logically independent events. Then, the set $\Pi$ of all coherent prevision assessments $(x, y, z)$ on $\mathcal{F} = \{A|H, (B|K), (A|H) \wedge_{a,b} (B|K)\}$, is*

$$\Pi_{a,b} = \{(x, y, z) : (x, y) \in [0, 1]^2, z \in [z', z'']\}, \quad (18)$$

*where $z'$ and $z''$ are defined in (13) and (14), respectively.*

We recall that, under logical independence, the set of all coherent assessment $(x, y, z)$ on $\{A|H, B|K, (A|H) \wedge (B|K)\}$ is the the Tetrahedron $\mathcal{T}$ of points $(1, 1, 1), (1, 0, 0), (0, 1, 0), (0, 0, 0)$, that is ([26]) $\mathcal{T} = \{(x, y, z) : (x, y) \in [0, 1]^2, z \in [\max\{x + y - 1, 0\}, \min\{x, y\}]\}$. As, from (13) and (14), $z' \leq \max\{x + y - 1, 0\}$ and $z'' \geq \min\{x, y\}$ for every $(a, b) \in [0, 1]^2$, it holds that

$$\mathcal{T} \subseteq \Pi_{a,b} \ \forall (a, b) \in [0, 1]^2.$$

**Remark 16** *Given a probability assessment $(x, y)$ on $\{A|H, B|K\}$, it holds that $z' = \max\{x + y - 1, 0\}$ and $z'' = \min\{x, y\}$, when $a = x, b = y$. That is, $(A|H) \wedge_{x,y} (B|K) = (A|H) \wedge (B|K)$ satisfies the Fréchet-Hoeffding bounds. In general, it may happen that, for some values of a and b, the extension z on $(A|H) \wedge_{a,b} (B|K)$ is coherent also when $z \notin [\max\{x+y-1, 0\}, \min\{x, y\}]$. That is the Fréchet-Hoeffding bounds (7) are not satisfied by the conjunction $(A|H) \wedge_{a,b} (B|K)$ for some values of $a, b$. For example, if $x = y = 1$, we have that $\max\{x + y - 1, 0\} = \min\{x, y\} = 1$; then, from (13) and (14) it holds that $z' = z'' = 1$ only when $a = b = 1$. However, if $x = y = 0$, we have that $\max\{x + y - 1, 0\} = \min\{x, y\} = 0$; then, from (13) and (14) it holds that $z' = z'' = 0$ for every $(a, b)[0, 1]^2$, that is th Fréchet-Hoeffding bounds (7) are satisfied by $(A|H) \wedge_{a,b} (B|K)$ for every $(a, b) \in [0, 1]^2$.*

## 4. Some Particular Cases

In this section we consider some particular cases of the generalized conjunction obtained under some logical relations and for some given values of $a$ and $b$. In particular we consider the case where the conditioning events $H$ and $K$ are incompatible. Then, we consider the cases where $a = b = 1$ and $a = b = 0$.

### 4.1. The Case $H$ and $K$ Incompatible

We analyze the particular case where the conditioning events $H$ and $K$ are incompatible, i.e. $HK = \emptyset$.

**Theorem 17** *Let $A|H$, $B|K$,be two conditional events with $H \neq \emptyset$, $K \neq \emptyset$, and $HK = \emptyset$. A prevision assessment $\mathcal{M} = (x, y, z)$ on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where*

$$z' = \min\{ay, bx\}, \quad z'' = \max\{ay, bx\}. \quad (19)$$

**Proof** The constituents and the points $Q_h$'s associated with the family $(\mathcal{M}, \mathcal{F})$ are $C_1 = AH\overline{K}$, $C_2 = \overline{H}BK$, $C_3 = \overline{A}H\overline{K}$, $C_4 = \overline{H}\,\overline{B}K$, $C_0 = \overline{H}\,\overline{K}$, and $Q_1 = (1, y, b)$, $Q_2 = (x, 1, a)$, $Q_3 = (0, y, 0), Q_4 = (x, 0, 0)$, $Q_0 = P = (x, y, z)$. We observe that $\mathcal{H}_n = H \vee K$ and that $\mathcal{I}$ is the convex hull of points $Q_1, \ldots Q_4$. For each given assessment $(x, y)$ based on Theorem 6 we determine the lower and upper bounds $z', z''$ for the coherent extension $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$. We distinguish two cases: $(i) ay \leq bx$, $(ii) bx < ay$. *Case* $(i)$. We show that the assessment $(x, y, z)$ on $\mathcal{F}$ is coherent if and only $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where $z' = ay$ and $z'' = by$.

**Lower Bound.** First we prove that $(x, y, ay)$, is coherent, and then that $z' = ay$ is the lower bound for $z$. We observe that $\mathcal{M} = (x, y, ay) = yQ_2 + (1 - y)Q_4$. Then, $\mathcal{M} \in \mathcal{I}$ with a solution of (16) given by $\Lambda = (0, y, 0, 1 - y)$. It follows that

$$\Phi_1(\Lambda) = \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_3 = 0,$$
$$\Phi_2(\Lambda) = \textstyle\sum_{h:C_h \subseteq K} \lambda_h = \lambda_2 + \lambda_4 = 1,$$
$$\Phi_3(\Lambda) = \textstyle\sum_{h:C_h \subseteq H \vee K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1.$$

As $\Phi_1(\Lambda) = 0$, it holds that $\mathcal{I}_0 \subseteq \{1\}$, with the sub-assessment $\mathcal{M}_0 = x$ on $\mathcal{F}_0 = \{A|H\}$ coherent because $x \in [0, 1]$. Then, by Theorem 4 the assessment $\mathcal{M} = (x, y, ay)$ on $\mathcal{F}$ is coherent. Now we show that $z' = ay$ is the lower bound. Of course, if $x = 0$, as $ay \leq bx = 0$, it holds that $ay = 0$. In this case we have that $z' = 0$ is the lower bound because $(A|H) \wedge_{a,b} (B|K) \in [0, 1]$. We assume now that $x \neq 0$. We observe that $Q_2, Q_3, Q_4$ belong to the plane $\pi : ayX + axY - xZ - axy = 0$. By setting $f(X, Y, Z) = ayX + axY - xZ - axy$, it holds that

$$f(Q_1) = ay - bx \leq 0, \quad f(Q_2) = f(Q_3) = f(Q_4) = 0.$$

Then, for every $\mathcal{M} \in \mathcal{I}$ it holds that $f(\mathcal{M}) = f(\sum_{h=1}^{4} \lambda_h Q_h) = \sum_{h=1}^{4} \lambda_h f(Q_h) \leq 0$. Then, by considering $\mathcal{M} = (x, y, z)$, with $z < ay$, it holds that $f(\mathcal{M}) = f(x, y, z) = x(ay - z) > 0$, and hence $\mathcal{M} \notin \mathcal{I}$. Thus, as $(x, y, ay)$ is coherent and $(x, y, z)$, with $z < ay$ is not coherent, it follows that $z' = ay$ it the lower bound for $(A|H) \wedge_{a,b} (B|K)$.

**Upper Bound.** We show that the assessment $\mathcal{M} = (x, y, bx)$ on $\mathcal{F}$ is coherent. We observe that $(x, y, bx) = xQ_1 + (1 - x)Q_3$. Then, $\mathcal{M} \in \mathcal{I}$ with a solution of (16) given by $\Lambda = (x, 0, 1 - x, 0)$. It follows that

$$\Phi_1(\Lambda) = \Phi_3(\Lambda) = 1, \; \Phi_2(\Lambda) = 0.$$

As $\Phi_2(\Lambda) = 0$, it holds that $\mathcal{I}_0 \subseteq \{2\}$, with the sub-assessment $\mathcal{M}_0 = y$ on $\mathcal{F}_0 = \{B|K\}$ coherent because $y \in [0, 1]$. Then, by Theorem 4 the assessment $\mathcal{M} = (x, y, bx)$ on $\mathcal{F}$ is coherent. We show that $z'' = bx$ is the upper bound for $z$. We distinguish two cases: $y > 0$ and $y = 0$. Let us suppose for now that $y > 0$. We observe that $Q_1, Q_3, Q_4$ belong to the plane $\pi : byX + bxY - yZ = bxy$. Considering the function $f(X, Y, Z) = byX + bxY - yZ - bxy$, it holds that

$$f(Q_1) = f(Q_3) = f(Q_4) = 0, \; f(Q_2) = bx - ay \geq 0.$$

Then, for every $\mathcal{M} \in \mathcal{I}$ it holds that $f(\mathcal{M}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \geq 0$. Then, by considering $\mathcal{M} = (x, y, z)$, with $z > bx$, it holds that $f(\mathcal{M}) = f(x, y, z) = y(bx - z) < 0$ and hence $\mathcal{M} \notin \mathcal{I}$. Thus, the assessment $\mathcal{M} = (x, y, z)$ with $z > bx$ is not coherent and hence $z'' = bx$.

If $y = 0$, we observe that $Q_1, Q_2, Q_3$ belong to the plane $\pi : -bX + (bx - a)Y + Z = 0$. We set $f(X, Y, Z) = -bX + (bx-a)Y + Z$ and it holds that $f(Q_1) = f(Q_2) = f(Q_3) = 0$, $f(Q_4) = -bx \leq 0$. Then, for every $\mathcal{M} \in \mathcal{I}$ it holds that $f(\mathcal{M}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leq 0$. Then, by considering $\mathcal{M} = (x, 0, z)$, with $z > bx$, it holds that $f(\mathcal{M}) = f(x, 0, z) = -bx + z > 0$, and hence $\mathcal{M} \notin \mathcal{I}$. Thus, the assessment $\mathcal{M} = (x, 0, z)$, with $z > bx$, is not coherent and hence $z'' = bx$.

*Case* $(ii)$. The proof can be obtained in a way similar to the proof in *Case* $(i)$, when switching $x$ with $y$ and $a$ with $b$. ■

We observe that the bounds obtained in Theorem 17 (under the logical relation $HK = \emptyset$) are more restrictive than the bounds obtained in Theorem 14 (under the assumption of logical independence of $A, H, B, K$). That is, $[\min\{ay, bx\}, \max\{ay, bx\}] \subseteq [z', z'']$, where $z'$ and $z''$ are given in (13) and (14), respectively. Of course, when $a < 1$ and $b < 1$, it holds that $z'' < 1$. This is in agreement to the fact that $(A|H) \wedge_{a,b} (B|K)$ can never be 1 in this case.

**Remark 18** *The result of Theorem 17 can be also obtained in a different way by exploiting the linearity of prevision. Indeed, when $HK = \emptyset$, it holds that $AHBK = \emptyset$, $\overline{H}BK = BK$ and $AH\overline{K} = AH$ and hence*

$$(A|H) \wedge_{a,b} (B|K) = (aBK + bAH)|(H \vee K). \qquad (20)$$

*We observe that $P(BK|(H \vee K)) = P(B|K)P(K|(H \vee K))$ and $P[AH|(H \vee K)] = P(A|H)P[H|(H \vee K)]$. Then, from (20), by the linearity of prevision, it follows that*

$$\begin{aligned} z &= \mathbb{P}[(A|H) \wedge_{a,b} (B|K)] = \\ &+ aP(\overline{H}BK|(H \vee K)) + bP(\overline{K}AH|(H \vee K)) = \\ &= aP(B|K)P(K|(H \vee K)) + bP(A|H)P(H|(H \vee K)). \end{aligned} \qquad (21)$$

*Denoting by $x = P(A|H)$, $y = P(B|K)$, $\alpha = P(H|(H \vee K)) = 1 - P(K|(H \vee K))$, formula (21) becomes*

$$z = ay(1 - \alpha) + bx\alpha. \qquad (22)$$

*It can be proved*[1] *that the assessment $(x, y, \alpha)$ on $\{A|H, B|K, H|(H \vee K)\}$, with $HK = \emptyset$, is coherent for every $(x, y, \alpha) \in [0, 1]^3$. Then, from (22) we obtain that $z \in [\min\{ay, bx\}, \max\{ay, bx\}]$, because $\alpha \in [0, 1]$.*

**Remark 19** *When $a = x$ and $b = y$, from (19) we obtain that $z' = z'' = ay = bx = xy$. That is, when $HK = \emptyset$, it holds that*

$$\begin{aligned} \mathbb{P}[(A|H) \wedge_{x,y} (B|K)] &= \mathbb{P}[(A|H) \wedge (B|K)] \\ &= P(A|H)P(B|K), \end{aligned}$$

*which is in agreement with the result given in [24] (see also [31]).*

### 4.2. The Case $a = b = 1$ and the Quasi Conjunction

It is interesting to notice that quasi conjunction is a particular case of the generalized conjunction $(A|H) \wedge_{a,b} (B|K)$ where $a = 1$ and $b = 1$. Indeed, by recalling (8)

$$\begin{aligned} (A|H) \wedge_{1,1} (B|K) &= (AHBK + \overline{H}BK + AH\overline{K})|(H \vee K) \\ &= (A|H) \wedge_S (B|K). \end{aligned}$$

[1] For proving that $(x, y, \alpha)$ on $\{A|H, B|K, H|(H \vee K)\}$, with $HK = \emptyset$, is coherent for every $(x, y, \alpha) \in [0, 1]^3$ it is sufficient to check that each of the eight vertices of the unit cube is coherent.

Then, when $H \vee K$ is true it holds that $(A|H) \wedge_S (B|K) \geq (A|H) \wedge_{a,b} (B|K)$. By applying [27, Theorem 6], it holds that $P(A|H) \wedge_S (B|K) \geq \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ and hence

$$(A|H) \wedge_S (B|K) \geq (A|H) \wedge_{a,b} (B|K)$$

in all cases. Moreover, based on Theorem 14, under logical independence, it follows that $(x, y, z)$ on $\{A|H, B|K, (A|H) \wedge_{1,1} (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where $z \in [z', z'']$ with

$$z' = \begin{cases} (x + y - 1), & \text{if } x + y - 1 > 0, \\ 0, & \text{otherwise} \end{cases}$$
$$= \max\{x + y - 1, 0\}$$

and

$$z'' = \begin{cases} \dfrac{x(b - ay) + y(a - bx)}{1 - xy}, & \text{if } (x, y) \neq (1, 1), \\ 1, & \text{if } (x, y) = (1, 1), \end{cases}$$
$$= \begin{cases} \dfrac{x + y - 2xy}{1 - xy}, & \text{if } (x, y) \neq (1, 1), \\ 1, & \text{if } (x, y) = (1, 1), \end{cases}$$

because $a(1-y)+b(1-x)+xy-1 = 1-y+1-x+xy-1 = 1-y-x+xy = 1-y-x(1-y) = (1-y)(1-x) \geq 0$. Then, the lower and upper bounds $z'$ and $z''$ coincide with $z'_S$ and $z''_S$ given in (9). We observe that $z''_S$ is the Hamacher t-conorm with the parameter $\lambda = 0$ . Then, for every $(x, y) \in [0, 1]^2$, as $\max(x, y)$ is the smallest t-conorm, it holds that

$$z''_S \geq \max\{x, y\} \geq \min\{x, y\},$$

and hence the quasi conjunction do not preserve the bounds in (7). Thus, differently from the unconditional case where $P(AB) = 0$ is the only coherent extension of $(1, 0)$ on $\{A, B\}$, for the probability of the quasi conjunction $(A|H) \wedge_S (B|K)$, any value $z_S \in [0, 1]$ is a coherent extension of $(1, 0)$ on $\{A|H, B|K\}$ because $z'_S = 0$ and $z''_S = 1$. This is not desirable from a probabilistic point of view because it allows to coherently assess probability 1 for the quasi conjunction even if one of the two conjuncts has probability 1 and the other one has probability zero. A similar comment can be also done in the particular case where $H$ and $K$ are incompatible. Indeed, when $HK = \emptyset$, by instantiating equation (19) with $a = b = 1$, we obtain

$$z'_S = \min\{x, y\}, \quad z''_S = \max\{x, y\}. \tag{23}$$

### 4.3. The Case $a = b = 0$

In the case where $a = b = 0$ the generalized conjunction reduces to a conditional event. Indeed,

$$(A|H) \wedge_{0,0} (B|K) = AHBK|(H \vee K).$$

Based on Theorem 14, under logical independence, it follows that $(x, y, z)$ on $\{A|H, B|K, (A|H) \wedge_{0,0} (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where $z' = 0$ and $z'' = min\{x, y\}$.

## 5. Interval-valued Prevision Assessments

We recall that under logical independence any assessment $(x, y) \in [0, 1]^2$ on $\{A|H, B|K\}$ is coherent. Based on Theorem 14, we denote by $[z'(x, y), z''(x, y)]$ the interval of coherent prevision extensions of $(x, y)$ to $(A|H) \wedge_{a,b} (B|K)$. Given the interval-valued assessment $[x_1, x_2]\times[y_1, y_2] \subseteq [0, 1]^2$ on $\{A|H, B|K\}$, we set $[z^*, z^{**}]$ the interval of coherent extensions $z$ on $(A|H) \wedge_{a,b} (B|K)$, where

$$z^* = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z'(x, y)$$

and

$$z^{**} = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z''(x, y).$$

The assessment $([x_1, x_2] \times [y_1, y_2] \times [z^*, z^{**}])$ on $\mathcal{F} = \{A|H, (B|K), (A|H) \wedge_{a,b} (B|K)\}$ is coherent w.r.t. Definition 8. Moreover, any assessment $[x_1, x_2]\times[y_1, y_2]\times[\alpha, \beta]$, with $[\alpha, \beta] \supset [z^*, z^{**}]$, is not coherent. Then, the interval $[z^*, z^{**}]$ is the least committal extension, that is the natural extension ([54], see also [45]), of the interval-valued assessment $[x_1, x_2] \times [y_1, y_2]$, which is equivalent to the lower probability assessment $(x_1, x_2, 1 - y_1, 1 - y_2)$ on $\{A|H, B|K, \overline{A}|H, \overline{B}|K\}$. In what follows we compute the lower and upper bounds, $z^*$ and $z^{**}$, by also considering the case where $HK = \emptyset$.

**Theorem 20** *Let A, B, H, K be any logically independent events and let $\mathcal{A} = ([x_1, x_2] \times [y_1, y_2])$ be an interval-valued assessment on $\{A|H, B|K\}$. Then, the interval of coherent extensions of $\mathcal{A}$ to $(A|H) \wedge_{a,b} (B|K)$ is the interval $[z^*, z^{**}] = [z'(x_1, y_1), z''(x_2, y_2)]$, where $z'(x, y)$ and $z''(x, y)$ are defined in formula (13) and formula (14), respectively.*

**Proof** *The proof is straightforward by showing that*

$$z'(x_1, y_1) = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z'(x, y)$$

*and*

$$z''(x_2, y_2) = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z''(x, y).$$

*Due to the lack of space the rest of the proof is omitted.* ■

We now generalize Theorem 17 for interval-valued prevision assessments.

**Theorem 21** *Let an interval-valued probability assessment* $\mathcal{A} = ([x_1, x_2] \times [y_1, y_2])$ *on* $\{A|H, B|K\}$*, with* $H \neq \emptyset$*,* $K \neq \emptyset$*, and* $HK = \emptyset$*, be given. Then, the interval of coherent extensions of* $\mathcal{A}$ *to* $(A|H) \wedge_{a,b} (B|K)$ *is the interval* $[z^*, z^{**}]$*, where*

$$z^* = \min\{ay_1, bx_1\}, \quad z^{**} = \max\{ay_2, bx_2\}. \tag{24}$$

**Proof** The proof is straightforward by recalling that $(x, y) \in [0, 1]^2$ on $\{A|H, B|K\}$ is coherent when $HK = \emptyset$ and by observing that both the lower and upper bounds $z'$ and $z''$ given in Theorem 17 are non-decreasing functions in the arguments $x$ and $y$. ■

## 6. Further aspects on $(A|H) \wedge_{a,b} (B|K)$

In this section we deepen two further aspects of the generalized conjunction $(A|H) \wedge_{a,b} (B|K)$. We first give an interpretation of $(A|H) \wedge_{a,b} (B|K)$ when we consider two individuals $O$ and $O'$. Then, we examine $(A|H) \wedge_{a,b} (B|K)$, when $A|H \subseteq B|K$.

Let us consider two individuals $O$ and $O'$. Suppose that $O'$ asserts $P'(A|H) = a$ and $P'(B|K) = b$. Then, based on Remark 13, for $O'$ the conjunction $(A|H) \wedge_{a,b} (B|K)$ coincides with its conjunction $(AHBK + P'(A|H)\overline{H}BK + P'(B|K)AH\overline{K})|(H \vee K)$, which we denote by $(A|H) \wedge' (B|K)$. Thus, by coherence, $\mathbb{P}'[(A|H) \wedge_{a,b} (B|K)]$ satisfies the Fréchet-Hoeffding bounds, that is:

$$\begin{aligned} &\mathbb{P}'[(A|H) \wedge_{a,b} (B|K)] \\ &= \mathbb{P}'[(A|H) \wedge' (B|K)] \in [\max\{a+b-1, 0\}, \min\{a, b\}]. \end{aligned}$$

Now, suppose that $O$ asserts $P(A|H) = x$ and $P(B|K) = y$. Then, under logical independence, the lower and upper bounds $z'$ and $z''$ on $(A|H) \wedge_{a,b} (B|K)$ computed in Theorem 14, for the individual $O$, represent the lower and upper bounds for the coherent extension $\mathbb{P}[(A|H) \wedge' (B|K)]$ of the assessment $(x, y)$ on $\{A|H, B|K\}$. Therefore, in general it holds that

$$\begin{aligned} &\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] = \\ &= \mathbb{P}[(A|H) \wedge' (B|K)] \neq \mathbb{P}[(A|H) \wedge (B|K)], \end{aligned}$$

where $(A|H) \wedge (B|K) = (A|H) \wedge_{x,y} (B|K)$.

We recall that, when $A|H \subseteq B|K$, it holds that $(A|H) \wedge (B|K) = A|H$ (see, e.g., [29, formula (16)]) and hence $\mathbb{P}[(A|H) \wedge (B|K)] = P(A|H)$. These relations are not preserved by $(A|H) \wedge_{a,b} (B|K)$. Indeed, if $A|H \subseteq B|K$, as $AHBK = AH$, $\overline{K}AH = \emptyset$, and $\overline{H}BK = \overline{H}K$, from (10) it holds that

$$(A|H) \wedge_{a,b} (B|K) = (AH + a\overline{H}K)|(H \vee K) \neq A|H, \tag{25}$$

because, when $\overline{H}K$ is true, it follows that $(A|H) \wedge_{a,b} (B|K) = a$, while $A|H = x$, where $x = P(A|H)$. Moreover, it holds that

$$\begin{aligned} &\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] = \\ &= P(A|H)P(H|H \vee K) + aP(\overline{H}K|H \vee K) = \\ &= \alpha x + (1 - \alpha)a, \end{aligned}$$

where $x = P(A|H)$, $\alpha = P(H|H \vee K)$. Thus, $\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] \in [\min\{x, a\}, \max\{x, a\}]$, when $A|H \subseteq B|K$. In particular when $a = b = 1$, that is $(A|H) \wedge_{a,b} (B|K) = (A|H) \wedge_S (B|K)$, from (25) it follows that $(A|H) \wedge_{a,b} (B|K) = (AH \vee \overline{H}K)|(A \vee K) \supseteq (A|H)$ (see [25, Remark 4]).

## 7. Conclusions

We recalled the notion of conjunction of two conditional events $A|H$ and $B|K$ studied in the framework of conditional random quantities defined as $(A|H) \wedge (B|K) = [ABHK + x\overline{H}BK + yAH\overline{K}]|(H \vee K)$, where $x = P(A|H)$ and $y = P(B|K)$. We also recalled that this notion preserves some basic probabilistic properties and in particular the Fréchet-Hoeffding bounds are satisfied. We generalized the conjunction by replacing the values $x$ and $y$ with two arbitrary numbers $a$ and $b$ in $[0, 1]$. The values $a$ and $b$ represent the values of the conjunction of two conditional events when one conditional event is true, while the other one is void. Thus, we defined $(A|H) \wedge_{a,b} (B|K)$ as the conditional random $[ABHK + a\overline{H}BK + bAH\overline{K}]|(H \vee K)$. Then, for each given pair $(a, b)$, we computed the set of all coherent assessments on the family $\{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$. We observed that in the general case the Fréchet-Hoeffding bounds for the conjunction are not preserved. Then, we analyzed the particular case where the conditioning events $H$ and $K$ are incompatible, by also computing the corresponding set of coherent assessments. We observed that $\wedge_{a,b}$ reduces to the quasi conjunction $\wedge_S$, when $(a, b) = (1, 1)$. We considered the imprecise case, by computing the lower and upper bounds, $z^*$ and $z^{**}$, for the prevision of $(A|H) \wedge_{a,b} (B|K)$ in the case of an interval valued probability assessment $[x_1, x_2] \times [y_1, y_2]$ on $\{A|H, B|K\}$. Then, we analyzed further aspects on the generalized conjunction, by observing that $(A|H) \wedge_{a,b} (B|K)$ reduces to the conjunction $(A|H) \wedge' (B|K)$ of an individual $O'$, who asserts $P'(A|H) = a$ and $P'(B|K) = b$, while it is different from the conjunction $(A|H) \wedge (B|K)$ of an individual $O$, who asserts $(P(A|H), P(B|K)) \neq (a, b)$. Finally, we observed that, differently from $(A|H) \wedge_{a,b} (B|K)$, the conjunction $(A|H) \wedge_{a,b} (B|K)$ does not coincide in general with $A|H$, when $A|H \subseteq B|K$ . Further work should concern the study of some other properties of the generalized conjunction under some other logical dependencies, for instance when $A = B$. Moreover, we recall that the random win of a *double-bet* in soccer betting can be interpreted as a suitable conjunction $(A|H) \wedge (B|K)$, with $\mathbb{P}[(A|H) \wedge (B|K)] = P(A|H)P(B|K)$ ([18, Example 2.1]). Then, a real application of $(A|H) \wedge_{a,b} (B|K)$ can be given in the soccer betting framework, when, for instance, we consider the bookmaker and the bettor as the individuals $O'$ and $O$, respectively, discussed in Section 6.

Based on the notion of conjunction of $n$ conditional events given in [27], a suitable generalized version of conjunction (and iterated conditional) can be introduced also for $n$ conditional events.

## Acknowledgments

We thank the four anonymous reviewers for their helpful comments and suggestions. We also thank A. Gilio for insightful discussions and comments. L. Castronovo and G. Sanfilippo are supported by the INdAM-GNAMPA Research Group. G. Sanfilippo is also supported by the FFR 2023 project of University of Palermo, Italy.

## Author Contributions

Both authors contributed equally to the article.

## References

[1] E. W. Adams. *The logic of conditionals*. Reidel, Dordrecht, 1975.

[2] V. Biazzo, A. Gilio, and G. Sanfilippo. Generalized coherence and connection property of imprecise conditional previsions. In *Proc. IPMU 2008, Malaga, Spain, June 22 - 27*, pages 907–914, 2008.

[3] P. G. Calabrese. An algebraic synthesis of the foundations of logic and probability. *Information Sciences*, 42(3):187 – 237, 1987.

[4] P. G. Calabrese. Deduction with uncertain conditionals (revised & simplified, with examples). *Heliyon*, 7(11): e08328, 2021. ISSN 2405-8440.

[5] J. Cantwell. Revisiting McGee's probabilistic analysis of conditionals. *Journal of Philosophical Logic*, 51(5): 973–1017, 2022. doi:10.1007/s10992-022-09657-5.

[6] L. Castronovo and G. Sanfilippo. Iterated conditionals, trivalent logics, and conditional random quantities. In M. Dupin, M. Öztürk, and N. Potyka, editors, *15th International Conference on Scalable Uncertainty Management (SUM 2022), October 17-19, 2022, Paris, France*, volume 13562 of *LNCS*, pages 47–63. Springer, 2022.

[7] D. Ciucci and D. Dubois. A map of dependencies among three-valued logics. *Information Sciences*, 250: 162 – 177, 2013. ISSN 0020-0255.

[8] G. Coletti and R. Scozzafava. Conditioning and inference in intelligent systems. *Soft Computing*, 3(3): 118–130, 1999.

[9] G. Coletti, R. Scozzafava, and B. Vantaggi. Coherent conditional probability, fuzzy inclusion and default rules. In *Soft Computing: State of the Art Theory and Novel Applications*, pages 193–208. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.

[10] G. Coletti, R. Scozzafava, and B. Vantaggi. Possibilistic and probabilistic logic under coherence: Default reasoning and System P. *Mathematica Slovaca*, 65 (4):863–890, 2015.

[11] G. Coletti, D. Petturiti, and B. Vantaggi. Dutch book rationality conditions for conditional preferences under ambiguity. *Ann. Oper. Res.*, 279(1-2):115–150, 2019.

[12] B. de Finetti. La Logique de la Probabilité. In *Actes du Congrès International de Philosophie Scientifique, Paris, 1935*, pages IV 1–IV 9. Hermann et C.ie, Paris, 1936.

[13] I. Douven, S. Elqayam, H. Singmann, and J. van Wijnbergen-Huitink. Conditionals and inferential connections: toward a new semantics. *Thinking & Reasoning*, pages 1–41, 2019.

[14] D. Dubois and H. Prade. Conditional objects as nonmonotonic consequence relationships. *IEEE Trans. on Syst. Man and Cybernetics,*, 24(12):1724 –1740, dec 1994. ISSN 0018-9472.

[15] D. Edgington. On conditionals. *Mind*, 104:235–329, 1995.

[16] P. Egré, L. Rossi, and J. Sprenger. De Finettian Logics of Indicative Conditionals Part I: Trivalent Semantics and Validity. *J Philos Logic*, 2020.

[17] T. Flaminio, L. Godo, and H. Hosni. Boolean algebras of conditionals, probability and logic. *Artificial Intelligence*, 286:103347, 2020.

[18] T. Flaminio, A. Gilio, L. Godo, and G. Sanfilippo. Compound Conditionals as Random Quantities and Boolean Algebras. In Gabriele Kern-Isberner, Gerhard Lakemeyer, and Thomas Meyer, editors, *Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning, KR 2022, Haifa, Israel. July 31 - August 5, 2022*, pages 141–151, 2022.

[19] T. Flaminio, A. Gilio, L. Godo, and G. Sanfilippo. On conditional probabilities and their canonical extensions to Boolean algebras of compound conditionals. *Int. J. Approx. Reason.*, 159:108943, 2023. doi:10.1016/j.ijar.2023.108943.

[20] A. Gilio. Probabilistic reasoning under coherence in System P. *Annals of Mathematics and Artificial Intelligence*, 34:5–34, 2002.

[21] A. Gilio. Generalizing inference rules in a coherence-based probabilistic default reasoning. *Int. J. Approx. Reason.*, 53(3):413–434, 2012. ISSN 0888-613X. doi:10.1016/j.ijar.2011.08.004.

[22] A. Gilio and S. Ingrassia. Totally coherent set-valued probability assessments. *Kybernetika*, 34(1):3–15, 1998.

[23] A. Gilio and G. Sanfilippo. Subjective probability, trivalent logics and compound conditionals. Under revision. Preprint: `http://arxiv.org/abs/2301.09327`.

[24] A. Gilio and G. Sanfilippo. Conjunction, disjunction and iterated conditioning of conditional events. In *Synergies of Soft Computing and Statistics for Intelligent Data Analysis*, volume 190 of *AISC*, pages 399–407. Springer, Berlin, 2013.

[25] A. Gilio and G. Sanfilippo. Quasi conjunction, quasi disjunction, t-norms and t-conorms: Probabilistic aspects. *Information Sciences*, 245:146–167, 2013.

[26] A. Gilio and G. Sanfilippo. Conditional random quantities and compounds of conditionals. *Studia Logica*, 102(4):709–729, 2014.

[27] A. Gilio and G. Sanfilippo. Generalized logical operations among conditional events. *Applied Intelligence*, 49(1):79–102, Jan 2019.

[28] A. Gilio and G. Sanfilippo. Algebraic aspects and coherence conditions for conjoined and disjoined conditionals. *Int. J. Approx. Reason.*, 126:98 – 123, 2020.

[29] A. Gilio and G. Sanfilippo. On compound and iterated conditionals. *Argumenta*, 6(2):241–266, 2021.

[30] A. Gilio and G. Sanfilippo. Iterated conditionals and characterization of p-entailment. In *Proc. of ECSQARU 2021*, volume 12897 of *LNCS*, pages 629–643. Springer, 2021.

[31] A. Gilio and G. Sanfilippo. Compound conditionals, Fréchet-Hoeffding bounds, and Frank t-norms. *Int. J. Approx. Reason.*, 136:168–200, 2021. ISSN 0888-613X.

[32] A. Gilio, N. Pfeifer, and G. Sanfilippo. Transitivity in coherence-based probability logic. *Journal of Applied Logic*, 14:46–64, 2016.

[33] I. R. Goodman and H. T. Nguyen. Conditional Objects and the Modeling of Uncertainties. In M. M. Gupta and T. Yamakawa, editors, *Fuzzy Computing*, pages 119–138. North-Holland, 1988.

[34] I. R. Goodman, H. T. Nguyen, and E. A. Walker. *Conditional Inference and Logic for Intelligent Systems: A Theory of Measure-Free Conditioning*. North-Holland, 1991. ISBN 978-0-444-88685-9.

[35] S. Holzer. On coherence and conditional prevision. *Bollettino dell'Unione Matematica Italiana*, 4(6):441–460, 1985.

[36] S. Kaufmann. Conditionals right and left: Probabilities for the whole family. *Journal of Philosophical Logic*, 38:1–53, 2009. ISSN 0022-3611.

[37] G. Kern-Isberner. *Conditionals in Nonmonotonic Reasoning and Belief Revision – Considering Conditionals as Agents*, volume 2087. LNAI, Springer, 2001.

[38] F. Lad. *Operational subjective statistical methods: A mathematical, philosophical, and historical introduction*. Wiley, New York, 1996.

[39] V. McGee. Conditional probabilities and compounds of conditionals. *Philosophical Review*, 98(4):485–541, 1989.

[40] P. Milne. Bruno de Finetti and the Logic of Conditional Events. *British Journal for the Philosophy of Science*, 48(2):195–232, 1997. URL `http://www.jstor.org/stable/687745`.

[41] E. Miranda and M. Zaffalon. Notes on desirability and conditional lower previsions. *Ann. Math. Artif. Intell.*, 60(3-4):251–309, 2010.

[42] A. Mura. Towards a new logic of indicative conditionals. *Logic and Philosophy of Science*, IX:17–31, 2011.

[43] D.E. Over and N. Cruz. Suppositional theory of conditionals and rationality. In Markus Knauff and Wolfgang Spohn, editors, *Handbook of rationality*, pages 395–404. MIT press, 2021.

[44] R. Pelessoni and P. Vicig. The Goodman-Nguyen relation within imprecise probability theory. *International Journal of Approximate Reasoning*, 55(8): 1694–1707, 2014. doi:10.1016/j.ijar.2014.06.002.

[45] D. Petturiti and B. Vantaggi. Modeling agent's conditional preferences under objective ambiguity in dempster-shafer theory. *Int. J. Approx. Reason.*, 119: 151–176, 2020. ISSN 0888-613X.

[46] N. Pfeifer and G. Sanfilippo. Probabilistic squares and hexagons of opposition under coherence. *International Journal of Approximate Reasoning*, 88: 282–294, 2017. doi:10.1016/j.ijar.2017.05.014.

[47] N. Pfeifer and G. Sanfilippo. Interpreting connexive principles in coherence-based probability logic. In *Proc. of ECSQARU 2021*, volume 12897 of *LNCS*, pages 672–687. Springer, 2021.

[48] N. Pfeifer and G. Sanfilippo. Connexive logic, probabilistic default reasoning, and compound conditionals. *Studia Logica*, page To appear, 2023.

[49] E. Regazzini. Finitely additive conditional probabilities. *Rendiconti del Seminario Matematico e Fisico di Milano*, 55:69–89, 1985.

[50] G. Sanfilippo. Lower and upper probability bounds for some conjunctions of two conditional events. In *SUM 2018*, volume 11142 of *LNCS*, pages 260–275. Springer, 2018. ISBN 978-3-030-00460-6.

[51] G. Sanfilippo, N. Pfeifer, D.E. Over, and A. Gilio. Probabilistic inferences from conjoined to iterated conditionals. *Int. J. Approx. Reason.*, 93(Supplement C):103 – 118, 2018. ISSN 0888-613X.

[52] G. Sanfilippo, A. Gilio, D.E. Over, and N. Pfeifer. Probabilities of conditionals and previsions of iterated conditionals. *Int. J. Approx. Reason.*, 121:150 – 173, 2020. ISSN 0888-613X.

[53] B.C. van Fraassen. Probabilities of conditionals. *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science*, 1:261–308, 1976. cited By 89.

[54] P. Walley. Coherent upper and lower previsions, 1997. The Imprecise Probabilities Project.

[55] P.M. Williams. Notes on conditional previsions. *International Journal of Approximate Reasoning*, 44(3): 366–383, 2007.