Indistinguishability through Exchangeability in Quantum Mechanics?

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Abstract

Arguments in quantum mechanics often involve systems of indistinguishable particles, such as electrons or photons. On the standard approach, the symmetrisation postulate is needed to model indistinguishable particles, and results in a theory of fermions and bosons. We investigate how indistinguishability can be implemented by incorporating structural assessments of symmetry in the sets of desirable measurements approach to uncertainty modelling in quantum mechanics, which is based on the theory of imprecise probabilities, and in particular on sets of desirable gambles. We show that an exchangeability assessment allows us to partially retrieve the concepts of fermions and bosons, but that in order to recover the complete fermion and boson framework, we need to rely on stronger symmetry assessments. We also lay bare the relationship between these stronger assessments and the count vector representation for sets of desirable measurements, which we argue corresponds to the commonly used second quantisation in quantum mechanics.

Keywords: exchangeability, quantum mechanics, indistinguishability, desirable measurements, strong symmetry, second quantisation

1. Introduction

Bose-Einstein condensates, superconductivity, Pauli's exclusion principle and Bose–Einstein vs. Fermi–Dirac statistics are key concepts in physics, with important technological applications [11, 19, 20]. They all result from a single *indistinguishability principle*, which states that it's physically impossible to distinguish certain particles from one another. This happens when they're identical, that is, when they have the same physical properties, such as charge, mass, spin, and so forth. Such particles include neutrons, electrons, protons, photons, ... in fact all fundamental particles.

Here, we examine how indistinguishability can be incorporated into the framework of desirable measurements—which goes back to Ref. [6] and which we recently explored and tried to justify in Ref. [18]—and to what conclusions it leads. In that earlier work on desirable measurements, we developed a decision-theoretic argument involving imprecise probabilities to model the uncertainty about a quantum system's state. This led to a similar mathematical framework as that first introduced by Benavoli et al. [6], but with a different interpretation. Our argument there proceeds along the following lines. The system, which is in an unknown state $|\Psi\rangle$ in the state space $\bar{\mathscr{X}}$, can be interacted with by performing measurements, represented by Hermitian operators. With any such measurement operator $\hat{A}$, we associate a utility function $u_{\hat{A}}\colon \bar{\mathscr{X}} \to \mathbb{R}$, which represents the reward associated with executing that measurement: if the system is in state $|\psi\rangle$, then $u_{\hat{A}}(|\phi\rangle)$ is the utility associated with performing the measurement $\hat{A}$, in the sense that performing measurement $\hat{A}$ on the system in the unknown state $|\Psi\rangle$ results in an uncertain reward $u_{\hat{A}}(|\Psi\rangle)$. Using a decision-theoretic approach that relies on the non-probabilistic quantum mechanical postulates, we then argue that this utility function must have the form $u_{\hat{A}}(|\phi\rangle) = \langle\phi|\hat{A}|\phi\rangle$ for all $|\phi\rangle \in \bar{\mathscr{X}}$. A rational subject—called You—expresses beliefs about the unknown state $|\Psi\rangle$ by expressing a preference between measurements $\hat{A}$ and $\hat{B}$, through a preference between their associated uncertain rewards $u_{\hat{A}}(|\Psi\rangle)$ and $u_{\hat{B}}(|\Psi\rangle)$. Such a (partial) preference ordering on measurements is therefore a model for Your uncertainty about $|\Psi\rangle$. Equivalently, You can use a so-called *set of desirable measurements*, which are those measurements You prefer to the status quo, or in other words, the null measurement. Modelling uncertainty in this way is a fairly direct application of the sets of desirable gambles approach that is by now common in imprecise probabilities research; see also Refs. [3, 12, 13, 24, 25, 28, 29, 30].

How does indistinguishability fit into this desirable measurements framework? Can we retrieve the standard framework of fermions and bosons by imposing structural symmetry assessments, similarly to what is done in the non-quantum-mechanical desirable gambles approach [14, 16, 23, 28]? After a concise introduction to the desirable measurements framework in Section 2, we begin to answer these questions in Section 3, where we incorporate exchangeability assessments into this framework. This concept of exchangeability has its roots in the work of de Finetti [17], and his well-known representation theorem, where an exchangeability assessment is taken to mean that the order of a sequence of random variables is irrelevant for inferences. In Ref. [9], de Finetti's representation theorem is generalised to quantum mechanics in the context of quantum-state tomography, a technique for estimating the system's state by performing measurements on multiple indistinguishable copies. On our approach, in contrast, an assessment of exchangeability dictates that the order in which the particles of a quantum system are considered should not matter for inferences and decision-making. We incorporate this assessment into the framework for desirable measurements using the approach first suggested by De Cooman and Quaeghebeur [15, 28]. In Section 4, we show that this allows us to retrieve some, but not all, aspects of the fermion and boson framework. Since exchangeability turns out to be too weak for this, we move in Section 5 to stronger symmetry assessments to achieve this goal. This stronger symmetry is somewhat reminiscent of, yet not identical to, the strong invariance under generalised permutations proposed by Benavoli et al. [7], who investigated a different approach to dealing with indistinguishable particles, also inspired by De Cooman and Quaeghebeur's [15, 28] work on exchangeability. We briefly discuss second quantisation in Section 6. Proofs are gathered in the supplementary materials.

2. Desirability in Quantum Mechanics

Let's first revisit some of the more relevant and important concepts in the sets of desirable measurements framework first introduced in Ref. [6], and which we provided a decision-theoretic justification for in Ref. [18]. A more extensive account of this framework for dealing with uncertainty in quantum mechanics is currently in the works.

2.1. Quantum Mechanics

The framework is based on combining ideas from decision theory with the non-probabilistic principles of quantum mechanics. For an account of the foundations of quantum mechanics, see Refs. [10, 27]. We base our argument on the following principles. The state $|\psi\rangle$ of a quantum system is a normalised element of a complex *Hilbert space* $\mathscr{X}$. In order to deal with certain aspects of quantum-mechanical systems, such as location, infinite-dimensional Hilbert spaces are essential. But to keep the discussion here as simple as possible, *we'll restrict ourselves to the case of finite-dimensional Hilbert spaces*, which can for instance be used to model such aspects as the spin or (with some extra assumptions, such as ignoring the higher energy levels) the energy of a bounded electron. Such finite-dimensional spaces are particularly useful in quantum computing and quantum cryptography [27]. We'll use the Dirac notation: a *ket* $|\psi\rangle$ is a vector in $\mathscr{X}$, and the bra $\langle\psi|$ its adjoint. The *state space* is the set $\bar{\mathscr{X}}$ of all normalised kets. A measurement on the system is represented by a Hermitian operator $\hat{A} := \sum_{k=1}^{n} \lambda_k |a_k\rangle\langle a_k|$ on $\mathscr{X}$, with $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ the eigenvalues and $|a_1\rangle, \dots, |a_n\rangle \in \bar{\mathscr{X}}$ corresponding pairwise orthogonal eigenkets. The possible results of such a measurement are the eigenvalues $\lambda_1, \dots, \lambda_n$. The *spectrum* $\operatorname{spec}(\hat{A}) := \{\lambda_1, \dots, \lambda_n\}$ of A is the set of its eigenvalues. We denote the real linear space of all such Hermitian operators by $\mathscr{H}(\mathscr{X})$, or simply $\mathscr{H}$ if no confusion is possible.

2.2. Utility Functions

We want a suitable representation for Your beliefs about the unknown quantum mechanical state $|\Psi\rangle$ of a system. You can interact with the system through measurements $\hat{A} \in \mathscr{H}$, which we can see as possible acts or options. As is common in decision theory [2, 4, 5, 13, 17, 26, 30, 31], Your uncertainty will be described by Your preferences between these different acts, and we attach to each such act/measurement $\hat{A}$ a utility function $u_{\hat{A}} \colon \bar{\mathscr{X}} \to \mathbb{R}$, where $u_{\hat{A}}(|\phi\rangle)$ is the reward associated with performing the measurement $\hat{A}$ when the system is in state $|\phi\rangle$, expressed in units of some linear utility.

Interestingly, we have argued [18] that a number of decision-theoretic principles that rely on the non-probabilistic postulates of quantum mechanics, leave You no choice about which utility functions $u_{\hat{A}}$ to use: they unequivocally determine them to take the form $u_{\hat{A}}(|\phi\rangle) = \langle\phi|\hat{A}|\phi\rangle$ for all $|\phi\rangle \in \bar{\mathscr{X}}$. The linear space $\mathscr{U}$ of all utility functions is therefore linearly isomorphic to the real linear space $\mathscr{H}$: we can identify measurements and their utility functions.

2.3. Desirability

Your uncertainty about the system's unknown state $|\Psi\rangle$ is now modelled through a partial strict preference ordering between uncertain rewards $u_{\hat{A}}(|\Psi\rangle)$, which is equivalent to a partial strict preference ordering on the linear space $\mathscr{H}$. A mathematically equivalent model for such a preference relation is a set of desirable utility functions: those uncertain rewards that You strictly prefer to the zero utility function 0, or equivalently, a *set of desirable measurements*: those measurements that You strictly prefer to the status quo $\hat{0}$.[1] Commonly, rationality criteria are then imposed on such a set of desirable utility functions [3, 13, 30]. As measurements can be identified with their utility functions, we can readily translate these to the desirable measurements framework. We call a set of desirable measurements $\mathscr{D}$ *coherent* if for all $\hat{A}, \hat{B} \in \mathscr{D}$ and all $\lambda \in \mathbb{R}_{>0}$:

D1. $\hat{0} \notin \mathscr{D}$; [strictness]

[1] The operator $\hat{0}$ is the unique Hermitian operator all of whose eigenvalues are zero, and whose utility function $u_{\hat{0}}$ is identically zero.

D2. $\mathscr{H}_{>\hat{0}} \subseteq \mathscr{D}$; [accepting sure gain[2]]

D3. $\hat{A}, \hat{B} \in \mathscr{D} \Rightarrow \hat{A} + \hat{B} \in \mathscr{D}$; [additivity]

D4. $\hat{A} \in \mathscr{D} \Rightarrow \lambda\hat{A} \in \mathscr{D}$. [positive scaling]

Here, $\mathscr{H}_{>\hat{0}} := \{\hat{A} \in \mathscr{H} : \hat{A} > \hat{0}\}$ is the set of *positive definite* measurements, whose eigenvalues are (strictly) positive. We'll denote by $\mathscr{H}_{<\hat{0}} := \{\hat{A} \in \mathscr{H} : \hat{A} < \hat{0}\} = -\mathscr{H}_{>\hat{0}}$ the set of *negative definite* measurements.

In Ref. [6], a similar framework involving sets of desirable measurements was used, with a different interpretation and justification, and with a slightly stronger version of the rationality criterion D2.

One of the interesting aspects of working with partial preference models in the form of coherent sets of desirable measurements, is that they allow for conservative inference; for details, see for instance Ref. [30, Section 3.7].

2.4. Coherent (Lower and Upper) Previsions

With a set of desirable measurements, we can associate a lower prevision $\underline{\varLambda}$ and an upper prevision $\overline{\varLambda}$ as follows:[3]

$$\underline{\varLambda}_{\mathscr{D}}(\hat{A}) := \sup\{\alpha \in \mathbb{R} : \hat{A} - \alpha\hat{I} \in \mathscr{D}\} \text{ for all } \hat{A} \in \mathscr{H}, \quad (1)$$

$$\overline{\varLambda}_{\mathscr{D}}(\hat{A}) := \inf\{\alpha \in \mathbb{R} : \alpha\hat{I} - \hat{A} \in \mathscr{D}\} \text{ for all } \hat{A} \in \mathscr{H}. \quad (2)$$

The lower prevision $\underline{\varLambda}_{\mathscr{D}}(\hat{A})$ is Your supremum buying price for the measurement $\hat{A}$ or equivalently, Your supremum buying price for the uncertain reward $u_{\hat{A}}(|\Psi\rangle)$. The upper prevision $\overline{\varLambda}_{\mathscr{D}}(\hat{A})$ is Your infimum selling price for the uncertain reward $u_{\hat{A}}(|\Psi\rangle)$. Observe that $\underline{\varLambda}_{\mathscr{D}}(\hat{A}) = -\overline{\varLambda}_{\mathscr{D}}(-\hat{A})$. It's well-known that the coherent lower prevision $\underline{\varLambda}_{\mathscr{D}}$ fully characterises the coherent set $\mathscr{D}$ up to border behaviour; see for instance Ref. [30, Section 3.8]. In this sense, lower previsions and sets of desirable measurements are (almost) equivalent mathematical models for Your beliefs.

A real functional $\underline{\varLambda}$ on $\mathscr{H}$ is called a *coherent lower prevision* if there's some coherent set of desirable measurements $\mathscr{D}$ such that $\underline{\varLambda} = \underline{\varLambda}_{\mathscr{D}}$. It's then a standard result that the coherence of a lower prevision is characterised by the following properties:[4] for any $\hat{A}, \hat{B} \in \mathscr{H}$ and $\lambda \in \mathbb{R}_{\geq 0}$,

LP1. $\underline{\varLambda}(\hat{A} + \hat{B}) \geq \underline{\varLambda}(\hat{A}) + \underline{\varLambda}(\hat{B})$; [super-additivity]

LP2. $\underline{\varLambda}(\lambda\hat{A}) = \lambda\underline{\varLambda}(\hat{A})$; [non-negative homogeneity]

LP3. $\underline{\varLambda}(\hat{A}) \geq \min\operatorname{spec}(\hat{A})$.[5] [accepting sure gains]

If we denote the *conjugate* upper prevision by $\overline{\varLambda}$, where $\overline{\varLambda}(\bullet) := -\underline{\varLambda}(-\bullet)$, then the following properties are also satisfied for all $\hat{A}, \hat{B} \in \mathscr{H}$ and all $\mu \in \mathbb{R}$:

LP4. $\min\operatorname{spec}(\hat{A}) \leq \underline{\varLambda}(\hat{A}) \leq \overline{\varLambda}(\hat{A}) \leq \max\operatorname{spec}(\hat{A})$);

LP5. $\underline{\varLambda}(\hat{A}) + \underline{\varLambda}(\hat{B}) \leq \underline{\varLambda}(\hat{A} + \hat{B}) \leq \underline{\varLambda}(\hat{A}) + \overline{\varLambda}(\hat{B})$;

LP6. $\underline{\varLambda}(\hat{A} + \mu\hat{I}) = \underline{\varLambda}(\hat{A}) + \mu$.

When a coherent lower prevision is self-conjugate, so $\underline{\varLambda} = \overline{\varLambda}$, then we call it a *linear prevision*, or a coherent prevision, and simply denote it as $\varLambda$. We can associate with every coherent lower prevision $\underline{\varLambda}$ the following closed[6] convex set of dominating linear previsions

$$\mathscr{M}_{\underline{\varLambda}} := \{\varLambda : (\forall\hat{A} \in \mathscr{H})\varLambda(\hat{A}) \geq \underline{\varLambda}(\hat{A})\},$$

also called the associated *credal set*. A straightforward application of the Hahn–Banach Theorem then tells us that a real bounded functional $\underline{\varLambda}$ is a coherent lower prevision if and only if it's the lower envelope of the associated credal set $\mathscr{M}_{\underline{\varLambda}}$, or in other words, if and only if $\underline{\varLambda}(\hat{A}) = \min\{\varLambda(\hat{A}) : \varLambda \in \mathscr{M}_{\underline{\varLambda}}\}$ for all $\hat{A} \in \mathscr{H}$; see Refs. [3, Propositions 2.3 and 2.4] and [13]. Since lower previsions are equivalent to desirable gambles up to border behaviour, and since credal sets are equivalent to lower previsions, all three types of models can be used to describe Your beliefs.

2.5. Density Operators

In the standard framework for dealing with probability in quantum mechanics, the (epistemic) uncertainty about a system's state $|\Psi\rangle$ is usually modelled by a (positive) probability mass function $p_1, \ldots, p_r$ over possible states $|\psi_1\rangle, \ldots, |\psi_r\rangle$. Such an 'uncertain state' is called a *mixed state*, and corresponds to a so-called *density operator* $\hat{\rho} := \sum_{k=1}^{r} p_k|\psi_k\rangle\langle\psi_k|$. The set of all such density operators is denoted by $\mathscr{R}$. The following is then a basic result; see Ref. [27, Theorem 2.5].

Proposition 1 *A linear operator $\hat{\rho}$ on $\mathscr{X}$ is a density operator if and only if it's a Hermitian operator such that* $\operatorname{Tr}(\hat{\rho}) = 1$[7] *and* $\hat{\rho} \geq \hat{0}$.[8]

According to the standard probabilistic postulate in quantum mechanics, *Born's rule*, the expected outcome of a measurement $\hat{A}$ is then $\mathrm{E}_{\hat{\rho}}(\hat{A}) = \operatorname{Tr}(\hat{\rho}\hat{A})$. While the sets of desirable measurements approach doesn't start from the assumption that there are probabilities in quantum mechanics, nor relies on anything remotely related to Born's rule, it does allow

[2] Often in similar contexts, a somewhat stronger requirement, such as accepting partial gains, is imposed. We need the weaker requirement here for our discussion on strong symmetry further on.

[3] $\hat{I}$ is the identity operator, defined by $\hat{I}|\psi\rangle := |\psi\rangle$ for all $|\psi\rangle \in \mathscr{X}$.

[4] See for example Refs. [3, C1–C3 and Proposition 2.2] and [13, 30]. There, this result is written in terms of gambles, which correspond to our utility functions, or equivalently, to their corresponding measurements.

[5] This is equivalent to C3 in Ref. [3], as $\min u_{\hat{A}} = \min\operatorname{spec}(\hat{A})$ [18].

[6] ... in the weak* topology (of point-wise convergence) [13].

[7] The trace $\operatorname{Tr}(\hat{A})$ of the Hermitian operator $\hat{A}$ is the sum of its eigenvalues. Given an orthonormal basis $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ of $\mathscr{X}$, the trace can also be written as $\operatorname{Tr}(\hat{A}) = \sum_{k=1}^{n}\langle\psi_k|\hat{A}|\psi_k\rangle$.

[8] By $\hat{\rho} \geq \hat{0}$, we mean that $\hat{\rho}$ is *positive semi-definite*, or in other words that its eigenvalues are non-negative.

us to recover density operators and the trace formula as a special case, as formalised in the following result; see Refs. [6, p. 19] and [18].

Theorem 2 *A real functional Λ on $\mathscr{H}$ is a linear prevision if and only if there's some (then* unique*) density operator $\hat{\rho}_\Lambda \in \mathscr{R}$ such that $\Lambda(\hat{A}) = \mathrm{Tr}(\hat{\rho}_\Lambda \hat{A})$ for all $\hat{A} \in \mathscr{H}$.*

Therefore, as Your beliefs about $|\Psi\rangle$ can be described by a credal set $\mathscr{M}_{\underline{\Lambda}}$, we can equivalently describe them using a convex closed[9] set of density operators:

$$\begin{aligned}\mathscr{R}_{\underline{\Lambda}} &:= \{\hat{\rho}_\Lambda \colon \Lambda \in \mathscr{M}_{\underline{\Lambda}}\}\\ &= \{\hat{\rho} \in \mathscr{R} \colon (\forall \hat{A} \in \mathscr{H})\, \mathrm{Tr}(\hat{\rho}\hat{A}) \geq \underline{\Lambda}(\hat{A})\},\end{aligned} \tag{3}$$

and $\underline{\Lambda}(\hat{A}) = \min\{\mathrm{Tr}(\hat{\rho}\hat{A}) \colon \hat{\rho} \in \mathscr{R}_{\underline{\Lambda}}\}$ for all $\hat{A} \in \mathscr{H}$.

3. Exchangeability

How does indistinguishability fit into this framework? For identical particles, such as electrons, that are close in space, there's no way to tell them apart, in the sense that there's no physical experiment able to distinguish between them. In standard quantum mechanics, a separate postulate is used to describe the effect of such particles, which leads to the well-known framework of bosons and fermions.

On our approach, we'll now first try and model indistinguishability through an *exchangeability* assessment. Conceptually, a collection of variables is exchangeable when the order in which they're observed has no bearing on the inferences based on these observations [17].

How can we apply this idea to dealing with indistinguishable particles? Instead of considering the order in a sequence of observations, we'll focus on the order of the particles in the mathematical description of Your belief model for the state $|\Psi\rangle$, and require that the order of the particles in this description should be irrelevant for inferences based on that model. Mathematically speaking, we'll use existing ideas [15, 16, 28] for dealing with exchangeability in conjunction with coherent sets of desirable gambles and lower previsions, and see how they can be brought to bear on the desirable measurements framework.

Concretely, consider a system with m indistinguishable particles, each of which is in an unknown state in a copy of the same n-dimensional state space $\bar{\mathscr{X}}^n$, with some basis $|\phi_1\rangle, \dots, |\phi_n\rangle$. The corresponding Hilbert space for this system, $\mathscr{X} := \otimes_{k=1}^m \mathscr{X}^n$,[10] is n^m dimensional. A generic element of this space is given by $|\psi\rangle = \sum_{\ell_1,\dots,\ell_m=1}^n \alpha_{\ell_1,\dots,\ell_m} \otimes_{k=1}^m |\phi_{\ell_k}\rangle$, $\alpha_{\ell_1,\dots,\ell_m} \in \mathbb{C}$, and the unknown system state $|\Psi\rangle$ is some normalised ket in $\bar{\mathscr{X}}$.

We denote the set of all permutations π of the index set $\{1, \dots, m\}$ by $\mathbb{P}$, and define the linear *permutation operator* $\hat{\Pi}_\pi$ on $\mathscr{X}$ corresponding to the permutation π of the indices, and thus the particles, through

$$\hat{\Pi}_\pi := \sum_{\ell_1,\dots,\ell_m=1}^n \otimes_{k=1}^m \left(|\phi_{\ell_{\pi(k)}}\rangle\langle\phi_{\ell_k}|\right).$$

Observe that, by the properties of the tensor product $\otimes$,

$$\begin{aligned}\hat{\Pi}_\pi^\dagger &= \sum_{\ell_1,\dots,\ell_m=1}^n \otimes_{k=1}^m \left(|\phi_{\ell_k}\rangle\langle\phi_{\ell_{\pi(k)}}|\right)\\ &= \sum_{\ell_1,\dots,\ell_m=1}^n \otimes_{k=1}^m \left(|\phi_{\ell_{\pi^{-1}(k)}}\rangle\langle\phi_{\ell_k}|\right) = \hat{\Pi}_{\pi^{-1}},\end{aligned}$$

so $\hat{\Pi}_\pi^\dagger \hat{\Pi}_\pi = \hat{I}$, and $\hat{\Pi}_\pi$ is unitary. Your assessment that the particles are exchangeable[11] means that You're indifferent between receiving the uncertain reward $u_{\hat{A}}(|\Psi\rangle)$ for any measurement $\hat{A}$ in the unknown state $|\Psi\rangle$ and the uncertain reward $u_{\hat{A}}(\hat{\Pi}_\pi|\Psi\rangle)$ for that measurement in the permuted unknown state $\hat{\Pi}_\pi|\Psi\rangle$, as the order of the particles should then be irrelevant to You. Now, for any $|\psi\rangle \in \bar{\mathscr{X}}$,

$$u_{\hat{A}}(\hat{\Pi}_\pi|\psi\rangle) = \langle\psi|\hat{\Pi}_\pi^\dagger \hat{A} \hat{\Pi}_\pi|\psi\rangle = u_{\hat{\Pi}_\pi^\dagger \hat{A} \hat{\Pi}_\pi}(|\psi\rangle),$$

and it'll be useful for what follows to define the linear operators π^t and $\overline{\pi}^t$ on the real linear space $\mathscr{H}$ by letting $\pi^t \hat{A} := \hat{\Pi}_\pi^\dagger \hat{A} \hat{\Pi}_\pi$ and $\overline{\pi}^t \hat{A} := \hat{\Pi}_\pi \hat{A} \hat{\Pi}_\pi^\dagger$ for all $\hat{A} \in \mathscr{H}$. So You're indifferent between the uncertain rewards $u_{\hat{A}}(|\Psi\rangle)$ and $u_{\pi^t \hat{A}}(|\Psi\rangle)$, or equivalently, You're indifferent between the uncertain reward $u_{\hat{A}-\pi^t\hat{A}}(|\Psi\rangle) = u_{\hat{A}}(|\Psi\rangle) - u_{\pi^t\hat{A}}(|\Psi\rangle)$ and the status quo 0. Our identification of measurements with their utility functions leads us to say that Your exchangeability assessment makes You indifferent between the measurements $\hat{A}$ and their permutations $\pi^t\hat{A}$, or equivalently, between the measurements $\hat{A} - \pi^t\hat{A}$ and the status quo $\hat{0}$, for all $\hat{A} \in \mathscr{H}$ and all $\pi \in \mathbb{P}$. This leads to a set of so-called *indifferent* measurements

$$\mathscr{I} := \left\{\hat{A} - \pi^t\hat{A} \colon \hat{A} \in \mathscr{H} \text{ and } \pi \in \mathbb{P}\right\}.$$

As in Ref. [15], we now call a set of desirable measurements $\mathscr{D}$ *exchangeable* if[12]

$$\mathscr{D} + \mathscr{I} \subseteq \mathscr{D}. \tag{4}$$

[9]... in the topology that is isomorphic to the weak* topology on the space of linear functionals.

[10]The tensor product $\otimes$ is typically used to describe composite systems, the reader can find details about this tensor product, its properties and its use in quantum mechanics in Ref. [27, Sections 2.1.7 and 2.2.8].

[11]We use the term 'exchangeable' for a finite collection of particles, as is common in the statistical and imprecise probabilities literature dealing with the exchangeability of sequences of observations, but less so in the literature on quantum state tomography; see for instance Ref. [9], where finitely exchangeable system copies are called 'symmetric', and where the term 'exchangeable' is reserved for infinite sequences of system copies.

[12]We use the Minskovski sum for sets. Furthermore, since $\hat{0} \in \mathscr{I}$, this condition is equivalent to $\mathscr{D} + \mathscr{I} = \mathscr{D}$.

Intuitively, this expresses that if a measurement is desirable, then adding a measurement that is indifferent—equivalent to the status quo—will preserve its desirability. See also Ref. [28] for a detailed justification of this requirement.

Running Example 1 *Consider a quantum system with two identical particles, each with two states, the ground state $|0\rangle$ and the excited state $|1\rangle$. The composite Hilbert space for the system is then $\mathscr{X} = \{\sum_{k,\ell=0}^{1} \alpha_{k\ell}|k\rangle \otimes |\ell\rangle : \alpha_{k\ell} \in \mathbb{C}\}$. We'll also use the notation $|k\ell\rangle := |k\rangle \otimes |\ell\rangle$.*

As there are only two particles, there are only two possible permutation operators, namely the identity $\hat{\Pi}_0 := \hat{I}$ and the swap operator

$$\hat{\Pi}_1 := |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|,$$

which maps $\sum_{k,\ell=0}^{1} \alpha_{k\ell}|k\ell\rangle$ to $\sum_{k,\ell=0}^{1} \alpha_{k\ell}|\ell k\rangle$. The set of indifferent operators is then

$$\mathscr{I} = \{\hat{0}\} \cup \{\hat{A} - \hat{\Pi}_1^\dagger \hat{A}\hat{\Pi}_1 : \hat{A} \in \mathscr{H}\}.$$

There is an alternative way to characterise exchangeability, which can give more insight into the concept. Inspired by the treatment in Ref. [15], we define the linear transformation $\mathrm{pr}_{\mathrm{ex}}$ of the real linear space $\mathscr{H}$ by

$$\mathrm{pr}_{\mathrm{ex}} : \mathscr{H} \to \mathscr{H} : \hat{A} \mapsto \frac{1}{m!} \sum_{\pi \in \mathbb{P}} \pi^t \hat{A}.$$

Interestingly, for all $\pi \in \mathbb{P}$ and all $\hat{A} \in \mathscr{H}$,

$$\mathrm{pr}_{\mathrm{ex}} \circ \pi^t(\hat{A}) = \mathrm{pr}_{\mathrm{ex}}(\hat{A}) = \pi^t \circ \mathrm{pr}_{\mathrm{ex}}(\hat{A}),$$

so clearly also $\mathrm{pr}_{\mathrm{ex}} \circ \mathrm{pr}_{\mathrm{ex}} = \mathrm{pr}_{\mathrm{ex}}$. This tells us that $\mathrm{pr}_{\mathrm{ex}}$ is a projection operator that maps any measurement to a corresponding measurement that is *permutation invariant*, so it projects $\mathscr{H}$ onto the permutation invariant subspace

$$\mathscr{H}_{\mathbb{P}} := \{\hat{A} \in \mathscr{H} : (\forall \pi \in \mathbb{P})\pi^t \hat{A} = \hat{A}\} = \mathrm{pr}_{\mathrm{ex}}(\mathscr{H}).$$

It's also not hard to see that $\mathrm{pr}_{\mathrm{ex}}(\mathscr{H}_{>\hat{0}}) \subseteq \mathscr{H}_{>\hat{0}}$, and that the kernel of this projection operator $\mathrm{pr}_{\mathrm{ex}}$,

$$\mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}} := \{\hat{A} \in \mathscr{H} : \mathrm{pr}_{\mathrm{ex}}(\hat{A}) = \hat{0}\},$$

characterises exchangeability in the same way that $\mathscr{I}$ does.

Proposition 3 *A coherent set of desirable measurements $\mathscr{D}$ is exchangeable if and only if $\mathscr{D} + \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}} \subseteq \mathscr{D}$. In particular, $\mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}}$ is the linear span of $\mathscr{I}$.*

We now know how to express Your beliefs about exchangeable particles in terms of coherent sets of desirable measurements. To relate this to the more standard approach in quantum mechanics, we'll express exchangeability in terms of coherent lower previsions and credal sets, and see what it entails in the special case of density operators.

The following proposition tells us how to characterise exchangeability for coherent lower previsions. The fourth characterisation shows that the behaviour of an exchangeable coherent lower prevision is completely determined by its behaviour on the typically much lower-dimensional subspace of all permutation invariant measurements $\mathscr{H}_{\mathbb{P}}$.

Proposition 4 *Consider any coherent lower prevision $\underline{\Lambda}$ on $\mathscr{H}$, then the following statements are equivalent:*

(i) *$\underline{\Lambda}$ is* exchangeable, *meaning that there's some exchangeable coherent set of desirable measurements $\mathscr{D}$ such that $\underline{\Lambda} = \underline{\Lambda}_{\mathscr{D}}$;*

(ii) *$\underline{\Lambda}(\hat{A} - \pi^t \hat{A}) = \overline{\Lambda}(\hat{A} - \pi^t \hat{A}) = 0$ for all $\hat{A} \in \mathscr{H}, \pi \in \mathbb{P}$;*

(iii) *$\underline{\Lambda}(\hat{A}) = \overline{\Lambda}(\hat{A}) = 0$ for all $\hat{A} \in \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}}$;*

(iv) *$\underline{\Lambda}(\hat{A}) = \underline{\Lambda}(\mathrm{pr}_{\mathrm{ex}}(\hat{A}))$ for all $\hat{A} \in \mathscr{H}$.*

Next, we investigate how exchangeability affects density operators. For any $\pi \in \mathbb{P}$, any $\hat{\rho} \in \mathscr{R}$ and any $\hat{A} \in \mathscr{H}$:

$$\begin{aligned}\mathrm{Tr}(\hat{\rho}(\pi^t \hat{A})) &= \mathrm{Tr}(\hat{\rho}\hat{\Pi}_\pi^\dagger \hat{A}\hat{\Pi}_\pi) = \mathrm{Tr}(\hat{\Pi}_\pi \hat{\rho}\hat{\Pi}_\pi^\dagger \hat{A}) \\ &= \mathrm{Tr}((\overline{\pi}^t \hat{\rho})\hat{A}),\end{aligned} \tag{5}$$

where the second equality follows from the cyclic property of the trace. This shows that a permutation π acts on a density operator via the operator $\overline{\pi}^t = (\pi^{-1})^t$—of course also a permutation operator—rather than through π^t. Permutation invariance of a density $\hat{\rho}$ is then strictly speaking expressed through '$\overline{\pi}^t \hat{\rho} = \hat{\rho}$ for all $\pi \in \mathbb{P}$', although the difference with '$\pi^t \hat{\rho} = \hat{\rho}$ for all $\pi \in \mathbb{P}$' is immaterial.

Corollary 5 *Consider any coherent lower prevision $\underline{\Lambda}$ on $\mathscr{H}$, and the corresponding closed convex set of density operators $\mathscr{R}_{\underline{\Lambda}}$. Then $\underline{\Lambda}$ is exchangeable if and only if $\mathscr{R}_{\underline{\Lambda}} \subseteq \mathscr{H}_{\mathbb{P}}$, or in other words, if all density operators $\hat{\rho}$ in $\mathscr{R}_{\underline{\Lambda}}$ are permutation invariant.*

Running Example 2 *In the case of two 2-dimensional particles, consider the states $|\phi_1\rangle := 1/\sqrt{2}(|00\rangle + |11\rangle)$ and $|\phi_2\rangle := 1/\sqrt{2}(|10\rangle - |01\rangle) \in \mathscr{X}$. The density operator*

$$\hat{\rho} = \frac{1}{2}|\phi_1\rangle\langle\phi_1| + \frac{1}{2}|\phi_2\rangle\langle\phi_2|,$$

or in other words,

$$\hat{\rho} = \frac{1}{4}(|00\rangle+|11\rangle)(\langle 00|+\langle 11|)+\frac{1}{4}(|10\rangle-|01\rangle)(\langle 10|-\langle 01|),$$

is permutation invariant, since $\hat{\rho} = \hat{\Pi}_1 \hat{\rho} \hat{\Pi}_1^\dagger$.

4. The Symmetrisation Postulate

How do the consequences of imposing exchangeability in our framework compare with the standard account of indistinguishable particles in quantum mechanics? To answer this question, we'll look at the special, so-called *precise*, case, where Your uncertainty about $|\Psi\rangle$ is described by a single permutation invariant density operator $\hat{\rho}$. As we'll see presently, this permutation invariance is a necessary condition for the density operator on the standard account, but a stronger symmetry condition is postulated there to describe fermions and bosons. To explain how this stronger postulate is formulated, we first need to recall a few extra details about permutations.

A permutation is called a *transposition* if it simply exchanges two indices and leaves the other indices invariant. Any permutation is then always a composition of transpositions, and the sign of a permutation π, denoted by $\mathrm{sgn}(\pi)$, is then equal to 1 if the number of such compositions is even, and equal to -1 if it's odd. We call a ket $|\psi\rangle \in \mathscr{X}$ *symmetric* if $\hat{\Pi}_\pi|\psi\rangle = |\psi\rangle$ for all $\pi \in \mathbb{P}$, and *antisymmetric* if $\hat{\Pi}_\pi|\psi\rangle = \mathrm{sgn}(\pi)|\psi\rangle$ for all $\pi \in \mathbb{P}$. The symmetrisation postulate now goes as follows [11].

Symmetrisation Postulate *When a system is made up of several identical particles, only certain kets in its state space can describe its physical states. Physical states are, depending on the nature of the identical particles, either symmetric or antisymmetric with respect to permutation of these particles. Those particles for which the physical states are symmetric are called* bosons*, and those for which they're antisymmetric,* fermions*.*

What, then, does this postulate imply for density operators, rather than states? In the case of fermions, the states are antisymmetric. Consider a density operator $\hat{\rho} := \sum_{k=1}^r p_k|\psi_k\rangle\langle\psi_k|$, where $r \in \mathbb{N}$ and $p_1, \dots, p_r$ is a (positive) probability mass function over the states $|\psi_1\rangle, \dots, |\psi_r\rangle \in \bar{\mathscr{X}}$. If the only possible states are antisymmetric, then so are $|\psi_1\rangle, \dots, |\psi_r\rangle$, and therefore $\hat{\Pi}_\pi\hat{\rho} = \hat{\Pi}_\pi \sum_{k=1}^r p_k|\psi_k\rangle\langle\psi_k| = \sum_{k=1}^r p_k\hat{\Pi}_\pi|\psi_k\rangle\langle\psi_k| = \sum_{k=1}^r p_k\,\mathrm{sgn}(\pi)|\psi_k\rangle\langle\psi_k| = \mathrm{sgn}(\pi)\hat{\rho}$ for all $\pi \in \mathbb{P}$. So for fermions, we find that density operators $\hat{\rho}$ must be *antisymmetric*, meaning that

$$\hat{\Pi}_\pi\hat{\rho} = \mathrm{sgn}(\pi)\hat{\rho} \text{ for all } \pi \in \mathbb{P}.$$

A similar reasoning shows that for bosons, density operators $\hat{\rho}$ must be *symmetric*, meaning that

$$\hat{\Pi}_\pi\hat{\rho} = \hat{\rho} \text{ for all } \pi \in \mathbb{P}.$$

These conditions are also sufficient.

Proposition 6 *A density operator $\hat{\rho}$ is (anti)symmetric if and only if there are (anti)symmetric states $|\psi_1\rangle, \dots, |\psi_r\rangle$ and a positive probability mass function $p_1, \dots, p_r$ over them such that $\hat{\rho} = \sum_{k=1}^r p_k|\psi_k\rangle\langle\psi_k|$. Moreover, if there are states $|\psi_1\rangle, \dots, |\psi_r\rangle$ and a positive probability mass function $p_1, \dots, p_r$ over them such that $\hat{\rho} := \sum_{k=1}^r p_k|\psi_k\rangle\langle\psi_k|$ is (anti)symmetric, then the states $|\psi_1\rangle, \dots, |\psi_r\rangle$ are necessarily (anti)symmetric.*

For any antisymmetric $\hat{\rho}$, we find that

$$\hat{\Pi}_\pi\hat{\rho}\hat{\Pi}_\pi^\dagger = \mathrm{sgn}(\pi)\hat{\rho}\hat{\Pi}_\pi^\dagger = \mathrm{sgn}(\pi)\left(\hat{\Pi}_\pi\hat{\rho}\right)^\dagger = \mathrm{sgn}(\pi)^2\hat{\rho} = \hat{\rho}$$

and similarly if $\hat{\rho}$ is symmetric. We conclude that, in both cases, $\overline{\pi}^t\hat{\rho} = \hat{\Pi}_\pi\hat{\rho}\hat{\Pi}_\pi^\dagger = \hat{\rho}$ for all $\pi \in \mathbb{P}$, so *all symmetric and antisymmetric density operators $\hat{\rho}$ are permutation invariant*: the exchangeability condition on the (elements of the) credal sets is implied by, but not necessarily equivalent to, the antisymmetry and symmetry conditions in the Standard Model of particle physics.

We can, nevertheless, still recover some of the aspects of the fermion and boson framework from an exchangeability assessment. To see how, we introduce three Hermitian operators on $\mathscr{X}$: the *symmetriser* $\hat{P}_\mathrm{s}$, the *antisymmetriser* $\hat{P}_\mathrm{a}$ and the *parasymmetriser* $\hat{P}_\mathrm{o} := \hat{I} - \hat{P}_\mathrm{a} - \hat{P}_\mathrm{s}$, with

$$\hat{P}_\mathrm{s} := \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\hat{\Pi}_\pi \text{ and } \hat{P}_\mathrm{a} := \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\mathrm{sgn}(\pi)\hat{\Pi}_\pi.$$

Proposition 7 $\hat{P}_k\hat{P}_\ell = \delta_{k\ell}\hat{P}_k$ *for* $k, \ell \in \{\mathrm{s}, \mathrm{a}, \mathrm{o}\}$. *Also,*

(i) *the symmetriser $\hat{P}_\mathrm{s}$ is a projection operator that projects any ket onto the* boson space $\mathscr{X}_\mathrm{s} := \{|\psi\rangle \in \mathscr{X} : (\forall\pi \in \mathbb{P})|\psi\rangle = \hat{\Pi}_\pi|\psi\rangle\}$;

(ii) *the antisymmetriser $\hat{P}_\mathrm{a}$ is a projection operator that projects any ket onto the* fermion space $\mathscr{X}_\mathrm{a} := \{|\psi\rangle \in \mathscr{X} : (\forall\pi \in \mathbb{P})|\psi\rangle = \mathrm{sgn}(\pi)\hat{\Pi}_\pi|\psi\rangle\}$;

(iii) *the parasymmetriser $\hat{P}_\mathrm{o}$ is a projection operator that projects any ket onto the* para space $\mathscr{X}_\mathrm{o} := (\mathscr{X}_\mathrm{s} \oplus \mathscr{X}_\mathrm{a})^\perp$.

Permutation invariance allows for an interesting decomposition of operators, which will allow us to retrieve some of the structure corresponding to fermions and bosons.

Proposition 8 *Consider any permutation invariant density operator $\hat{\rho} \in \mathscr{R}$. Then $\hat{\rho} = \hat{\omega}_\mathrm{s} + \hat{\omega}_\mathrm{a} + \hat{\omega}_\mathrm{o}$, with $\hat{\omega}_\mathrm{s} := \hat{P}_\mathrm{s}\hat{\rho}\hat{P}_\mathrm{s}$, $\hat{\omega}_\mathrm{a} := \hat{P}_\mathrm{a}\hat{\rho}\hat{P}_\mathrm{a}$ and $\hat{\omega}_\mathrm{o} := \hat{P}_\mathrm{o}\hat{\rho}\hat{P}_\mathrm{o}$.[13] Moreover, $\hat{\rho}$ is symmetric if and only if $\hat{\rho} = \hat{P}_\mathrm{s}\hat{\rho}\hat{P}_\mathrm{s}$, and antisymmetric if and only if $\hat{\rho} = \hat{P}_\mathrm{a}\hat{\rho}\hat{P}_\mathrm{a}$.*

Clearly, $\hat{\omega}_\mathrm{s}$ corresponds to bosons and $\hat{\omega}_\mathrm{a}$ to fermions, but what does $\hat{\omega}_\mathrm{o}$ represent? The answer, as can be found in Ref. [22], is *para-particles*: particles that are neither

[13] $\hat{\omega}_\mathrm{s}$ is Hermitian, $\hat{\omega}_\mathrm{s} \geq 0$ and $\mathrm{Tr}(\hat{\omega}_\mathrm{s}) \leq 1$. If $\hat{\omega}_\mathrm{s} \neq \hat{0}$ we can renormalise $\hat{\omega}_\mathrm{s}$ into a density operator $\hat{\rho}_\mathrm{s} := \hat{\omega}_\mathrm{s}/\mathrm{Tr}(\hat{\omega}_\mathrm{s})$ (on $\mathscr{X}_\mathrm{s}$). Similarly for $\hat{\omega}_\mathrm{a}$ and $\hat{\omega}_\mathrm{o}$.

bosons nor fermions, but which obey a weaker permutation symmetry. As shown in Ref. [22], it's impossible to distinguish a collection of para-particles from a collection of standard particles by means of a measurement, which is the main reason why in the Standard Model of particle physics, para-particles are never considered [21], even though their existence is not excluded. Nevertheless, the topic of para-particles is still being explored, and experimental confirmation of their existence has been attempted [1].

Running Example 3 *In our running example,*

$$\hat{P}_\mathrm{s} = \frac{1}{2}\big(\hat{I} + \hat{\Pi}_1\big) \text{ and } \hat{P}_\mathrm{a} = \frac{1}{2}\big(\hat{I} - \hat{\Pi}_1\big) \text{ and } \hat{P}_\mathrm{o} = \hat{0}.$$

For the permutation invariant density operator

$$\hat{\rho} = \frac{1}{4}(|00\rangle+|11\rangle)(\langle 00|+\langle 11|)+\frac{1}{4}(|10\rangle-|01\rangle)(\langle 10|-\langle 01|),$$

we get

$$\hat{\omega}_\mathrm{s} = \hat{P}_\mathrm{s}\hat{\rho}\hat{P}_\mathrm{s} = \frac{1}{4}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$$
$$\hat{\omega}_\mathrm{a} = \hat{P}_\mathrm{a}\hat{\rho}\hat{P}_\mathrm{a} = \frac{1}{4}(|10\rangle - |01\rangle)(\langle 10| - \langle 01|)$$

and $\hat{\omega}_\mathrm{o} = \hat{P}_\mathrm{o}\hat{\rho}\hat{P}_\mathrm{o} = \hat{0}$*, and therefore* $\hat{\rho} = \hat{\omega}_\mathrm{s} + \hat{\omega}_\mathrm{a}$*. Clearly, since for two* 2*-dimensional particles* $\mathscr{X}_\mathrm{s} \oplus \mathscr{X}_\mathrm{a} = \mathscr{X}$*, we need to go to larger state spaces to find examples of non-trivial parasymmetric density operators.*

Let's for the sake of the argument assume that there are such para-particles—although whether or not they exist will not affect our conclusions. What, then, does Proposition 8 imply? On the standard view, which we're trying to recover here, indistinguishable particles are either bosons or fermions (or para-particles). The set of possible states is therefore $\bar{\mathscr{X}}_\mathrm{a} \cup \bar{\mathscr{X}}_\mathrm{s} \cup \bar{\mathscr{X}}_\mathrm{o}$, and Your uncertainty about the state is then represented by some mixed state $\hat{\rho} = \sum_{k=1}^{r} p_k |\psi_k\rangle\langle\psi_k|$, with probabilities $p_1, \dots, p_r \in \mathbb{R}$ over the respective states $|\psi_1\rangle, \dots, |\psi_r\rangle \in \bar{\mathscr{X}}_\mathrm{a} \cup \bar{\mathscr{X}}_\mathrm{s} \cup \bar{\mathscr{X}}_\mathrm{o}$. Since then clearly $\hat{\rho} = \hat{\omega}_\mathrm{a}+\hat{\omega}_\mathrm{s}+\hat{\omega}_\mathrm{o}$, we see that every density operator in our alternative, credal set under exchangeability approach obeys this condition, and could therefore be interpreted as in accordance with the standard view.

But, there's a defect in our exchangeability approach. A permutation invariant density operator $\hat{\rho} = \hat{\omega}_\mathrm{s} + \hat{\omega}_\mathrm{a} + \hat{\omega}_\mathrm{o}$ may correspond to a probability distribution $p_1, \dots, p_r$ over states $|\phi_1\rangle, \dots, |\phi_r\rangle \in \bar{\mathscr{X}}$, where at least some $|\phi_k\rangle$ don't belong to $\bar{\mathscr{X}}_\mathrm{a} \cup \bar{\mathscr{X}}_\mathrm{s} \cup \bar{\mathscr{X}}_\mathrm{o}$ and must therefore be *superpositions* of symmetric, antisymmetric and/or para-particle states. In other words, that $\hat{\rho} = \hat{\omega}_\mathrm{s} + \hat{\omega}_\mathrm{a} + \hat{\omega}_\mathrm{o}$ doesn't imply that the only possible states are the states of para-particles, fermions or bosons and that superpositions of these are impossible, but it does imply that we can always interpret $\hat{\rho}$ as coming from a probability distribution over such states.

Often, of course, You'll know what kind of particles You're dealing with, and it will be useful to be able to express this kind of knowledge also in the sets of desirable measurements framework, by imposing a stronger type of symmetry assessment, which expresses that the particles under consideration are fermions, or that they are bosons.

Before we attempt to find such a stronger symmetry assessment, let's briefly mention the approach followed by Benavoli et al. [7], as they also managed to achieve this goal in their different framework, using generalised permutations. Their approach is rather different from ours, if only because they, in contrast to what we do here, don't follow the practice, standard in quantum mechanics, of using the tensor product space $\otimes_{k=1}^{m} \mathscr{X}^n$ of the particle state spaces $\mathscr{X}^n$ in order to represent the system state. Instead, they essentially use only a subset of this space, namely the Cartesian product $\times_{k=1}^{m} \mathscr{X}^n$. On this alternative, smaller set of states of the type $x := (|\phi_1\rangle, \dots, |\phi_m\rangle) \in \times_{k=1}^{m} \mathscr{X}^n$, they consider all quadratic gambles, which are defined as functions of the form

$$g_{\hat{A}}(|\phi_1\rangle, \dots, |\phi_m\rangle) := \big(\otimes_{k=1}^{m} \langle\phi_k|\big)\hat{A}\big(\otimes_{k=1}^{m} |\phi_k\rangle\big),$$

also symbolically written as $g_{\hat{A}}(x) = x^\dagger \hat{A} x$, corresponding to the Hermitian operators $\hat{A}$ on $\otimes_{k=1}^{m} \mathscr{X}^n$. The authors' concept of algorithmic rationality leads to coherence axioms and a framework of desirability that closely resembles ours in spirit, but is rather different in the mathematical details. Their gambles $g_{\hat{A}}$ correspond to our utility functions, but essentially restricted to the smaller Cartesian product; it's easy to see that there's a one-to-one correspondence between them, as they are both isomorphic to the space of Hermitian operators. In order to model indistinguishable particles, however, they use a different definition of a permuted gamble than we do. They argue that, since all gambles are quadratic, they can define a permutation symmetry by treating a gamble $g_{\hat{A}}(x)$ as $g_{\hat{A}}(x,x)$, a function with "two different variables", and letting permutations act on each of these "two variables" separately. A general permuted gamble then takes the form

$$\pi_l g_{\hat{A}}(x,x)\pi_r := \frac{1}{2}\big(g_{\hat{A}}(\pi_l x, \pi_r x) + g_{\hat{A}}(\pi_r x, \pi_l x)\big),$$

which is clearly different from our permuted utility function

$$u_{\hat{A}}\big(\hat{\Pi}_\pi |\psi\rangle\big) = u_{\hat{\Pi}_\pi^\dagger \hat{A} \hat{\Pi}_\pi}(|\psi\rangle).$$

Such permuted gambles are then used to retrieve a framework for bosons, and, by introducing the sign of the permutation into the definition of a permuted gamble, the fermion framework can be retrieved. This means that they assume from the outset that You know what kind of particle You are dealing with, which leaves no room for uncertainty about the nature of the particle, something we have just shown is still there on our approach in the previous sections. Benavoli et al. thus end up with a framework for modelling uncertainty about the state of bosonic systems and about the state of fermionic systems separately. We'll see that it shows some similarity to the framework we're about to explore now, if we ignore the not unimportant difference between the state spaces used.

5. Stronger Symmetry

A symmetry assessment of exchangeability in our desirable measurements framework already allows us to usefully identify different kinds of particle system states: fermion states, boson states and para-states, but still allows a particle system to be in a superposition of such states. And, even if we were to exclude such superpositions, it would still allow You to be uncertain about whether the particles under consideration are fermions, bosons, or para-particles.

We'll now show that the desirable measurements framework allows for stronger symmetry assessments that express, from the outset, that a particle system consists of bosons, or of fermions. We'll discuss the boson and fermion cases in one fell swoop, and use the flag $\star \in \{\mathrm{s},\mathrm{a}\}$ to identify the type of symmetry we're imposing. Let

$$\operatorname{sgn}^\star(\pi) := \begin{cases} \operatorname{sgn}(\pi) & \star = \mathrm{a} \\ 1 & \star = \mathrm{s} \end{cases} \quad \text{for } \pi \in \mathbb{P}.$$

The symmetry operators π^t we've been considering, act in a two-sided fashion on a measurement $\hat{A}$: $\pi^t \hat{A} = \hat{\Pi}_\pi^\dagger \hat{A} \hat{\Pi}_\pi$. The discussion of boson and fermion symmetry in the previous section inspires us to look for symmetry operators that let go of this two-sided approach, and suggests looking at maps of the kind $\mathscr{H} \to \mathscr{H}\colon \hat{A} \mapsto \operatorname{sgn}^\star(\pi)\hat{\Pi}_\pi^\dagger \hat{A}$ for $\pi \in \mathbb{P}$. This is, of course, invalid as $\hat{\Pi}_\pi^\dagger \hat{A}$ is not necessarily Hermitian, but we can easily fix this by taking the average of $\hat{\Pi}_\pi^\dagger \hat{A}$ and its adjoint. This idea of averaging to recover Hermitianity is also used by Benavoli et al. [7] for the permutation operators on their alternative state spaces. We're thus led to the symmetry operators[14]

$$S_\pi^\star\colon \mathscr{H} \to \mathscr{H}\colon \hat{A} \mapsto \frac{\operatorname{sgn}^\star(\pi)}{2}\big(\hat{\Pi}_\pi^\dagger \hat{A} + \hat{A}\hat{\Pi}_\pi\big), \text{ for } \pi \in \mathbb{P}.$$

Using analogous arguments as with the (weaker) exchangeability condition, we construct a set of indifferent operators

$$\mathscr{I}^\star := \big\{\hat{A} - S_\pi^\star(\hat{A})\colon \hat{A} \in \mathscr{H}, \pi \in \mathbb{P}\big\},$$

and call a coherent set of desirable measurements $\mathscr{D}$ *strongly $\star$-symmetric* if $\mathscr{D} + \mathscr{I}^\star \subseteq \mathscr{D}$.[15]

In order to find a simpler representation, we introduce the following linear transformation of the linear space $\mathscr{H}$

$$\mathrm{pr}_\star\colon \mathscr{H} \to \mathscr{H}\colon \hat{A} \mapsto \hat{P}_\star \hat{A} \hat{P}_\star,$$

with kernel $\mathscr{I}_{\mathrm{pr}_\star} := \{\hat{A} \in \mathscr{H}\colon \mathrm{pr}_\star(\hat{A}) = \hat{0}\}$.[16]

It follows from Proposition 7 that $\mathrm{pr}_\star \circ \mathrm{pr}_\star = \mathrm{pr}_\star$, so $\mathrm{pr}_\star$ is a linear projection operator. We also list a few other of its properties.

Proposition 9 *The following statements hold for all $\hat{A} \in \mathscr{H}$ and all $\pi \in \mathbb{P}$:*

(i) $\hat{\Pi}_\pi^\dagger \operatorname{pr}_\star(\hat{A})\hat{\Pi}_\pi = \operatorname{pr}_\star(\hat{A})$;

(ii) $\operatorname{pr}_\star(\hat{A}) = \operatorname{sgn}^\star(\pi)\hat{\Pi}_\pi^\dagger \operatorname{pr}_\star(\hat{A}) = \operatorname{sgn}^\star(\pi)\operatorname{pr}_\star(\hat{A})\hat{\Pi}_\pi$;

(iii) *$\mathscr{I}_{\mathrm{pr}_\star}$ is the linear span of $\mathscr{I}^\star$.*

Similarly to the results in Section 3, we can now express strong $\star$-symmetry in terms of $\mathscr{I}_{\mathrm{pr}_\star}$, and then find requirements for the corresponding lower prevision, as formalised in the following results.

Proposition 10 *A coherent set of desirable measurements in strongly $\star$-symmetric if and only $\mathscr{D} + \mathscr{I}_{\mathrm{pr}_\star} \subseteq \mathscr{D}$.*

Running Example 4 *In our running example, let's look at the case of fermions. Consider the joint state $|11\rangle$, where the two particles reside in the same particle state $|1\rangle$, and the measurement $\hat{A} := |11\rangle\langle 11|$ that returns 1 if the system resides in state $|11\rangle$ and 0 for any system state orthogonal to $|11\rangle$. Since $\hat{P}_\mathrm{a}\hat{A}\hat{P}_\mathrm{a} = \hat{0}$, we see that $\hat{A} \in \mathscr{I}_{\mathrm{pr}_\mathrm{a}}$, so You are indifferent between receiving nothing and receiving the uncertain reward $u_{\hat{A}}(|\Psi\rangle)$. This is tantamount to Your believing that the system can't reside in the state $|11\rangle$. This also turns out to be the case for any system state that has multiple particles residing in the same particle state. This reflects the well-known* Pauli principle*, which prohibits multiple fermions from being in the same particle state.*

Proposition 11 *Consider any coherent lower prevision $\underline{\Lambda}$ on $\mathscr{H}$, then the following statements are equivalent:*

(i) *$\underline{\Lambda}$ is* strongly $\star$-symmetric*, meaning that there's some strongly $\star$-symmetric and coherent set of desirable measurements $\mathscr{D}$ such that $\underline{\Lambda} = \underline{\Lambda}_\mathscr{D}$;*

(ii) *$\underline{\Lambda}(\hat{A} - S_\pi^\star(\hat{A})) = \overline{\Lambda}(\hat{A} - S_\pi^\star(\hat{A})) = 0$ for all $\hat{A} \in \mathscr{H}$ and $\pi \in \mathbb{P}$;*

[14] Observe that his symmetry operator $S_\pi^\star$ corresponds to a single permutation, in contradistinction with the generalised permutations in Ref. [7], where a second permutation enters the definition.

[15] We use the term 'strongly' $\star$-symmetric as a reference to and reminder of the strong invariance in Refs. [14, 29]; observe that exchangeability could also be called 'strong permutation symmetry'. See also the quite relevant discussion in Ref. [28, Section 6].

[16] In fact, this set $\mathscr{I}_{\mathrm{pr}_\star}$ is related to the set defined in Ref. [7, Corollary 1], but we retrieve it through a different symmetry operator.

(iii) $\underline{\Lambda}(\hat{A}) = \overline{\Lambda}(\hat{A}) = 0$ *for all* $\hat{A} \in \mathscr{I}_{\mathrm{pr}_\star}$;

(iv) $\underline{\Lambda}(\hat{A}) = \underline{\Lambda}(\mathrm{pr}_\star(\hat{A}))$ *for all* $\hat{A} \in \mathscr{H}$.

If we now turn to the sets of density operators that correspond to strongly $\star$-symmetric coherent sets of desirable measurements, we recover the expected boson and fermion symmetry conditions, as in the standard framework for quantum mechanics the density operators for indistinguishable particles are exactly the ones that satisfy the symmetry conditions $\hat{\rho} = \mathrm{sgn}^\star(\pi)\hat{\Pi}_\pi\hat{\rho}$ for all $\pi \in \mathbb{P}$, with $\star = \mathrm{s}$ for bosons and $\star = \mathrm{a}$ for fermions.

Corollary 12 *Consider any coherent lower prevision* $\underline{\Lambda}$ *on* $\mathscr{H}$ *and its corresponding set of density operators* $\mathscr{R}_{\underline{\Lambda}}$*. Then* $\underline{\Lambda}$ *is strongly* $\star$*-symmetric if and only if* $\hat{\rho} = \mathrm{sgn}^\star(\pi)\hat{\Pi}_\pi\hat{\rho}$ *for all* $\pi \in \mathbb{P}$ *and* $\hat{\rho} \in \mathscr{R}_{\underline{\Lambda}}$*.*

Clearly, the stronger symmetry assessments don't suffer from the same defect as an assessment of exchangeability: due to Proposition 6, the stronger symmetry of the density operators implies that the only possible states are fermions in the antisymmetric case and bosons in the symmetric case, and that no superpositions of these states are possible.

6. Second Quantisation

So far, we've been describing fermions and bosons using *first quantisation*, which means we've been considering symmetric and antisymmetric density operators. However, as You believe the particles to obey a strong symmetry, we can expect there to be some redundancy in this type of description. In standard quantum mechanics, *second quantisation* is then often used: instead of describing the state of each particle, the occupation numbers of the different possible states are considered. This is reminiscent of the use of count vectors in an exchangeability context, as described in Ref. [15]. In fact, Benavoli et al. [8] used this concept of count vectors to describe second quantisation for bosons in the quantum expectation operator approach. Let's now briefly show how we can implement the ideas of Ref. [15] to give an account of second quantisation in the more general sets of desirable measurements framework for both bosons and fermions.

We start with a concise overview of how second quantisation is implemented in standard quantum mechanics. For a more thorough and detailed account, see Section XIV.C in Ref. [11]. We'll adopt the notations from Section 3 for a system of m indistinguishable n-dimensional particles, with n^m-dimensional state space $\bar{\mathscr{X}}$.

The boson space $\mathscr{X}_{\mathrm{s}} = \hat{P}_{\mathrm{s}}(\mathscr{X})$ is a lower dimensional subspace of symmetric kets. An orthonormal basis for this boson space is given by

$$|\mathbf{m}\rangle_{\mathrm{s}} = |m_1 \ldots m_n\rangle_{\mathrm{s}} := \sqrt{\frac{m!}{\prod_{\ell=1}^n m_\ell!}}\,\hat{P}_{\mathrm{s}} \otimes_{\ell=1}^n \otimes_{k_\ell=1}^{m_\ell} |\phi_\ell\rangle,$$

for all count vectors $\mathbf{m} = (m_1, \ldots m_n)$ in the set

$$\mathscr{N}_{\mathrm{s}}^m := \left\{(m_1, \ldots m_n) \in \mathbb{N}_0^n \colon \sum_{k=1}^n m_k = m\right\}.$$

Similarly, an orthonormal basis for the fermion space $\mathscr{X}_{\mathrm{a}} = \hat{P}_{\mathrm{a}}(\mathscr{X})$ of antisymmetric kets is given by

$$|\mathbf{m}\rangle_{\mathrm{a}} = |m_1 \ldots m_n\rangle_{\mathrm{a}} := \sqrt{m!}\,\hat{P}_{\mathrm{a}} \otimes_{\ell=1}^n \otimes_{k_\ell=1}^{m_\ell} |\phi_\ell\rangle,$$

for all count vectors $\mathbf{m} = (m_1, \ldots m_n)$ in the set

$$\mathscr{N}_{\mathrm{a}}^m := \left\{(m_1, \ldots, m_n) \in \{0,1\}^n \colon \sum_{k=1}^n m_k = m\right\}.$$

Interestingly, these count vectors consist only of zeroes and ones, as the antisymmetry of fermionic states doesn't allow more than one particle to be in the same state; this is a translation of the *Pauli principle*.

While this representation may seem complicated, the following example shows that it's quite intuitive.

Running Example 5 *In our running example, the basis states for* $\mathscr{X}_{\mathrm{s}}$ *are*

$$|20\rangle_{\mathrm{s}} := |00\rangle, |11\rangle_{\mathrm{s}} := \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle, |02\rangle_{\mathrm{s}} := |11\rangle.$$

Conversely, $\mathscr{X}_{\mathrm{a}}$ *only has the single basis state*

$$|11\rangle_{\mathrm{a}} := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

We can now use the ideas and argumentation in Ref. [15] to prove that under strong $\star$-symmetry it's enough to consider sets of desirable measurements on the reduced state space $\bar{\mathscr{X}}_\star$, corresponding to the Hilbert space $\mathscr{X}_\star$ spanned by the basis vectors $|\mathbf{m}\rangle_\star$, $\mathbf{m} \in \mathscr{N}_\star^m$. To this end, we take a closer look at the set of symmetrised measurements $\mathscr{H}_\star := \mathrm{pr}_\star(\mathscr{H}) = \{\mathrm{pr}_\star(\hat{A}) \colon \hat{A} \in \mathscr{H}\}$. The following result shows that for a strongly $\star$-symmetrical model, in order to check that a measurement $\hat{A}$ is desirable it's sufficient to look at its $\star$-symmetrised counterpart $\mathrm{pr}_\star(\hat{A}) = \hat{P}_\star\hat{A}\hat{P}_\star$.

Proposition 13 *Consider a strongly* $\star$*-symmetric and coherent set of desirable measurements* $\mathscr{D}$*. Then for all* $\hat{A} \in \mathscr{H}$*,* $\hat{A} \in \mathscr{D} \Leftrightarrow \mathrm{pr}_\star(\hat{A}) \in \mathscr{D}$*.*

Indeed, any measurement $\hat{A}$ is the sum of $\mathrm{pr}_\star(\hat{A}) \in \mathscr{H}_\star$ and $\hat{A} - \mathrm{pr}_\star(\hat{A}) \in \mathscr{I}_{\mathrm{pr}_\star}$. Since, by Your strong $\star$-symmetry assessment, You deem the measurements in $\mathscr{I}_{\mathrm{pr}_\star}$ to be equivalent to $\hat{0}$, only the measurements $\hat{A}$ such that $\mathrm{pr}_\star(\hat{A}) = \hat{A}$, or in other words, only the symmetrised measurements in $\mathscr{H}_\star$, matter in describing desirability.

However, since $\mathrm{pr}_\star(\hat{A}) = \hat{P}_\star \hat{A} \hat{P}_\star$, we see that the effect of $\mathrm{pr}_\star$ is to restrict $\hat{A}$ to $\mathscr{X}_\star$. We can now introduce the notation $\mathrm{red}_\star$ for the linear mapping from $\mathscr{H}$ to $\mathscr{H}(\mathscr{X}_\star)$ uniquely[17] defined by $\langle\psi|\,\mathrm{red}_\star(\hat{A})|\psi\rangle = \langle\psi|\hat{A}|\psi\rangle$ for all $|\psi\rangle \in \bar{\mathscr{X}}_\star$ and all $\hat{A} \in \mathscr{H}$. On the other hand, we define the (cylindrical) extension of a measurement $\hat{A} \in \mathscr{H}(\mathscr{X}_\star)$ to $\mathscr{H}$ as $\mathrm{ext}_\star(\hat{A}) := \hat{A}\hat{P}_\star$.

Our final result makes it clear that there's a simpler representation for sets of desirable measurements on boson or fermion systems, and assures us that an assessment of strong $\star$-symmetry leads to sets of desirable measurements that are fully compatible with the standard quantum mechanical models based on the symmetrisation postulate, but which also allow us to deal with partial preferences.

Theorem 14 *A set of desirable measurements $\mathscr{D}$ for $\bar{\mathscr{X}}$ is coherent and strongly $\star$-symmetric if and only if there's some coherent set of desirable measurements $\mathscr{D}_o$ for $\bar{\mathscr{X}}_\star$ such that $\mathscr{D} = \{\hat{A} \in \mathscr{H} \colon \mathrm{red}_\star(\hat{A}) \in \mathscr{D}_o\}$. In that case necessarily, $\mathscr{D}_o = \mathscr{D}_\star := \{\hat{C} \in \mathscr{H}(\mathscr{X}_\star) \colon \mathrm{ext}_\star(\hat{C}) \in \mathscr{D}\}$.*

7. Conclusion

The desirable measurements framework has the advantage that it allows us at the same time to justify using Born's rule (in terms of density operators) and to extend it to situations where Your beliefs only lead to partial preferences, and therefore to working with sets of density operators. The general question then naturally arises if and how well-known standard quantum mechanical concepts are expressible in this more general language. The specific question we have tried to answer here, is how to deal with indistinguishable particles.

Using exchangeability in the sets of desirable measurements framework for dealing with uncertainty about a quantum system with multiple particles allows us to recover some of the structural symmetry of bosons, fermions and para-states, but we need to impose stronger symmetry requirements to get to bosonic or fermionic system descriptions. This observation explains and justifies the question mark in our title, and the answer to the resulting question is, then, a clear 'no'. The uncertainty models that exhibit this stronger symmetry are then amenable to a lower-dimensional representation, akin to what happens for second quantisation on the standard quantum mechanical formalism.

To also describe in the desirable measurements framework the important consequences of indistinguishability in quantum systems, such as why bosons tend to bunch together or electrons tend to resist being close together, we'll need to take into account the dynamical aspects of quantum mechanics, which we hope to do in future work.

We have thus far only dealt with quantum systems with a fixed number of particles. However, it's often useful to consider the case where the number of particles is not determined up front, and for that reason, we also intend to examine in future work the implications of allowing, in the sets of desirable measurements framework, for uncertainty about the number of particles in a quantum system.

[17] A Hermitian operator $\hat{A}$ is uniquely determined by its corresponding utility function, or in other words, by $\langle\psi|\hat{A}|\psi\rangle$ for all $|\psi\rangle \in \bar{\mathscr{X}}$.

Author Contributions

The initiative for the research in this paper was taken by Keano, who also worked on a first draft, based on extensive discussions with the other authors. Later drafts were based on revision efforts by Gert and Jasper, and writing efforts by Gert and Keano.

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