Proceedings of Machine Learning Research 215:165–176, 2023 ISIPTA 2023

The Set Structure of Precision

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Abstract

In literature on imprecise probability little attention is paid to the fact that imprecise probabilities are precise on some events. We show that this *system of precision* forms, under mild assumptions, a so-called (pre-)Dynkin-system. Interestingly, there are several settings, ranging from machine learning on partial data over frequential probability theory to quantum probability theory and decision making under uncertainty, in which a priori the probabilities are only desired to be precise on a specific underlying set system. Here, (pre-)Dynkin-systems have been adopted as systems of precision, too. Under extendability conditions those pre-Dynkin-systems equipped with probabilities can be embedded into algebras of sets. Surprisingly, the extendability conditions elaborated in a strand of work in quantum physics are equivalent to coherence. Thus, we link the literature on probabilities on pre-Dynkin-systems to the literature on imprecise probability. In fact, the system of precision and imprecise probabilities live in structural duality.

Keywords: pre-Dynkin-system, Dynkin-system, coherence, extendability, quantum probability, intersectability

1. Introduction

Scholarship in imprecise probability largely focuses on the imprecision of probabilities. However, imprecise probability models often lead to *precise* probabilistic statements on certain events or gambles. In this work, we follow a hitherto not taken route investigating the *system of precision*, i.e. what is the set structure on which an imprecise probability[1] is precise? It turns out that (pre-)Dynkin-systems[2] describe the set of events with precise probabilities (cf. § 3). This event structure is a neglected object in the literature on imprecise probability. In particular, it constitutes a parametrized choice somewhat "orthogonal" to the standard. Roughly stated, approaches to imprecise probability generalize the probability measure μ_σ in a classical probability space $(\Omega, \mathcal{F}_\sigma, \mu_\sigma)$.[3] We start by generalizing $\mathcal{F}_\sigma$ from a σ-algebra to a pre-Dynkin-system.

This suggestion is practically motivated: What do the following scenarios have in common?

(a) A machine learning algorithm has access to a restricted subset of attributes. It cannot jointly query all attributes simultaneously. This is called "learning on partial, aggregated information" [17]. The reasons might be manifold: for privacy preservation, "not-missing-at-random" features, restricted data base access for acceleration or multi-measurement data sets.

(b) Quantum physical quantities, e.g. location and impulse, are (statistically) incompatible [22].

(c) A preference ordering on a set of acts gives rise to precise beliefs on a set of events, whereas this belief is not necessarily precise for intersections of such events [18, 56].

In all of these scenarios, there does not exist a precise probability over all attributes and events. Or, there is no such precise probability accessible. Two attributes might each on their own exhibit a precise probabilistic description, while a joint precise probabilistic description does not exist. On a more fundamental level, no intersectability is provided. A precise probabilistic description of two events does *not* imply that the intersection of those events possesses a precise probability. The set system for the description of the events with precise probabilities which independently turned up in the various, previously mentioned fields of research is, again, the (pre-)Dynkin-system.

The question of intersectability (or "intersectionality") is of considerable interest in the social sciences where it is

[1] We elaborate the exact definition of imprecise probabilities used here in Section 3.

[2] Pre-Dynkin-systems appear under plenty of names: pre-Dynkin-system [44], additive-class [41, p. 2], concrete logic [37, 12], partial field [19], quantum-mechanical algebra [47], semi-algebra [31, p. 13], set-representable orthomodular poset (SROMP) [39]. Dynkin-systems are equally variable in their naming: Dynkin-system [28], d-system [55, p. 193], λ-class [5, p. 7], quantum-mechanical σ-algebra [47], σ-class [23].

[3] Following Kolmorogov's classical setup Ω is the base set, $\mathcal{F}_\sigma$ a σ-algebra and μ_σ a countably additive probability on $\mathcal{F}_\sigma$. Approaches to imprecise probability often do not even presuppose an underlying measure space (e.g. [52]). However, they are often linked to finitely additive measure spaces $(\Omega, \mathcal{F}, \mu)$, where μ is a finitely additive probability and $\mathcal{F}$ is an algebra of sets (sometimes called field).

used as a label to describe the problem of the *joint* effect of various individual attributes on social outcomes [6, 45, 54]. This notion of intersectionality clearly has *something* to do with set systems. Needless to say, the concept as used in the social sciences is rich, complex, and somewhat vague, which is not necessarily held to be a weakness: "at least part of its success has been attributed to its vagueness" [26, page 260]. Our interest is in under what circumstances precise probabilities can be ascribed to events; we speculate that such formal results may well contribute to a deeper empirical understanding of social intersectionality, without resorting to fuzzy logic [25] with its renowned lack of operational definition [7].

All of the preceding considerations bring us to the main question of this paper: **What is the system of precision and how does it relate to an imprecise probability on "all" events?** We approach this question from two perspectives.

First, we show that, under mild assumptions, a pair of lower and upper probabilities assign precise probabilities, i.e. lower and upper probability coincide, to events which form a pre-Dynkin-system or even a Dynkin-system.

Second, we define probabilities on pre-Dynkin-systems in accordance with the literature on quantum probability, in particular [20]. We argue that probabilities on pre-Dynkin-systems, as well as their inner and outer extension, exhibit little desirable properties, e.g. subadditivity cannot be guaranteed. Hence, extendability, the ability to extend a probability from a pre-Dynkin-system to a larger set structure, turns out to be crucial, as it implies coherence of the probability defined on the pre-Dynkin-system. This observation links together the research from probabilities defined on weak set structures [20, 56, 44] to imprecise probabilities [52, 1]. Furthermore, extendability guarantees the existence of a nicely behaving, so-called coherent extension. We finally show that the inner and outer extension of a probability defined on a pre-Dynkin-system is always more pessimistic than its corresponding lower and upper coherent extension.

The two perspectives reflect a duality: we can map an imprecise probability to a pre-Dynkin-system equipped with a precise probability. Respectively, we can map a pre-Dynkin-system equipped with a precise probability to an imprecise probability. This duality can be rigorously stated (Section 5). It uncovers a family of imprecise probabilities parametrized and structurally interpolated via pre-Dynkin-systems.

For more additional details on this work, a complete list of proofs and some remarks concerning countably additive probabilities and σ-algebras see our companion preprint [16]. Before we begin the structural investigation of pre-Dynkin-systems, we first introduce the used notation and fix the mathematical framework.

1.1. Notation and Technical Details

As we deal with a lot of sets, sets of sets, and rarely even sets of sets of sets in this paper, we agree on the following notation: Sets are written with capital latin or greek letter, e.g. A or Ω. Sets of sets are denoted $\mathcal{A}$. Sets of sets of sets obtain the notation $\mathfrak{A}$. As usual, $\mathbb{N}$ is reserved for the set of natural numbers. The power set of a set A is written as 2^A. We denote the indicator function of a set A as χ_A.

In the course of this work, we require a base set Ω and an algebra $\mathcal{F}$ on Ω, i.e. a set system which is closed under complement, finite union and contains the empty set.[4] For example, we can choose $\mathcal{F} = 2^\Omega$. Probability measures are denoted by lowercase greek letters, e.g. μ, ν and ψ (except for σ). Generally, we use "σ" to emphasize the countable nature of a mathematical object. This becomes clear when we define (pre-)Dynkin-systems. Equipped with these notions and tools we are ready for a first preliminary question.

2. What Is a (Pre-)Dynkin-System?

In this work, the main objects under consideration are pre-Dynkin-systems and Dynkin-systems. A (pre-)Dynkin-system is a set system on Ω. It contains the empty set, is closed under complement and (countable) disjoint union. More formally:

Definition 1 ((Pre-)Dynkin-system) *We say $\mathcal{D} \subseteq 2^\Omega$ is a* pre-Dynkin-system *on some set Ω if and only if all of the following conditions hold:*

(a) $\emptyset \in \mathcal{D}$,

(b) $D \in \mathcal{D}$ implies $D^c := \Omega \setminus D \in \mathcal{D}$,

(c) $C, D \in \mathcal{D}$ with $C \cap D = \emptyset$ implies $C \cup D \in \mathcal{D}$.

We call $\mathcal{D}_\sigma \subseteq 2^\Omega$ a Dynkin-system, *if and only if the conditions (a), (b) and*

(c') Let $\{D_i\}_{i\in\mathbb{N}} \subseteq \mathcal{D}_\sigma$. If for all $i, j \in \mathbb{N}$ with $i \neq j$ it holds $D_i \cap D_j = \emptyset$ then $\bigcup_{i\in\mathbb{N}} D_i \in \mathcal{D}_\sigma$,

are fulfilled.

We will denote pre-Dynkin-systems by the use of $\mathcal{D}$, in contrast to $\mathcal{D}_\sigma$ for Dynkin-systems. This should not be confused with $\mathrm{D}(\mathcal{A})$ for $\mathcal{A} \subseteq 2^\Omega$, which is the intersection of all pre-Dynkin-systems which contain $\mathcal{A}$, i.e. the smallest pre-Dynkin-system containing $\mathcal{A}$.[5] In other words, $\mathrm{D}(\mathcal{A})$

[4] Our notion of an algebra should not be confused with the notion of an algebra over a field. We use the term algebra to emphasize the similarity to its cousin, the σ-algebra, an algebra which is closed under countable union [55, Definition 1.1].

[5] For $\mathcal{A} = \emptyset$ we define $\mathrm{D}(\mathcal{A}) = \{\emptyset, \Omega\}$. We remark, furthermore, that one can easily check that the intersection of pre-Dynkin-systems form a pre-Dynkin-system.

is the pre-Dynkin-hull generated by $\mathcal{A}$. The following short lemma will be helpful in later proofs.

Lemma 2 (Closedness under Set Difference) *Let $\mathcal{D} \subseteq 2^{\Omega}$ be a pre-Dynkin-system, if $A, B \in \mathcal{D}$ and $A \subseteq B$, then $B \setminus A \in \mathcal{D}$.*

Proof We show that $B^c \cup A \in \mathcal{D}$, because then by closedness under complement (Definition 1 (b)) $(B^c \cup A)^c = B \setminus A \in \mathcal{D}$. Clearly, $A, B^c \in \mathcal{D}$ and $A \cap B^c = \emptyset$. Thus, $B^c \cup A \in \mathcal{D}$. ■

In classical probability theory, Dynkin-systems appear as a technical object required for the measure-theoretic link between cumulative distribution functions and probability measures (cf. [55, Proof of Lemma 1.6]). In particular, every σ-algebra, the well-known domain of probability measures, is a Dynkin-system. Thus, all statements within this work are generalizations of classical probability theoretical results. We give a short example of a pre-Dynkin-system, which is not an algebra in the following. This example gets reused to illustrate forthcoming statements.

Example 1 *The smallest pre-Dynkin-system which is not an algebra can be defined on $\Omega = \{1, 2, 3, 4\}$. It is given by $\mathcal{D} = \{\emptyset, 12, 34, 13, 24, \Omega\}$, where we write 12 as a shorthand for $\{1, 2\}$. See Figure 1.*

Pre-Dynkin- and Dynkin-systems naturally arise in probability theory. For instance, the set of all subsets $A \subseteq \mathbb{N}$, such that the natural density $\mu(A) = \lim_{n\to\infty} \frac{|A\cap\{1,\ldots,n\}|}{n}$ exists (cf. [44]) is a pre-Dynkin-system $\mathcal{D}_{\mathbb{N}}$, but not an algebra[6]. It is sometimes called the density logic [40] and constitutes the foundation of von Mises' century-old frequential theory of probability [49] (refined and summarized in [50]).

Another class of Dynkin-systems occur in so-called marginal scenarios [8]. Marginal scenarios are settings in which marginal probability distributions for a subset of a set of random variables are given, but not the entire joint distribution. This restricted "joint measurability" of the involved random variables can be expressed via Dynkin-systems [23, Example 4.2] [51].

Pre-Dynkin-systems are so helpful because they structurally align with finitely additive probability measures. The same statement holds for Dynkin-systems and countably additive probabilities. If we know the probability of an event, then we know the probability of the complement, i.e. the event does not happen. If we know the probability of several events which are disjoint, then we know the probability of the union, which is just the sum. Probabilities following their standard definition go hand in hand with Dynkin-systems. We see this observation manifested in many following statements.

Remarkably, (pre-)Dynkin-systems appeared under a variety of names (cf. Footnote 2). Fundamental to all its regular, independent occurrences in many research areas is the need for a set structure which does not allow for arbitrary intersections.

2.1. Compatibility

(Pre-)Dynkin-systems are not necessarily closed under intersections. However, when the intersection of two sets is contained in the (pre-)Dynkin-system, we call the two events *compatible*.

Definition 3 (Compatibility) *Let A, B be elements in a pre-Dynkin-system $\mathcal{D}$ on Ω, then A and B are* compatible*, if and only if $A \cap B \in \mathcal{D}$.*

This definition follows the definitions given in e.g. [20, 21, 23].[7] Compatibility in pre-Dynkin-systems is a symmetric relation, but it is not necessarily transitive. Furthermore, it is complement inherited, i.e. if A, B are compatible in a pre-Dynkin-system then so are A, B^c [22, Lemma 3.6]. Lastly, compatibility, even though expressed as intersectability, i.e. "closed under intersection", can be equivalently expressed as unifiability, i.e. "closed under union".

Lemma 4 (Cup gives Cap gives Cup) *Let $\mathcal{D}$ be a pre-Dynkin-system on Ω and $A, B \in \mathcal{D}$. Then*

$$A \cap B \in \mathcal{D} \Leftrightarrow A \cup B \in \mathcal{D}$$

Proof Using Lemma 2 for pre-Dynkin-systems we can quickly see that the following two decompositions give the desired equivalence:
For the "$\Rightarrow$"-direction: $A \cup B = (A \setminus (A \cap B)) \cup B$. The fact $A, A \cap B, B \in \mathcal{D}$ implies $(A \setminus (A \cap B)) \cup B \in \mathcal{D}$.
For the "$\Leftarrow$"-direction: $A \cap B = A \setminus ((A \cup B) \setminus B)$. The fact $A, A \cup B, B \in \mathcal{D}$ implies $A \setminus ((A \cup B) \setminus B) \in \mathcal{D}$. (A related result for Dynkin-systems is given in [20, 5.1].) ■

Example 2 *We reconsider the set Ω and pre-Dynkin-system $\mathcal{D}$ from Example 1. The elements 12 and 34 are intersectable $12 \cap 34 = \emptyset \in \mathcal{D}$ and unifiable $12 \cup 34 = \Omega \in \mathcal{D}$. The elements 12 and 13 are not intersectable $12 \cap 13 = 1 \notin \mathcal{D}$ and not unifiable $12 \cup 13 = 123 \notin \mathcal{D}$.*

[6] Intriguingly, this was used as an example by Kolmogorov [34] of a measure defined on a restricted set system for which it is desired to extend the measure to the power set $2^{\mathbb{N}}$ (cf. § 4); see the discussion in [31, pages 11-14] who observed (page 14) that "the main problem is *non-uniqueness* of an extension" and that such extended measures are impossible to verify from observed frequencies, because the relative frequencies do not converge for events in $2^{\mathbb{N}} \setminus \mathcal{D}_{\mathbb{N}}$. The non-uniqueness is naturally handled in the present paper by working with lower and upper previsions (or lower and upper probabilities).

[7] It should not be confused with the very similar, and sometimes equivalent, notion of commutativity in logical structures [35, Definition 14].

The term "compatibility" underlines that closedness under intersection gets loaded with further meaning in the context of theories of probability. As we define in the next section, $\mathcal{D}$ is the set of events which get assigned a precise probability. Hence, two events A, B are called compatible, if and only if a joint probabilistic description, i.e. a precise probability of $A \cap B$ exists.[8]

Compatibility is not only a property of elements in a pre-Dynkin-systems. One can take compatibility as a primary notion, i.e. one requires the statements of Lemma 4 and [22, Lemma 3.6] to hold. Then, a set structure which contains the empty set and the entire base set and is equipped with this notion of compatibility is a pre-Dynkin-system [31, Definition 5.1].[9]

Interestingly, the assumption of arbitrary compatibility is fundamental to most parts of probability theory. σ-algebras, the domain of classical probability measures, are exactly those Dynkin-systems in which all events are compatible with all others [21, Theorem 2.1]. Surprisingly, it turns out that, as well, all pre-Dynkin-systems can be dissected into such "blocks" of full compatibility. Every pre-Dynkin-system consists of a set of maximal algebras which we call *blocks*. In particular, maximality here stands for: there is no algebra contained in $\mathcal{D}$ such that some $\mathcal{F}_i$ is a strict sub-algebra of this algebra. Similar and related results can be found in [29, 46, 3, 48].

Theorem 5 (Pre-Dynkin-System Consists of Algebras) *Let $\mathcal{D}$ be a pre-Dynkin-system on Ω. Then there is a unique family of maximal algebras $\{\mathcal{F}_i\}_{i\in I}, \mathcal{F}_i \subseteq 2^\Omega$ such that $\mathcal{D} = \bigcup_{i\in I} \mathcal{F}_i$. We call these algebras the* blocks *of $\mathcal{D}$.*

Proof We consider the set $\mathfrak{A} := \{\mathcal{F} \subseteq \mathcal{D} \colon \mathcal{F}$ is an algebra$\}$. Certainly, every element $D \in \mathcal{D}$ is in at least one of the algebras in this set, because $\{\emptyset, D, D^c, \Omega\} \subseteq \mathcal{D}$ is an algebra. The set $\mathfrak{A}$ is ordered by set inclusion. In particular, every chain, i.e. every totally ordered subset, e.g. the singleton $\{\{\emptyset, D, D^c, \Omega\}\}$, in $\mathfrak{A}$ has an upper bound in $\mathfrak{A}$, which is the union of the elements in this chain. This provably forms a proper algebra. Thus, Zorn's lemma applies [43, p. 144], i.e. there is a unique set of maximal elements in $\mathfrak{A}$, which we define as $\{\mathcal{F}_i\}_{i\in I}$. The union of these elements necessarily covers the entire pre-Dynkin-system $\mathcal{D}$, as every element in $\mathcal{D}$ is in at least one maximal sub-algebra. ■

Example 3 *The pre-Dynkin-system $\mathcal{D}$ of Example 1 consists of the algebras* $\{\emptyset, 12, 34, \Omega\}$ *and* $\{\emptyset, 13, 24, \Omega\}$.

[8] For a more thorough discussion of the nature of compatibility (and its cousin commutativity) we point to the literature on quantum probability, e.g. [32, Definition 3.12], or [42].

[9] Pre-Dynkin-systems are called semi-algebras in [31, Definition 5.1].

Theorem 5 enables a new perspective. Instead of pre-Dynkin-systems, one can equivalently consider a set of algebras. However, not every union of algebras is a pre-Dynkin-system.

2.2. Probabilities on Pre-Dynkin-Systems

We require a notion of probability on a pre-Dynkin-system to elaborate the relationship of imprecise probability and the system of precision in the following. Probabilities are classically defined on σ-algebras. We generalize this definition, as e.g. stated in [55, p. 18f], to pre-Dynkin-systems.

Definition 6 (Probability on Pre-Dynkin-System) *Let $\mathcal{D}$ be a pre-Dynkin-system on Ω. We call a function $\mu \colon \mathcal{D} \to [0, 1]$ a* countably additive probability measure *on $\mathcal{D}$, if and only if it fulfills the following two conditions:*

(a) Normalization*: $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$.*

(b) σ-Additivity*: Let $I \subseteq \mathbb{N}$ and $\{A_i\}_{i\in I}$ such that $A_i \in \mathcal{D}$ for all $i \in I$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Then $\mu(\bigcup_{i\in I} A_i) = \sum_{i\in I} \mu(A_i)$.*

If condition (b) holds at least for finite I, we say that μ is a finitely additive probability measure.

For the sake of readability, we use "probability" and "probability measure" exchangeably.[10] Probabilities on pre-Dynkin-systems are monotone, i.e. for $A, B \in \mathcal{D}$, if $A \subseteq B$, then $\mu(A) \leq \mu(B)$.[11] But, in contrast to a probability defined on a σ-algebra, a probability on a pre-Dynkin-system is not necessarily *modular*, i.e. for $A, B \in \mathcal{D}$, $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$ [15, p. 16]).[12] It is that sophisticated interplay of set structure and probability function which leads us through this paper.

3. Imprecise Probabilities Are Precise on a Pre-Dynkin-System

As we now demonstrate, pre-Dynkin-systems are, under mild assumptions, the systems of precision. To make this formal, we solely require a normed, conjugate pair of lower and upper probability which fulfill super (resp. sub)-additivity and possibly a continuity assumption.

Theorem 7 (Probability Induces (Pre-)Dynkin-System) *Let $\mathcal{F}$ be an algebra on Ω. Let $\ell \colon \mathcal{F} \to [0, 1]$ and $u \colon \mathcal{F} \to [0, 1]$ be two set functions, for which all the following properties hold:*

[10] With "probability" we mean precise probability, in contrast to the later introduced inner and outer probability (Proposition 8) and coherent lower and upper probability (Corollary 13).

[11] This can be seen when applying Lemma 2 and Definition 6.

[12] It is, surely, possible to demand probabilities on pre-Dynkin-systems to be modular. This leads to a fixed parametrization of probability functions already on simple examples [36, p. 125].

(a) Normalization: $u(\emptyset) = \ell(\emptyset) = 0$.

(b) Conjugacy: $u(A) = 1 - \ell(A^c)$ *for* $A, A^c \in \mathcal{F}$.

(c) Subadditivity of u: for $A, B \in \mathcal{F}$ *such that* $A \cap B = \emptyset$ *then* $u(A \cup B) \leq u(A) + u(B)$.

(d) Superadditivity of ℓ*: for* $A, B \in \mathcal{F}$ *such that* $A \cap B = \emptyset$ *then* $\ell(A \cup B) \geq \ell(A) + \ell(B)$.

Then u and ℓ *define a finitely additive probability measure* $\mu := u|_{\mathcal{D}} = \ell|_{\mathcal{D}}$ *on a pre-Dynkin-system* $\mathcal{D} \subseteq \mathcal{F}$*. If either u fulfills*

(e) Continuity from below: for $A_n \in \mathcal{F}$ *with* $A_n \subseteq A_{n+1}$ *such that* $\bigcup_{n=1}^{\infty} A_n = A \in \mathcal{F}$*, then* $\lim_{n\to\infty} u(A_n) = u(A)$*,*

or ℓ *fulfills*

(e') Continuity from above: for $A_n \in \mathcal{F}$ *with* $A_{n+1} \subseteq A_n$ *such that* $\bigcap_{n=1}^{\infty} A_n = A \in \mathcal{F}$*, then* $\lim_{n\to\infty} \ell(A_n) = \ell(A)$*,*

then u and ℓ *define a countably additive probability measure* $\mu_\sigma := u|_{\mathcal{D}_\sigma} = \ell|_{\mathcal{D}_\sigma}$ *on a Dynkin-system* $\mathcal{D}_\sigma \subseteq \mathcal{F}$*.*

Proof We start proving the first part of the theorem. Let

$$\mathcal{D} := \{A \in \mathcal{F} : \ell(A) = u(A)\}, \tag{1}$$

and show that $\mathcal{D}$ is a pre-Dynkin-system. First, $\emptyset \in \mathcal{D}$ by assumption (a). Second, let $D \in \mathcal{D}$. Then $u(D^c) = 1 - \ell(D) = 1 - u(D) = \ell(D^c)$ by the conjugacy relation. Third, let $\{A_i\}_{i\in I} \subseteq \mathcal{D}$ for finite $I \subseteq \mathbb{N}$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, then

$$\begin{aligned}\sum_{i\in I} \ell(A_i) &\overset{(d)}{\leq} \ell\left(\bigcup_{i\in I} A_i\right) \\ &\overset{(\star)}{\leq} u\left(\bigcup_{i\in I} A_i\right) \\ &\overset{(c)}{\leq} \sum_{i\in I} u(A_i) \\ &\overset{Eq.1}{=} \sum_{i\in I} \ell(A_i).\end{aligned}$$

For $(\star)$, remark that $\ell(A) \leq u(A)$ for all $A \in \mathcal{F}$, since

$$\begin{aligned}\ell(A) + \ell(A^c) &\leq \ell(A \cup A^c) \\ &= 1 = u(A \cup A^c) \leq u(A) + u(A^c),\end{aligned}$$

we have,

$$\begin{aligned}&\ell(A) + \ell(A^c) \leq u(A) + u(A^c) \\ \Leftrightarrow\ &\ell(A) + 1 - u(A) \leq u(A) + 1 - \ell(A) \\ \Leftrightarrow\ &\ell(A) \leq u(A).\end{aligned}$$

Concluding, we define $\mu := \ell|_{\mathcal{D}} = u|_{\mathcal{D}}$ for which it is trivial to show that it is a finitely additive probability on $\mathcal{D}$.

For the second part, we first notice that continuity from below and from above are equivalent for conjugate set functions on set systems which are closed under complement [15, Proposition 2.3]. Next, we show that subadditivity and continuity from below of u imply σ-subadditivity of u: for $\{A_i\}_{i\in I} \subseteq \mathcal{F}$ such that $I \subseteq \mathbb{N}$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$ with $i, j \in I$ then $u\left(\bigcup_{i\in I} A_i\right) \leq \sum_{i\in I} u(A_i)$. In case that I is finite, subadditivity is provided by assumption. For infinite I we can construct an increasing sequence of sets, namely $B_j = \bigcup_{i\leq j} A_i$, so that $B_j \subseteq B_{j+1}$. Furthermore, $\bigcup_{j=1}^{\infty} B_j = \bigcup_{i\in I} A_i$. Thus,

$$\begin{aligned}u\left(\bigcup_{i\in I} A_i\right) &= u\left(\bigcup_{j=1}^{\infty} B_j\right) \\ &\overset{(e)}{=} \lim_{j\to\infty} u\left(B_j\right) \\ &= \lim_{j\to\infty} u\left(\bigcup_{i\leq j} A_i\right) \\ &\overset{(d)}{\leq} \lim_{j\to\infty} \sum_{i\leq j} u\left(A_i\right) \\ &= \sum_{i\in I} u\left(A_i\right).\end{aligned}$$

The same argument holds analogously for superadditivity and continuity from above of ℓ which is implied by continuity from below and the conjugacy relationship [15, Proposition 2.3]. In summary, the proof of the first part can then be applied again, now without the restriction that $I \subseteq \mathbb{N}$ is finite. Instead it potentially is countable. ■

Example 4 *Let* $\mathcal{F} = 2^{\Omega}$ *for* $\Omega = \{1, 2, 3, 4\}$*. We define* $\ell : \mathcal{F} \to [0, 1]$ *with* $\ell := \underline{\mu}_{\mathcal{D}}$ *where* $\underline{\mu}_{\mathcal{D}}$ *is defined as given in Figure 1. Futhermore,* $u : \mathcal{F} \to [0, 1]$ *by* $u(A) := 1 - \ell(A^c)$*. It is easy to show that* ℓ *and u fulfill the assumptions (a), (b), (c) and (d) in Theorem 7. The imprecise probabilities u and* ℓ *coincide on* $\{\emptyset, 12, 34, 13, 24, \Omega\}$*, the pre-Dynkin-system described in Example 1 and highlighted in Figure 1.*

In summary, imprecise probabilities are, under mild assumptions, precise on a pre-Dynkin-system or even a Dynkin-system. This, importantly, is as well the case if the system of precision is strictly larger than the trivial pre-Dynkin-systems $\{\emptyset, \Omega\}$. Exemplarily, a pair of conjugate, coherent lower and upper probability (e.g. [52, Section 2.7.4]) fulfills the conditions (a) - (d). Hence, the system of precision is a pre-Dynkin-system $\mathcal{D} \subseteq \mathcal{F}$. What if we first define precise,

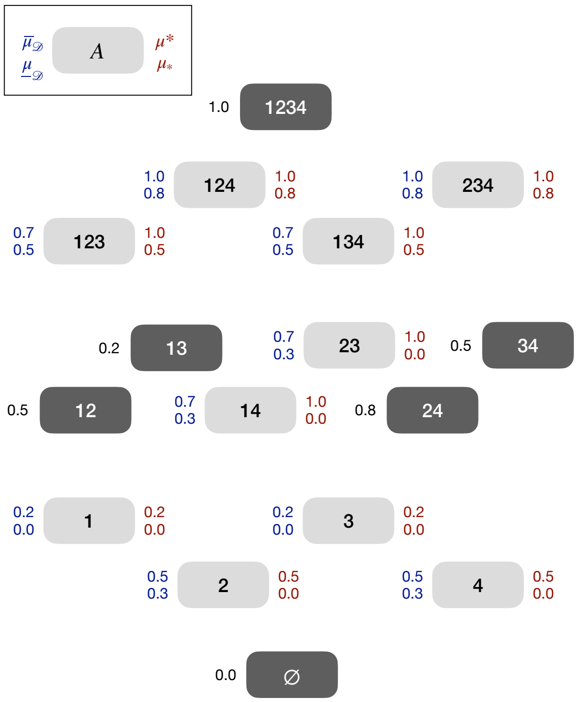

Figure 1: Illustration of the running example. The dark elements are contained in the pre-Dynkin-system $\mathcal{D}$ on $\Omega = \{1, 2, 3, 4\}$. The lower and upper coherent extension, respectively the inner and outer extension are denoted at the sides of the elements in the set system as shown in the example in the left upper corner. Elements in $\mathcal{D}$ possess a precise probability.

finitely additive probabilities on a pre-Dynkin-system, i.e. we fix a system of precision? We can then ask for "imprecise probabilities" deduced from this probability which are defined on a larger set structure, e.g. an algebra in which the pre-Dynkin-system is contained.

4. Extending Probabilities on Pre-Dynkin-Systems

Precise probabilities on pre-Dynkin-system naturally arise in many, very distinct, applied scenarios as we argued in the Introduction (§ 1). However, we acknowledge that the definition of probabilities on pre-Dynkin-systems is mathematically cumbersome. The possibilities to prove standard theorems is very limited as the approaches by Gudder [21, 22], Gudder and Zerbe [24] demonstrate. However, if we consider a probability defined on a pre-Dynkin-system as an imprecise probability on a larger set system with a fixed system of precision, we possibly obtain a richer, mathematical toolkit to work with. In this case, the encompassing set system preferably is an algebra. It remains to clarify how we construct the imprecise probability on the algebra from the precise probability on the pre-Dynkin-system.

4.1. Inner and Outer Extension

A simple but, as we show, unsatisfying solution is the use of an inner and outer measure extension. It does not rely on imposing any conditions on the probability defined on the pre-Dynkin-system. We pay for this generality with the few properties that we can derive for the obtained extension.

Proposition 8 (Inner and Outer Extension) *(see [56, Lemma 2.2]) Let $\mathcal{F}$ be an algebra on Ω, $\mathcal{D} \subseteq \mathcal{F}$ a pre-Dynkin-system and μ a finitely additive probability measure on $\mathcal{D}$. The inner probability*

$$\mu_*(A) := \sup\{\mu(B) : A \supseteq B \in \mathcal{D}\}, \ \forall A \in \mathcal{F},$$

and outer probability

$$\mu^*(A) := \inf\{\mu(B) : A \subseteq B \in \mathcal{D}\}, \ \forall A \in \mathcal{F},$$

define $\mu_*, \mu^* : \mathcal{F} \to [0, 1]$, for which the following conditions are fulfilled:

(a) Normalization: $\mu^*(\emptyset) = 0$, $\mu_*(\Omega) = 1$.

(b) Conjugacy: $\mu^*(A) = 1 - \mu_*(A^c)$, $\forall A \in \mathcal{F}$.

(c) Monotonicity: for $A, B \in \mathcal{F}$, if $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$.

Furthermore, μ_* is superadditive, for $A, B \in \mathcal{F}$ if $A \cap B = \emptyset$ then $\mu_*(A \cup B) \geq \mu_*(A) + \mu_*(B)$. But μ^* is not generally subadditive.

Example 5 *The inner and outer extension of μ on $\mathcal{D}$ as defined in Example 6 is given in Figure 1. The inner and outer extension are not coherent (Definition 11). In particular, the outer extension is not subadditive: $\mu^*(14) = 1 - \mu_*(23) = 1 > 0.2 + 0.5 = (1 - \mu_*(234)) + (1 - \mu_*(123)) = \mu^*(1) + \mu^*(4)$.*

In conclusion, the inner and outer extension provides an imprecise probability, which is not necessarily coherent (cf. Definition 11) and it does not fulfill the conditions required for Theorem 7 to post-hoc guarantee that the set of precision is a pre-Dynkin-system. For this reason we now explore another, more powerful extension method.

4.2. Extendability

In the following, we try to entirely embed pre-Dynkin-systems equipped with a probability into larger algebras. Then, we extend the probability defined on the pre-Dynkin-system in all possible ways to probabilities on the algebra. It turns out that this embedding is only possible under certain conditions on the probability defined on the pre-Dynkin-system. We call this condition *extendability*. Formally:

Definition 9 (Extendability) *Let $\mathcal{F}$ be an algebra on Ω and $\mathcal{D} \subseteq \mathcal{F}$ a pre-Dynkin-system. We call a finitely additive probability measure μ on $\mathcal{D}$* extendable to $\mathcal{F}$, *if and only if there is a finitely additive probability measure $\nu : \mathcal{F} \to [0, 1]$ such that $\nu|_{\mathcal{D}} = \mu$.*

The definition is non-vacuous [23, 12]. For instance, a finitely additive probability measure on a pre-Dynkin-System is not generally extendable to a measure on the generated algebra (e.g. Example 3.1 in [23]). If a probability is extendable, its extension is in general non-unique.

Extendability of probabilities on (pre-)Dynkin-systems has already been part of discussions in quantum probability since 1969 [20] up to more current times [11]. Several necessary and/or sufficient conditions on the structure of $\mathcal{D}$ and/or the values of μ are known [23, 12, 11]. We present here a sufficient and necessary condition discovered by Horn and Tarski [27].[13]

Theorem 10 (Extendability Condition) *(see Horn and Tarski [27]) Let $\mathcal{F}$ be an algebra on Ω and $\mathcal{D} \subseteq \mathcal{F}$ a pre-Dynkin-system. A finitely additive probability measure μ on $\mathcal{D}$ is extendable to $\mathcal{F}$, if and only if*

$$\sum_{k=1}^{m} \chi_{B_k}(\omega) - \sum_{j=1}^{n} \chi_{A_j}(\omega) \geq 0, \ \forall \omega \in \Omega$$
$$\implies \sum_{k=1}^{m} \mu(B_k) - \sum_{j=1}^{n} \mu(A_j) \geq 0$$

for all finite families of sets in $\mathcal{D}$: $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathcal{D}$.

Example 6 *For $\mathcal{D}$ as in Example 4 let $\mu \colon \mathcal{D} \to [0, 1]$ be defined as $\mu(\emptyset) = 0, \mu(12) = 0.5, \mu(34) = 0.5, \mu(13) = 0.2, \mu(24) = 0.8, \mu(\Omega) = 1$ (cf. Figure 1). The probability μ on $\mathcal{D}$ meets the extendability condition for $\mathcal{F} = 2^{\Omega}$.*

The reader familiar with [52] might already notice the remarkable similarity of this extendability condition with Walley's more general formulation of coherence. We confirm this intuition in the following.

[13] In fact, Theorem 10 can be stated for a more general definition of probabilities on arbitrary set systems e.g. [41, Theorem 3.2.10] [10, Proposition 2.2]. For the sake of consistency, we restrict this result to pre-Dynkin-systems and probabilities defined on pre-Dynkin-systems.

4.3. Extendability is Equivalent to Coherence

Extendability proves to be more than a helpful mathematical property for embedding pre-Dynkin-systems and their respective probabilities into algebras. Whether a probability defined on $\mathcal{D}$ can be extended to a probability on $\mathcal{F}$ is directly connected to the question whether the probability measure on $\mathcal{D}$ is coherent in the sense of [52, p. 68, p. 84] or not. Coherence is a minimal consistency requirement for probabilistic descriptions which has been introduced in de Finetti's fundamental work [9] and developed in Walley's book [52]. Shortly summarizing, an incoherent imprecise probability is tantamount to an irrational betting behavior, thus the name. Thus, extendability is, besides its mathematical convenience, a desirable property of probabilities in pre-Dynkin settings.

We adapt here the definition of coherence of previsions in [52, Definition 2.5.1] to probabilities.

Definition 11 (Coherent Probability) *Let $\mathcal{A} \subseteq 2^{\Omega}$ be an arbitrary collection of subsets. A set function $\underline{\nu} \colon \mathcal{A} \to [0, 1]$ is a* coherent lower probability *if, and only if*

$$\sup_{\omega \in \Omega} \sum_{i=1}^{j} (\chi_{A_i}(\omega) - \underline{\nu}(A_i)) - m(\chi_{A_0}(\omega) - \underline{\nu}(A_0)) \geq 0,$$

for non-negative $n, m \in \mathbb{N}$ and $A_0, A_1, \ldots A_n \in \mathcal{A}$. If $\mathcal{A}$ is closed under complement, the conjugate coherent upper probability *is given by $\overline{\nu}(A) := 1 - \underline{\nu}(A^c)$ for all $A \in \mathcal{A}$. If furthermore $\overline{\nu}(A) = \underline{\nu}(A)$ for all $A \in \mathcal{A}$, we call $\nu := \overline{\nu}$ a* coherent additive probability.

At first sight the Horn-Tarski condition given in Theorem 14 and the coherence condition presented here already appear similar. This becomes even more apparent in Walley's reformulation of coherence for additive probabilities [52, Theorem 2.8.7]. In the following, we show that this superficial similarity is indeed based on a rigorous link. Surprisingly, Walley did not mention Horn and Tarski's work in his foundational book.

Proposition 12 (Extendability Equals Coherence) *Let $\mathcal{F}$ be an algebra on Ω and $\mathcal{D} \subseteq \mathcal{F}$ a pre-Dynkin-system. A finitely additive probability measure μ on $\mathcal{D}$ is extendable to $\mathcal{F}$ if, and only if it is a coherent additive probability on $\mathcal{D}$.*

Proof If μ is a coherent additive probability on $\mathcal{D}$, then the linear extension theorem [52, Theorem 3.4.2] applies. Hence, a coherent additive probability $\nu \colon \mathcal{F} \to [0, 1]$ exists, such that $\nu|_{\mathcal{D}} = \mu$. In particular, ν is a finitely additive probability following Definition 6 on $\mathcal{F}$ [52, Theorem 2.8.9].

For the converse direction we observe that if μ possess an extension following Definition 9, then such an extension is a finitely additive probability on $\mathcal{F}$ following Definition 6. Hence, Walley [52, Theorem 2.8.9] guarantees that the extension is a coherent additive probability (Definition 11). Any restriction to a subdomain $\mathcal{D} \subseteq \mathcal{F}$ keeps the probability coherent and additive. ■

The linear extension theorem in Walley [52, Theorem 3.4.2] used here is a generalization of de Finetti's fundamental theorem of probability [9, Theorems 3.10.1 & 3.10.7]. De Finetti's theorem is furthermore interesting, as he explicitly states that a coherent additive probability defined on an arbitrary collection of sets can be extended in a precise way (so lower and upper probability coincide) to some sets. De Finetti does not characterize this collection. Our Theorem 7, however, gives an answer to this question: the collection forms a pre-Dynkin-system.

Proposition 12 provides a missing link between two strands of work: on the one hand, probabilities on pre-Dynkin-systems and related weak set structures have been closely investigated in foundational quantum probability theory [20, 22] and decision theory [18, 56]. On the other hand, coherent probabilities are central to imprecise probability, in particular, the more general formulations of coherent previsions and risk measures [52, 13, 38]. Not far from this relation, Casanova et al. [4] bridged desirability to marginal problems. Desirability is an even more general framework for imprecise probability [53]. Marginal problems can equivalently expressed in terms of probabilities on pre-Dynkin-systems [51, 30]

A probability on a pre-Dynkin-system $\mathcal{D}$, even when extendable, only allows for probabilistic statements on $\mathcal{D}$ itself. However, extendability guarantees that a "nice" embedding into a larger system of measurable sets exists. More specifically, extendability expressed in terms of credal sets provides a well-known tool for the worst-case extension of a probability from a pre-Dynkin-system to a larger algebra.

4.4. Coherent Extension

If a finitely additive probability on a pre-Dynkin-system is extendable, then we can obtain lower and upper probabilities of events which are not in the pre-Dynkin-system but on a larger algebra. We follow the idea of natural extensions, e.g. as described by [52, p. 136]. In particular, [52, Theorem 3.3.4 (b)] directly applies as long as a probability on a pre-Dynkin-system is extendable.

Corollary 13 (Coherent Extension of Probability) *Let $\mathcal{F}$ be an algebra on Ω and $\mathcal{D} \subseteq \mathcal{F}$ a pre-Dynkin-system. We define $M(\mu, \mathcal{D}) := \{\nu \in P \colon \nu(A) = \mu(A),\ \forall A \in \mathcal{D}\}$, the credal set. If a finitely additive probability measure μ on $\mathcal{D}$ is extendable to $\mathcal{F}$, then $\forall A \in \mathcal{F}$,*

$$\underline{\mu}_{\mathcal{D}}(A) := \inf_{\nu \in M(\mu,\mathcal{D})} \nu(A), \quad \overline{\mu}_{\mathcal{D}}(A) := \sup_{\nu \in M(\mu,\mathcal{D})} \nu(A).$$

define a coherent lower respectively upper probability on $\mathcal{F}$ following Definition 11.

Example 7 *The coherent extension of μ on $\mathcal{D}$ as defined in Example 6 is shown in Figure 1 (cf. [52, p. 122]). Even though coherent, $\underline{\mu}_{\mathcal{D}}$ is neither super- nor submodular:*

$$\underline{\mu}_{\mathcal{D}}(12) + \underline{\mu}_{\mathcal{D}}(13) = 0.7 > 0.5 = \underline{\mu}_{\mathcal{D}}(123) + \underline{\mu}_{\mathcal{D}}(1),$$
$$\underline{\mu}_{\mathcal{D}}(1) + \underline{\mu}_{\mathcal{D}}(2) = 0.3 < 0.5 = \underline{\mu}_{\mathcal{D}}(12) + \underline{\mu}_{\mathcal{D}}(\emptyset).$$

This implies that as well $\overline{\mu}_{\mathcal{D}}$ is neither super- nor submodular [15, Proposition 2.3].

These lower and upper probabilities allow for at least two interpretations: We can assume that a precise probability on a pre-Dynkin-systems $\mathcal{D} \subseteq \mathcal{F}$ just reveals its values on $\mathcal{D}$, but is actually defined over $\mathcal{F}$. Then the lower and upper probability constitute lower and upper bounds of the precise "hidden probability" on $\mathcal{F}$, which is solely accessible on $\mathcal{D}$. On the other hand, we can even reject the existence of such precise "hidden probability". Then lower and upper probability *are* the inherently imprecise probability of an event in $\mathcal{F}$ but not in $\mathcal{D}$.[14]

The obtained lower and upper probabilities represent the imprecise interdependencies between all events of precise probabilities. We illustrate this statement: in the variety of updating methods in imprecise probability we pick the generalized Bayes' rule [52, Section 6.4] to exemplarily compute the conditional probability of two events for the coherent extension of a probability from a pre-Dynkin-system . For $A, B \in \mathcal{D}$ such that $\mu(B) > 0$ the generalized Bayes' rule gives [52, Theorem 6.4.2]:

$$\overline{\mu}_{\mathcal{D}}(A|B) := \sup_{\nu \in M(\mu,\mathcal{D})} \frac{\nu(A \cap B)}{\nu(B)}$$
$$= \frac{\sup_{\nu \in M(\mu,\mathcal{D})} \nu(A \cap B)}{\mu(B)} = \frac{\overline{\mu}_{\mathcal{D}}(A \cap B)}{\mu(B)}.$$

We can easily rewrite $\overline{\mu}_{\mathcal{D}}(A \cap B) = \overline{\mu}_{\mathcal{D}}(A|B)\mu(B)$. In this case the imprecision of the probability of the intersected event is *purely* controlled by the conditional probability $\overline{\mu}(A|B)$ and *not* by the marginal, which is precise. So, the imprecision captured by the lower and upper probabilities locates solely in the interdependency of the events. We remark that Dempster's rule gives the same conditional probability here [14].

[14] As remarked by Walley [52, p. 138], De Finetti [9] surprisingly only considered the first mentioned interpretation.

4.5. Inner and Outer Extension Is More Pessimistic Than Coherent Extension

We presented two extension methods for probabilities defined on pre-Dynkin-systems in this paper. We relate the methods in the following. In the case of an extendable probability we can guarantee the following inequalities to hold.

Theorem 14 (Extension Theorem) *Let $\mathcal{F}$ be an algebra on Ω, $\mathcal{D} \subseteq \mathcal{F}$ a pre-Dynkin-system and μ a finitely additive probability on $\mathcal{D}$ which is extendable to $\mathcal{F}$. Then*

$$\mu_*(A) \leq \underline{\mu}_{\mathcal{D}}(A) \leq \overline{\mu}_{\mathcal{D}}(A) \leq \mu^*(A), \qquad \forall A \in \mathcal{F}.$$

Proof Since $\mathcal{D} \subseteq \mathcal{F}$, we easily obtain

$$\begin{aligned}\mu_*(A) &= \sup\{\mu(B) \colon A \supseteq B \in \mathcal{D}\}\\ &= \sup\{\underline{\mu}_{\mathcal{D}}(B) \colon A \supseteq B \in \mathcal{D}\}\\ &\leq \sup\{\underline{\mu}_{\mathcal{D}}(B) \colon A \supseteq B \in \mathcal{F}\}\\ &= \underline{\mu}_{\mathcal{D}}(A),\end{aligned}$$

for all $A \in \mathcal{F}$. The other inequalities follow by the conjugacy of inner and outer measure, and lower and upper coherent extension. ■

In words, Theorem 14 states that the inner and outer extension is more "pessimistic" than the coherent extension. We allude to "pessimistic" in the sense of giving a looser bound for the probabilities assigned to elements not in the pre-Dynkin-system $\mathcal{D}$ but in $\mathcal{F}$.

5. The Credal Set and its Relation to Pre-Dynkin-System Structure

In the first section of this paper, we derived pre-Dynkin-systems as the system of precision for relatively general imprecise probabilities. Then, we showed that, under extendability conditions, a precise probability on a pre-Dynkin-system gives rise to a neat imprecise probability on an encompassing algebra. In other words, imprecise probabilities can be "mapped" to pre-Dynkin-systems and vice-versa. We show that it is possible to concretize these mappings in the following.

In addition to the notations of previous sections, we fix $\mathcal{F}$ to be an algebra on Ω and P to be the set of all finitely additive probabilities on $\mathcal{F}$. Additionally, we introduce $\psi \in P$ as a "reference measure" on $\mathcal{F}$. The probability ψ replaces μ to emphasize the difference that μ as has been used earlier is not necessarily in P. First, we define a function which maps set systems to so-called credal sets. Credal sets are in one-to-one correspondence to coherent lower and upper previsions, i.e. generalizations of coherent lower and upper probabilities [52, Theorem 3.6.1].[15]

Definition 15 (Credal Set Function) *For a fixed finitely additive probability $\psi \in P$ we call*

$$m \colon 2^{\mathcal{F}} \to 2^P, m(\mathcal{A}) := \{\nu \in P \colon \nu(A) = \psi(A),\ \forall A \in \mathcal{A}\},$$

the credal set function.

Second, we introduce the dual function which maps subsets of the set of probabilities on $\mathcal{F}$ to set systems.

Definition 16 (Dual Credal Set Function) *For a fixed finitely additive probability $\psi \in P$ we call*

$$m^\circ \colon 2^P \to 2^{\mathcal{F}}, m^\circ(Q) := \{A \in \mathcal{F} \colon \nu(A) = \psi(A), \forall \nu \in Q\},$$

the dual credal set function.

Those two introduced functions possess a series of helpful (and expectable) properties.

Proposition 17 (Properties of Credal Set Function) *Let $\mathcal{A} \subseteq \mathcal{F}$ and $Q \subseteq P$. The following properties hold:*

(a) *The set $m(\mathcal{A})$ is a pre-Dynkin-system.*

(b) *The set $m^\circ(Q)$ is a credal set, i.e. non-empty, weak*-closed and convex subset of P (see [52, Theorem 2.8.9 & 3.6.1]).*[16]

(c) *The credal set function m and the dual credal set function m° form a so-called Galois connection, i.e. $\mathcal{A} \subseteq m^\circ(Q) \Leftrightarrow Q \subseteq m(\mathcal{A})$.*

Galois connections partially map order structure between sets of sets (or more generally between posets). The mappings involved in the Galois connection are antitone, i.e. for $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{F}$ and $Q_1, Q_2 \subseteq P$, $\mathcal{A}_1 \subseteq \mathcal{A}_2 \Rightarrow m(\mathcal{A}_2) \subseteq m(\mathcal{A}_1)$ and $Q_1 \subseteq Q_2 \Rightarrow m^\circ(Q_2) \subseteq m^\circ(Q_1)$ [2, V.8]. Their pairwise application is extensive, i.e. for $Q \subseteq P$ and $\mathcal{A} \subseteq \mathcal{F}$ it holds $Q \subseteq m(m^\circ(Q))$ and $\mathcal{A} \subseteq m^\circ(m(\mathcal{A}))$ [2, V.8].

However, Galois connections are weaker than structural isomorphisms, but induce exactly such: in fact, every Galois connection defines closure operators, i.e. extensive, monotone and idempotent maps [43, Definition 4.5.a]. Bipolar-closed sets are sets $\mathcal{A} \subseteq \mathcal{F}$ such that $\mathcal{A} = m^\circ(m(\mathcal{A}))$, respectively $Q \subseteq P$ such that $Q = m(m^\circ(Q))$. Most importantly, the bipolar-closed sets form two antitone isomorphic lattices ordered by set inclusion [2, Theorem V.8.20]. For illustration see Figure 2. Since a bipolar-closed subset of $\mathcal{F}$ is a pre-Dynkin-system and a bipolar-closed subset of P is a credal set (Proposition 17), the relationship between bipolar-closed sets give us a lattice duality between set systems and imprecise probabilities.

[15] We argue that the domain is indeed the set of credal sets in Proposition 17.

[16] We refer to closedness with respect to the weak$\star$-topology on $ba(\Omega, \mathcal{F})$, i.e. the space of all bounded finitely additive measures on $\mathcal{F}$, here.

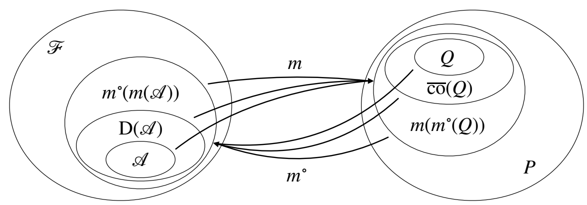

Figure 2: Galois connection between the lattice of pre-Dynkin-systems and the set of credal sets. In the illustrated case, we have $m^{\circ}(Q) = m^{\circ}(m(m^{\circ}(\mathcal{A}))$ respectively $m(\mathcal{A}) = m(m^{\circ}(m(Q))$. The set containment on both sides follow from the properties of the defined Galois connection.

5.1. Interpolation from Algebra to Trivial Pre-Dynkin-System

In standard probability theory there is a choice of (σ-)algebra to be made to define the set of events which get assigned probabilities. This significant choice has (mathematically) been standardized (cf. standard probability space). However, we remind the reader of the introductory examples (§ 1). It cannot be taken for granted that all events are compatible with all others, as implied by a (σ-)algebra (cf. Section 2.1). As e.g. argued by Khrennikov [33], a more appropriate probabilistic modeling should appeal to weaker domains for probabilities. For instance, von Mises' axiomatization of probability inherently reflects potential incompatible events in terms of a pre-Dynkin-system [50, 44]. In other words, there is a choice to be made about the system of precision. Which sets should be compatible to each other, which should not? How do the choices of the systems of precision relate to each other?

In the above Section 5, we have shown that, without loss of generality, the range of choices is captured by the system of pre-Dynkin-systems $\mathfrak{D} := \{\mathcal{D} \subseteq \mathcal{F} : \mathcal{D} = \mathrm{D}(\mathcal{D})\}$. Ordered by set inclusion $(\mathfrak{D}, \subseteq)$ forms a lattice with $\mathcal{D}_1 \vee \mathcal{D}_2 = \mathrm{D}(\mathcal{D}_1 \cup \mathcal{D}_2)$ and $\mathcal{D}_1 \wedge \mathcal{D}_2 = \mathcal{D}_1 \cap \mathcal{D}_2$. This lattice spans a range of choices from $\mathcal{D} = \mathcal{F}$, i.e. the domain of events with precise probability is the entire algebra, to $\mathcal{D} = \{\emptyset, \Omega\}$, i.e. except of $\emptyset$ and Ω all sets in $\mathcal{F}$ get assigned the vacuous lower probability 0 and upper probability 1. How "close" $\mathcal{D}$ is to the algebra $\mathcal{F}$ determines how "classical" the probability behaves. In other words, $(\mathfrak{D}, \subseteq)$ parametrizes a family of imprecise probabilities. Unfortunately, the mentioned interpolation is slightly improper. It turns out that there are pre-Dynkin-systems $\mathcal{D}_1 \neq \mathcal{D}_2$ such that $m(\mathcal{D}_1) = m(\mathcal{D}_2)$. One can easily show that if one focuses on pre-Dynkin-systems $\mathcal{D} \subseteq \mathfrak{D}$ which are bipolar-closed $m^{\circ}(m(\mathcal{D})) = \mathcal{D}$, then the mapping m is injective and the interpolation is proper. Each bipolar-closed pre-Dynkin-system is assigned a unique imprecise probability. How this family of imprecise probabilities links to other parametrized families of imprecise probabilities is an open question. A first glimpse of an answer is given in our extended companion paper [16].

6. Conclusion

We have explicated relations between the system of precision and imprecise probabilities. Under weak assumptions the system of precision of an imprecise probability is a pre-Dynkin-system. From an opposite perspective, we detailed under which circumstances probabilities defined on pre-Dynkin-systems can be extended to an algebra. Extendability, a crucial property for obtaining nicely behaving extensions, was revealed to be equivalent to coherence of the probability. This observation links together hitherto unrelated areas of research: coherent imprecise probabilities and probabilities defined on sparse set structures. In the course of this investigation, we found the inner and outer measure construction to bound the lower and upper coherent extension. Finally, we shortly discussed a family of imprecise probabilities which follow an order structure provided by the lattice of pre-Dynkin-systems. Generally, we stayed within the framework of set systems and probability measures. How does the system of precision look like for general lower and upper prevision defined on more general sets of gambles? We conjecture that the analogues of (pre-)Dynkin-systems are closed, convex subsets of the set of all gambles. We leave this question open to future work contributing to an understanding of the system of precision and the imprecise probability model.

Acknowledgments

This work was funded in part by the Deutsche Forschungsgemeinschaft under Germany's Excellence Strategy — EXC number 2064/1 — Project number 390727645.

The authors thank the International Max Planck Research School for Intelligent System for supporting Rabanus Derr. Many, many thanks to Christian Fröhlich for helpful discussions and feedback.

We thank all anonymous reviewers for the detailed and helpful feedback. In particular, with their help the statements and proofs of Proposition 12 and Theorem 14 have been cleaned and shortened substantially.

Author Contributions

Robert C. Williamson supervised and steered this project. He wrote short paragraphs and took over the internal editing. Rabanus Derr wrote the majority of the paper.

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