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A Pointfree Approach to Measurability and Statistical Models

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Abstract

In this work we approach the problem of finding the most *natural* algebraic structure of the set of all possible random variables on a measurable space, inspired by Nelson's point of view. We build our work on previous papers by the same authors and set our investigation in the framework of MV-algebras and algebraic logic. We approach the problem from the perspective of pointfree topology, in order to take the notion of random variable as the primitive one. In the final part of the paper we approach statistical models from the point of view of algebra and category theory, providing a different and perhaps more insightful justification for our logico-algebraic approach to the notion.

Keywords: MV-algebras, pointfree topology, probability, measurable functions, statistical models

1. Introduction

Many treatises of probability start from Kolmogorov triples (Ω, Σ, P) where Ω is a set, Σ is a σ-algebra of subsets of Ω and P is a probability measure on Σ. This approach is heavily set-theoretic, because the foundation of everything is the sample space Ω, which is a set devoid of any structure. However, the interesting properties of probability spaces do not really depend on Ω. Starting from this remark, a pointfree approach to probability has been advocated, for instance, by mathematicians like B. de Finetti [4], E. De Giorgi [5], T. Tao [23] and G.C. Rota [21].

Based on the idea that probability can be founded on abstract algebra, one might search for the most *natural* algebraic structure of the set of all possible random variables on a measurable space, as can be seen in Nelson's approach, see [19]. This route is taken by Mundici in [18], where he shows that a quite natural algebraic structure for the set of all continuous random variables on a compact and Hausdorff space is given by MV-algebras. These are the algebras that serve as models for Łukasiewicz logic, one of the most prominent many-valued logics. The approach taken by Mundici is based on the fact that Łukasiewicz logic and MV-algebras have been proven to be a fruitful ground to deal with probability theory in a *logico-algebraic* setting. Indeed, it was Mundici in [15] that showed how to define a notion of probability, under the name of *state* since it is inspired by the theory of quantum structures. Moreover, Mundici proved in [16] that states are coherent *à la de Finetti*, meaning that any assignment on *many-valued* events satisfies a coherence criterion if and only if it can be extended to a state. By many-valued event we mean an event that is codified using the language of Łukasiewicz logic. This is because logical formulas (possibly modulo logical equivalence) give a natural way of modeling an event that is being described to us by sentences of natural language.

Thus, Mundici's work on states has been the starting point for what is nowadays understood as a way to model uncertainty by allowing for more complex events. This offers an alternative to the idea of dealing with imprecise probabilities via, for example, possibility and necessity measures, belief and plausibility functions, upper and lower probabilities.

This paper has its roots in the remark that the language of MV-algebras (and Łukasiewicz logic) is able to capture only the class of continuous random variables, instead of taking into account the totality of all (possibly non-continuous) measurable functions. The first step to move beyond continuity is taken in [8], where the authors define *observables* to be generalized random variables. Loosely speaking, the main point of [8] is to generalize the idea that a classical random variable induces, via preimages, a homomorphism between σ-algebras. Thus, to obtain a generalized random variable, instead of σ-algebras we use a certain class of MV-algebras. The right choice for this class has provided a setting which, on the one hand allows to obtain a logico-algebraic setting for a wider class of *classical* random variables (with respect to Mundici's result) and on the other hand gives a setting that allows to *generalize* classical random variables to homomorphisms in an appropriate variety of algebras. This type of model gives a sort of layering of the uncertainty that allows to handle at the same time the uncertainty due to classical phenomena and the vagueness do to non-classical phenomena.

Our framework will be, in particular, one having events that depend on countably many variables and that are codified in a *rich* logical system that expands the one of Łukasiewicz logic. By this we mean a logic whose algebraic models have enough structure to allow us to compute, at least, limits of sequences of functions and countable suprema. These models are called σ-complete Riesz MV-algebras and one can simply think of (some of) them as a universal-algebraic counterpart of measurable functions: this will be the first result on this paper.

To be slightly more precise, we build our investigation on one of the main results of [8], that is Theorem 3.3. Such a theorem yields a correspondence between classical $[0,1]$-valued random variables and *generalized* random variables having a certain class of algebras (called σ-semisimple in [9]) as co-domain. The theorem gives a way of describing the uncertainty of an event in two distinct layers: the first layer is a purely probabilistic one, while the second is the fuzzy component (or vagueness) of the event itself. Thus, we want to study classical random variables and generalized random variables from a *pointfree* point of view and at once; we do so by studying from different perspectives the class of algebras that appear in [8], see also Theorem 3.

More specifically, the goal of Section 3 is to find adequate topological restrictions that allow to generalize Mundici's result on continuous random variables. In particular, we prove in Proposition 8 that only a subclass of compact Hausdorff spaces is suitable to discuss measurability and we give a topological description of the spaces X such that $C(X)$ is an algebra of measurable functions from some σ-algebra to $[0,1]$.

In Section 4 we show that our framework affords, via categorical equivalence, a completely point free approach. Indeed in Corollary 11 and Proposition 18 we prove that both Borel-measurable functions and generalized random variables have suitable counterparts in the setting of point-free topology. This shows that the theory of σ-complete Riesz MV-algebras provides an adequate setting to discuss random variables in terms of abstract universal algebra both in the classical and generalized case.

Section 5 takes its inspiration from [22], where a categorical point of view is given for random variables. The main goal of the section is to show that the notion of logico-algebraic statistical model given in [12] can be lifted to a pre-sheaf (loosely speaking, a contravariant functor into the category to sets) similarly to the work of A. Simpson for random variables. We show in Theorem 21 that states on our algebras induce a mapping of statistical models to states on free algebras.

2. Preliminary Notions

As this paper touches topology, universal algebra, functional analysis, (non-classical) probability and category theory, in this section and through the paper we try to give as many preliminary notions as possible. For more details on the category-theoretical notions, we refer the interested reader to [13].

2.1. Algebras of Łukasiewicz logic

This work mainly deals with a class of expansions of MV-algebras, called σ-complete Riesz MV-algebras. These are (up to isomorphism) MV-algebras of continuous functions from a compact Hausdorff space to $[0,1]$. We now give the fundamental notions needed here, but we urge the interested reader to consult the cited bibliography for further detail.

Recall that an MV-algebra is a structure $(A, \oplus, \neg, 0)$ such that $(A, \oplus, 0)$ is a commutative monoid and $\neg$ is a unary operation such that

$$\begin{aligned} \neg\neg x &= x, \\ x \oplus \neg 0 &= \neg 0, \\ \neg(\neg x \oplus y) \oplus y &= \neg(\neg y \oplus x) \oplus x. \end{aligned}$$

Further operations can be defined by setting

$$\begin{aligned} 1 &:= \neg 0, \\ x \odot y &:= \neg(\neg x \oplus \neg y), \\ x \ominus y &:= x \odot \neg y. \end{aligned}$$

Every MV-algebra is a distributive lattice under the order given by $x \leq y$ if and only if there is some z such that $x \oplus z = y$, or equivalently, $x \ominus y = 0$. The prototypical example of an MV-algebra is the real interval $[0,1]$ where

$$x \oplus y = \min(x+y, 1) \text{ and } \neg x = 1 - x.$$

It generates MV-algebras both as a variety and as a quasi-variety.

Enriching MV-algebras with additional structure has been a fruitful research direction of the past two decades. Among the outcomes of this approach we find the class of Riesz MV-algebras, (two sorted) structures (A, ∇) where A is an MV-algebra and $\nabla\colon [0,1] \times A \to A$. We write $rx = \nabla(r, x)$. Additional compatibility conditions must hold, see [6], but for the purpose of this work we simply mention that ∇ is meant to model a (real vector space) scalar multiplication. Note that Riesz MV-algebras are an equational variety, like MV-algebras, and such variety is again generated by $[0,1]$.

This work is framed in the setting of σ-complete (i.e., closed under countable suprema) Riesz MV-algebras, introduced in [7] as the infinitary variety $\mathbf{RMV}_\sigma$ obtained by enriching the language of Riesz MV-algebras with an operation of countable arity meant to model the countable disjunction.

In $\mathbf{RMV}_\sigma$ we consider the notion of ideal as usual given in MV-algebras: ideals are downward closed sets that are closed under $\oplus$ and contain 0. For any $A \in \mathbf{RMV}_\sigma$, MV-maximal ideals are simply ideals that are maximal (with respect to the inclusion) for the MV-reduct of A. Furthermore, a σ-ideal is an ideal closed under countable suprema. We denote by $\mathsf{Max}\,(A)$ the set of all MV-maximal ideals of A, while $\mathsf{Max}_{\sigma}\,(A)$ denotes the set of MV-maximal ideals of A that are also σ-ideals. We recall that $\mathsf{Max}\,(A)$ is a compact and Hausdorff topological space for any MV-algebra, see [17, Proposition 4.15].

By [7, Theorem 4.6], each σ-complete Riesz MV-algebra is of the form $C(X)$, where X coincides with $\mathsf{Max}\,(A)$ and it is a compact Hausdorff and *basically disconnected* topological space. A compact and Hausdorff topological space is called basically disconnected if the closure of any countable union of clopen set is open.

In [7, 9], it was proved that, when κ is countable, the free κ-generated algebra in $\mathbf{RMV}_{\sigma}$ coincides with the algebra $\mathsf{Borel}([0,1]^{\kappa})$. This is the algebra of all Borel-measurable functions $f\colon [0,1]^{\kappa} \to [0,1]$. We recall that, for a given topological space X, the Borel σ-algebra of X is the σ-algebra generated by the open subsets of X and it will be subsequently denoted by $\mathcal{BO}(X)$. Consequently, a function $f\colon X \to [0,1]$ is Borel measurable if the preimage of a Borel subset of $[0,1]$ (with the Euclidean topology) is a Borel subset of X.

We denote by I, S and P the standard universal-algebraic operators in $\mathbf{RMV}_{\sigma}$, that is, taking isomorphic images, taking subalgebras and taking products. We call Riesz tribes (or simply tribes) those algebras that belong to $SP([0,1])$, while algebras in $ISP([0,1])$ have been characterized in [9] as follows. Note that belonging to $SP([0,1])$ implies belonging to $ISP([0,1])$.

Theorem 1 *[9, Theorem 4.11 and Corollary 4.16] Let $A \in \mathbf{RMV}_{\sigma}$. The following are equivalent:*

(i) $A \in ISP([0,1])$.

(ii) The intersection of all MV-maximal σ-ideals of A is trivial, in symbols

$$\bigcap \{M \mid M \in \mathsf{Max}_{\sigma}\,(A)\} = \{0\}.$$

In addition, if A has a countable set of generators, the previous conditions are also equivalent to the following.

(iii) There exist a countable cardinal κ and a subset $\mathcal{V} \subseteq \mathcal{BO}([0,1]^{\kappa})$ such that $A \simeq \mathsf{Borel}(V)$, where $V = \bigcap_{W \in \mathcal{V}} W$ and $\mathsf{Borel}(V)$ is the algebra of restrictions to V of elements of $\mathsf{Borel}([0,1]^{\kappa})$.

We call an algebra *σ-semisimple* if it satisfies one of the equivalent conditions of Theorem 1.

Finally, given two categories $\mathbf{C}$ and $\mathbf{D}$, a functor $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ yields an equivalence of categories if and only if the following three conditions hold:

- it is *full*, that is, for any two objects A, B in $\mathbf{C}$ and any $f_1, f_2\colon A \to B$, if $\mathcal{F}(f_1) = \mathcal{F}(f_2)$, then $f_1 = f_2$;
- it is *faithful*, that is, for any $g\colon \mathcal{F}(A) \to \mathcal{F}(B)$ in $\mathbf{D}$ there exists $f\colon A \to B$ such that $\mathcal{F}(f) = g$;
- it is *essentially surjective*, that is each object $D \in \mathbf{D}$ is isomorphic to an object of the form $\mathcal{F}(A)$, $A \in \mathbf{C}$.

Equivalently, there exists a functor $\mathcal{G}$ that is inverse of $\mathcal{F}$ in a precise sense, see [13]. Furthermore, for any A, B objects in $\mathbf{C}$, denoted by $\mathsf{Hom}_{\mathsf{C}}(A,B)$ the set of all morphisms $f\colon A \to B$, the first two conditions of the above definition are equivalent to saying that $\mathsf{Hom}_{\mathsf{C}}(A,B)$ and $\mathsf{Hom}_{\mathsf{D}}(\mathcal{F}(A),\mathcal{F}(B))$ are isomorphic for any A and B in $\mathbf{C}$, with the obvious definition for $\mathsf{Hom}_{\mathsf{D}}(-,-)$. A *duality* is an equivalence induced by a contravariant functor $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$, that is a functor that reverses domain and codomain of the arrows, so that $\mathsf{Hom}_{\mathsf{C}}(A,B)$ and $\mathsf{Hom}_{\mathsf{D}}(\mathcal{F}(B),\mathcal{F}(A))$ are isomorphic.

2.2. Probability on Many-Valued Events

Probability measures are encoded in MV-algebras using the notion of a *state*, as introduced by D. Mundici with the idea of obtaining an averaging process for formulas in Łukasiewicz logic.

Any MV-algebra A can be endowed with a partial operation, denoted by $+$, defined when $x \odot y = 0$, for $x, y \in A$. In this case $x + y := x \oplus y$. Using this partial operation, a *state* of a Riesz MV-algebra A is a map $s : A \to [0,1]$ satisfying the following conditions:

(1) $s(1) = 1$,

(2) for all $x, y \in A$ such that $x \odot y = 0$, $s(x \oplus y) = s(x) + s(y)$.

A σ-state is a state that, in addition, preserves countable suprema of increasing sequences, that is,

(3) If $\{a_n\}_{n\in\mathbb{N}}$ is an increasing sequence of elements of A, then $s(\bigvee_n a_n) = \bigvee_n s(a_n)$.

In the case of tribes, we can obtain an integral representation for σ-states, which was firstly proved by Butnariu and Klement, see [3, Chapter II, Section 6]. We give here the version more suited to our framework, see also [8, Theorem 2.2].

Theorem 2 *For every Riesz tribe $A \subseteq [0,1]^X$, set $\mathcal{S}(A) = \{B \subseteq X \mid \chi_B \in A\}$. For every σ-state s of A, there exists a measure $\mu_s : \mathcal{S}(A) \to [0,1]$, given by $\mu(B) = s(\chi_B)$, such that for every $f \in A$,*

$$s(f) = \int_X f \,\mathrm{d}\mu_s.$$

Note that a more general version of Theorem 2 is the so-called Kroupa-Panti theorem, see [17, Theorem 10.5] for a precise statement. Furthermore, for any Riesz MV-algebra A and for every state s on A, we have $rs(a) = s(ra)$, for all $a \in A$ and $r \in [0,1]$. We also remark that the correspondence between σ-states and measures detailed in Theorem 2 is actually an isometry. Indeed, in [1, Theorem 3.2.1] it is proved that the set of states on a tribe A and the set of measures on the corresponding $\mathcal{S}(A)$ can both be endowed with a metric structure (making them Banach spaces) in a way that the assignment above defined becomes an isometry.

In [11] the author shows that also in the infinitary setting of σ-complete Riesz MV-algebras, a coherence criterion à la de Finetti holds, allowing to think of σ-states on σ-semimple algebras as subjective probabilities. Furthermore, the author also proves that for any σ-semisimple algebra presented (in the sense of Theorem 1(3)) by some Polish space, the notions of state and σ-state coincide. Hence, in this particular case, the requirement of σ-additivity is a consequence of the other axioms in the definition of states.

States are the algebraic counterpart of probability measures, while in literature it is often used the term *observable* to denote the algebraic counterpart of a random variable. We mention here the point of view that is relevant to our development.

In [8] observables are defined as homomorphisms from the free algebra in $\mathbf{RMV}_\sigma$ to any other σ-complete Riesz MV-algebra. In more detail, if κ is any cardinal and $A \in \mathbf{RMV}_\sigma$, a κ-dimensional observable posed in A is a σ-homomorphism

$$\mathcal{X}\colon \mathsf{Borel}([0,1]^\kappa) \to A$$

When $A \in ISP([0,1])$, that is, a σ-semisimple algebra, Theorem 3 gives a bijection between classical random variables and generalized random variables. Hence, the focus of Mundici's point of view is on continuous random variables taken as element of some algebra, while in [8] random variables are mappings between algebras. Thus, we can think of an observable as a *two-layer random variable* in the sense of the following theorem.

Theorem 3 *[8, Theorem 3.3] Let X be a nonempty set, $A \subseteq [0,1]^X$ and $f : X \to [0,1]^\kappa$ a measurable function w.r.t. $\mathcal{S}(\mathcal{T}) = \{Y \subseteq X \mid \chi_Y \in A\}$. Then the function*

$$\mathcal{X}_f : \mathsf{Borel}([0,1]^\kappa) \to A, \quad \mathcal{X}_f(a) = a \circ f$$

is a κ-dimensional observable on A.

Conversely, for any κ-dimensional observable $\mathcal{X}$ on A, there exists a unique $f : X \to [0,1]^\kappa$ such that $\mathcal{X} = \mathcal{X}_f$.

It is also worth recalling that any σ-complete Riesz MV-algebra A is naturally closed under taking uniform limits, where the norm on $A := C(\mathsf{Max}\,(A))$ is the uniform norm. Furthermore, by [9, Lemma 5.6] every element of such an A is a countable supremum of multiples of continuous characteristic functions. Hence, any $f \in C(X)$ can be approximated by functions that take a finite number of values. These are desirable properties for an algebra of random variables.

3. A Topological Characterization of σ-semisimplicity

If $\mathcal{B}$ is any σ-algebra on a set B, let us denote by $\mathsf{Meas}(B,[0,1])$ the Riesz MV-algebra of measurable functions from B to $[0,1]$, where we take the unit interval endowed with the σ-algebra of its Borel subsets $\mathcal{BO}([0,1])$.

Theorem 4 *A σ-complete Riesz MV-algebra A is isomorphic to a Riesz MV-algebra of the form $\mathsf{Meas}(B,[0,1])$ if and only if A is σ-semisimple.*

Proof By Theorem 1, an algebra A is σ-semisimple if and only if $A \in ISP([0,1])$. By [17, Lemma 11.8(i)], algebras in $SP([0,1])$ are algebras of measurable functions. Conversely, it is well known that any algebra of the form $\mathsf{Meas}(B,[0,1])$ is closed under pointwise countable suprema. Thus, it is in $SP([0,1])$ and, consequently, it is σ-semisimple. ■

Theorem 4 requires some considerations. Although its proof is an immediate consequence of the authors's previous work, it provides a characterization of each algebra of measurable functions whose codomain is $[0,1]$ (endowed with its Borelian subsets). Hence, we argue that Theorem 4 gives to measurable functions a universal-algebraic description: algebras of the form $\mathsf{Meas}(B,[0,1])$ are exactly elements of the pre-variety generated by $[0,1]$ in $\mathbf{RMV}_\sigma$.

With this in mind, the aim of this section is to obtain a topological characterization of $\mathsf{Meas}(B,[0,1])$, for any measurable space $(B,\mathcal{B})$, by characterizing (topologically) the notion of σ-semisimplicity.

Let (X,τ) be a topological space. We denote by $\overline{S}$ the closure of a set S, while $\mathsf{coz}\,(f)$ denotes the cozero of a function $f\colon X \to [0,1]$, that is, the set of points of X where f is nonzero. Note that, in basically disconnected compact Hausdorff spaces, clopens form a basis of open sets. Furthermore, cozeros are exactly countable unions of clopens and their closure is clopen, see [9, Remark 2.3]. Moreover, since clopens are cozeros (as the characteristic function of a clopen is continuous), we deduce that every open contains a cozero.

Cozeros of countable joins satisfy the following inclusion property.

Lemma 5 *Let X be a compact, Hausdorff and basically disconnected space. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $C(X)$, then $\mathsf{coz}\,(\bigvee_n f_n) \subseteq \overline{\bigcup_n \mathsf{coz}\,(f_n)}$.*

Proof We reason by contraposition. Suppose for a contradiction that there is an open set O such that $O \cap \mathsf{coz}\,(\bigvee_n f_n) \neq \emptyset$ and $O \cap \mathsf{coz}\,(f_n) = \emptyset$ for every n.

Since X has an open basis of clopen sets, we can suppose that O is a clopen. So its characteristic function χ_O is continuous. Let $g = \bigvee_n f_n \wedge \neg\chi_O$. We have

$$g(x) = \begin{cases} 0 & x \in O \\ \bigvee_n f_n(x) & x \notin O \end{cases}$$

Since $O \cap \mathsf{coz}\,(f_n) = \emptyset$, it follows that $f_n(x) = 0$ for any $x \in O$ and therefore $g \geq f_n$ for every n. Moreover, since $O \cap \mathsf{coz}\,(\bigvee_n f_n) \neq \emptyset$, it follows that $\bigvee_n f_n$ is not identically zero on O, while g is so. Consequently, g is not greater than $\bigvee_n f_n$ and this contradicts the definition of supremum of the functions f_n. Hence, the claim follows. ∎

As already mentioned, given Theorem 4, a natural question is whether it is possible to characterize the topological spaces X such that $A := C(X)$ is σ-semisimple. We already know that X is compact, Hausdorff and basically disconnected. Furthermore, points of X are in bijective correspondence with MV-maximal ideals of A. The correspondence is given by the following stipulation: using the notations of [9], any MV-maximal ideal of A has the form

$$\mathbb{I}(y) := \{f \in A \mid f(y) = 0\}, \text{ where } y \in X.$$

The previous characterization implies that a MV-maximal ideal $\mathbb{I}(y)$ is a σ-ideal if for any sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, from $f_n(y) = 0$ for every n, it follows $\bigvee_n f_n(y) = 0$. If T is any subset of X, we write $\mathbb{I}(T) := \bigcap_{t\in T} \mathbb{I}(t)$.

In the remaining part of this section, X is always assumed to be compact, Hausdorff and basically disconnected. A direct translation of σ-semisimplicity is the following.

Lemma 6 *The algebra $A := C(X)$ is σ-semisimple if and only if there is a dense set Y such that $\mathbb{I}(y) \in \mathsf{Max}_\sigma\,(A)$ for any $y \in Y$.*

Proof By definition of σ-semisimplicity, A is σ-semisimple if, and only if, $\bigcap \mathsf{Max}_\sigma\,(A) = \{0\}$. This is equivalent to say that for any $g \neq 0$ there exists $J \in \mathsf{Max}_\sigma\,(A)$ such that $g \notin J$. Since J is in particular a maximal ideal, there exists $y \in X$ such that $J = \mathbb{I}(y)$. Hence, for any $g \in A$ there exists $y \in X$ such that $y \in \mathsf{coz}\,(g)$ and $\mathbb{I}(y) \in \mathsf{Max}_\sigma\,(A)$. Since cozeros are a base of open subsets for X, by selecting a point for each non-zero continuous function $g \in C(X)$ we obtain a dense set Y.

Conversely, assume that there exists a dense set Y as required by the hypothesis. Then, $\mathbb{I}(y) \in \mathsf{Max}_\sigma\,(C(X))$ for any $y \in Y$ and by the properties of $\mathbb{I}$, see [14, Lemma 2.6 and Lemma 3.6], it follows

$$\bigcap \mathsf{Max}_\sigma\,(C(X)) \subseteq \bigcap_{y\in Y} \mathbb{I}(y) = \mathbb{I}(Y) = \mathbb{I}(\overline{Y}) = \mathbb{I}(X) = \{0\}.$$

∎

Proposition 7 *The algebra $A := C(X)$ is σ-semisimple if and only if there exists a dense set $Y \subseteq X$ such that for every cozero C, $Y \cap \overline{C} \subseteq C$.*

Proof From right-to-left, by hypothesis we can assume that, for any $y \in Y$ and for every cozero C if $y \in \overline{C}$ then $y \in C$.

Without loss of generality (see [9, Lemma 5.6]), assume that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of multiples of continuous characteristic functions. Take $y \in \mathsf{coz}\,(\bigvee_n f_n)$. By Lemma 5, $y \in \overline{\bigcup_n \mathsf{coz}\,(f_n)}$.

For any $n \in \mathbb{N}$, the set $\mathsf{coz}\,(f_n)$ is countable union of clopens, and consequently $C = \bigcup_n \mathsf{coz}\,(f_n)$ is a countable union of clopens and therefore it is a cozero. Hence $y \in \overline{C}$ implies $y \in C$ and therefore $y \in \mathsf{coz}\,(f_n)$ for some n. We have proved that $y \in \mathsf{coz}\,(\bigvee_n f_n)$ implies $y \in \mathsf{coz}\,(f_n)$ for some n, which is equivalent to the claim by Lemma 6.

Conversely, by hypothesis and Lemma 6, there exists a dense set Y such that, for any sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ and for any $y \in Y$, if $y \in \mathsf{coz}\,(\bigvee_n f_n)$ then $y \in \mathsf{coz}\,(f_n)$ for some n. Let C be a cozero. Then, there exists a countable set of clopens, $\{C_n\}_{n\in\mathbb{N}}$, such that $C = \bigcup_n C_n$.

Suppose $y \in \overline{C}$ and let $f_n = \chi_{C_n}$. Since C_n is clopen, f_n is continuous and $C_n = \mathsf{coz}\,(f_n)$. By [9, Lemma 5.2], it follows that

$$\mathsf{coz}\left(\bigvee_n f_n\right) = \overline{\bigcup_n \mathsf{coz}\,(f_n)}.$$

Therefore, if $y \in \overline{C}$, then $y \in \mathsf{coz}\,(\bigvee_n f_n)$ and by hypothesis there exists n such that $y \in \mathsf{coz}\,(f_n)$. Since $\mathsf{coz}\,(f_n) = C_n$ for some n, we get $y \in C$. ∎

Let us give a further characterization that will be more useful in what follows.

Proposition 8 *The algebra $A := C(X)$ is σ-semisimple if and only if $\bigcap_{C\in\mathsf{coz}(X)} (\mathsf{int}(X \setminus C) \cup C)$ is dense, where $\mathsf{int}(X \setminus C)$ denotes the interior of $X \setminus C$.*

Proof The claim is a rephrasing of the condition given in Proposition 7. Indeed,

$$\begin{aligned} & Y \cap \overline{C} \subseteq C, \text{ for all } C \in \mathsf{coz}\,(X) \\ \Leftrightarrow\, & Y \subseteq (X \setminus \overline{C}) \cup C, \text{ for all } C \in \mathsf{coz}\,(X) \\ \Leftrightarrow\, & Y \subseteq \mathsf{int}(X \setminus C) \cup C, \text{ for all } C \in \mathsf{coz}\,(X) \end{aligned}$$

$$\Leftrightarrow Y \subseteq \bigcap_{C \in \mathsf{coz}(X)} (\mathsf{int}(X \setminus C) \cup C).$$

Therefore, there exists a dense Y such that $Y \cap \overline{C} \subseteq C$ is equivalent to say that there exists a dense Y such that $Y \subseteq \bigcap_{C \in \mathsf{coz}(X)} (\mathsf{int}(X \setminus C) \cup C)$. The latter, in turn, is equivalent to the claim. ∎

Finally, a further characterization is the following.

Lemma 9 *The algebra $A := C(X)$ is σ-semisimple if and only if there exists a dense set $Y \subseteq X$ such that every continuous function is constant in a neighborhood of every $y \in Y$.*

Proof First recall that, by Proposition 7, $C(X)$ is σ-semisimple if and only if there is a dense set Y such that for every cozero C, if $y \in \overline{C} \cap Y$ then $y \in C$.

Let y be a point of Y and let A be σ-semisimple. Let f be continuous and let us assume first that $f(y) = 0$. Then $y \notin \mathsf{coz}\,(f)$. By hypothesis we have $y \notin \overline{\mathsf{coz}\,(f)}$. By definition of closure, there is a neighborhood U of y disjoint from $\mathsf{coz}\,(f)$, so $f(U) = 0$. More generally, if $f(y) = c$, we apply the previous argument to $|f(y) - c|$, and we can find a neighborhood U of y where $f(U) = c$.

Conversely, suppose every continuous function is locally constant on every point of Y. Let C be a cozero, say $C = \mathsf{coz}\,(f)$ for some $f \in C(X)$, and suppose $y \in \overline{C}$. Then in every neighborhood U of y there exists $z \in U$ such that $f(z) \neq 0$. Then, it follows by hypothesis that $f(y) \neq 0$ and therefore $y \in C$. ∎

4. On the Connection between Frames and Measurable Functions

Building on Proposition 8, in this section we aim at characterizing the notion of measurability in a pointfree setting. Our point of view is inspired by the categorical duality for $\mathbf{RMV}_\sigma$ obtained in [9].

Traditionally, a topology is defined starting from a set X and the collection O of its open subsets. After Stone's famous duality result between certain topological spaces and boolean algebras, it became more clear that one could discuss topology taking as primitive the notion of open set, rather than the one of point. With this idea in mind, topology was reconsidered *pointfree*, using the notions we now introduce.

A *frame* L is a bounded complete lattice $(L, \vee, \wedge, \perp, \top)$ such that

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i). \tag{D}$$

The symbols $\perp$ and $\top$ denote respectively the minimum and the maximum of the bounded lattice.

Similarly, a σ-frame is a σ-complete lattice that satisfies a distributive property analogous to (D), under the assumption that I is countable.

The prototypical example of a frame is the set of all open subsets of a topological space X. In what follows we will always denote such a frame by $\Omega(X)$.

A first immediate remark is that any $A \in \mathbf{RMV}_\sigma$ is a σ-frame. Nonetheless, we take a different route that is inspired by [2], where a connection between frames and MV-algebras is investigated. We start by introducing some additional notion and notation from frame theory and category theory.

Let L be a frame. For any $x \in L$, its *pseudo-complement* is defined as follows

$$x^* := \bigvee \{y \in L \mid x \wedge y = \perp\}.$$

We say that x is *way below* y, written $x \prec y$, if and only if $x^* \vee y = \top$. This is equivalent to saying that there exists $a \in L$ such that $x \wedge a = \perp$ and $a \vee y = \top$. Furthermore, we notice that in the frame of open subsets of a topological space X, the pseudocomplement of U is $\mathsf{int}(X \setminus U)$. An element $a \in L$ is dense if $a^{**} = \top$.

A frame L is called *regular* if and only if for any $x \in L$,

$$x = \bigvee \{y \in L \mid y \prec x\}.$$

An element a of a frame is called *compact* if whenever $a \leq \bigvee S$ for some $S \subseteq L$, there exists a finite subset T of S such that $a \leq \bigvee T$. The frame L is called *compact* when $\top$ is compact.

Frame homomorphisms are (bounded) lattice homomorphisms that preserve finite meets and arbitrary joins. As the open sets of a topological space offer the prototypical example of a frame, this correspondence has been investigated in all of its facets and from a categorical point of view. One of the most celebrated outcomes of these investigations is the so-called Isbell's duality, see [10].

Theorem 10 (Isbell's duality) *The category* **KRF** *of compact and regular frames with frame homomorphisms is dual to the category* **KH** *of compact and Hausdorff topological spaces with continuous functions between them.*

Let us denote by $\Omega(I)$ the frame of open subsets of $[0, 1]$ with the Euclidean topology. Next corollary says that any σ-complete Riesz MV-algebra has a natural translation in terms of pointfree topology.

Corollary 11 *For any σ-complete Riesz MV-algebra A there exists a frame L such that $A \simeq \mathsf{Hom}_{\mathsf{KRF}}(\Omega(I), L)$.*

Proof Let $L = \Omega(\mathsf{Max}\,(A))$ be the frame of opens of $\mathsf{Max}\,(A)$. Since $\mathsf{Max}\,(A)$ is a compact and Hausdorff topological space, Isbell duality yields that $\mathsf{Hom}_{\mathsf{KRF}}(\Omega(I), L)$ is in bijection with $\mathsf{Hom}_{\mathsf{KH}}(\mathsf{Max}\,(A)\,, [0,1])$, which is the set of all continuous functions $C(\mathsf{Max}\,(A))$. Such a set has a natural structure of MV-algebra and by [7, Theorem 4.5] $A \simeq C(\mathsf{Max}\,(A))$, and the claim follows. ■

Building on Theorem 10 we provide a frame-theoretical description of σ-semisimple algebras, improving Corollary 11.

For any frame L, a *cozero* is an element $c \in L$ such that there exists a frame homomorphism $h : \Omega(I) \to L$ and $c = h((0,1])$. Denote the set of cozeros by $\mathsf{coz}\,(L)$. Notice that in [1] cozero elements are defined as images, through frame homomorphisms, of $\mathbb{R} \setminus \{0\}$. The definitions are equivalent, as next lemma shows.

Lemma 12 *Let L be a frame and $c \in \mathsf{coz}\,(L)$. There exists $h : \Omega(\mathbb{R}) \to L$ such that $h(\mathbb{R} \setminus \{0\}) = c$ if, and only if there exists $g : \Omega(I) \to L$ such that $g((0,1]) = c$.*

Proof From left-to-right, let c be a cozero in the sense of [1]. Let $g : \mathbb{R} \to [0,1]$ given by $g(x) = x^2 \wedge 1$, so that $g(x) > 0$ if and only if $x \neq 0$. As g is continuous, it induces a frame homomorphism $\eta : \Omega(I) \to \Omega(\mathbb{R})$ and $\eta((0,1]) = \mathbb{R} \setminus \{0\}$. Then, $c = h \circ \eta((0,1])$.

Conversely, the we apply the same proof argument to the continuous embedding $\iota : [0,1] \hookrightarrow \mathbb{R}$. ■

A frame L is called *basically disconnected* if $c^* \vee c^{**} = \top$ for any $c \in \mathsf{coz}\,(L)$. Notice that, again by Isbell's duality, $C \in \Omega(X)$ is a cozero if and only if $C \subseteq X$ is a cozero in topological terms.

Building on [9, Definition 5.4], we give the following definition.

Definition 13

1. *Let X, Y be compact and Hausdorff topological spaces. We call a continuous function $f : X \to Y$* cozero-closed *if, for any cozero set U, $f^{-1}(\overline{U}) = \overline{f^{-1}(U)}$, see [9].*

2. *A frame homomorphisms $h : L \to M$ is called* cozero-closed *if for any $c \in \mathsf{coz}\,(L)$, $h(c^{**}) = (h(c))^{**}$.*

In what follows we denote by **BDKH** the category of basically disconnected, compact and Hausdorff spaces with cozero-closed continuous functions.

Theorem 14 *Isbell duality can be restricted to basically disconnected compact Hausdorff spaces with cozero-closed maps and basically disconnected regular compact frames with cozero-closed frame homomorphisms.*

Proof The fact that basically disconnected, compact and Hausdorff spaces are in bijection with basically disconnected, compact regular frames is known, see for example [1, Proposition 8.4.3].

We prove that topological cozero-closed functions correspond exactly to frame-theoretical cozero-closed homomorphisms. Let $X, Y \in$ **BDKH**. If $f : X \to Y$ is a continuous function in **BDKH**, the induced frame homomorphism is $f^{-1} : \Omega(Y) \to \Omega(X)$. We prove that f is cozero-closed in **BDKH** if and only if $h := f^{-1}$ is cozero-closed as frame morphism.

Notice first that for any $C \in \mathsf{coz}\,(X)$, $\overline{C}$ is a clopen. Thus, in $\Omega(X)$,

$$C^{**} = \mathsf{int}(\overline{C}) = \overline{C}.$$

Moreover, for any pair of frames L, M and any frame homomorphism $g : L \to M$, if $c = h((0,1]) \in \mathsf{coz}\,(L)$, then

$$g(c) = (g \circ h)((0,1]) \in \mathsf{coz}\,(M)\,.$$

Similarly, on topological spaces, if $C \in \mathsf{coz}\,(Y)$ there exists $g \in C(Y)$ such that $C = \mathsf{coz}\,(g)$. Hence, if $f : X \to Y$ is continuous, we have that

$$f^{-1}(C) = \{x \in X \mid g(f(x)) \neq 0\} \in \mathsf{coz}\,(X)\,.$$

Whence, $h(C^{**}) = h(C)^{**}$ is exactly a reformulation of $f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$, settling the claim. ■

Together with [9, Theorem 5.10], Theorem 14 yields the following.

Corollary 15 *The algebraic category* $\mathbf{RMV}_\sigma$ *is equivalent to the category of basically disconnected, regular and compact frames with cozero-closed homomorphisms.*

Moving then to σ-semisimple algebras, the following corollary, together with Corollary 15, can be seen as the wanted *frame-theoretical* characterization of measurable functions.

Corollary 16 *An algebra $A \in \mathbf{RMV}_\sigma$ is σ-semisimple if and only if the associate frame $L = \Omega(\mathsf{Max}\,(A))$ satisfies*

$$\left(\bigwedge_{c \in \mathsf{coz}(L)} (c^* \vee c) \right)^{**} = \top. \qquad (\sigma\text{-ss})$$

Proof It is a straightforward consequence of Corollary 15 and Proposition 7. ■

Let us denote by **ssFrm** the category whose objects are basically disconnected, regular and compact frames satisfying Equation (σ-ss) with cozero-closed homomorphisms. It is immediate to deduce the following from Corollary 16.

Theorem 17 *The category* **ssFrm** *is equivalent to the full subcategory* $\mathbf{ssRMV}_\sigma$ *of* $\mathbf{RMV}_\sigma$ *whose objects are* σ*-semisimple algebras.*

This equivalence is the composition of two dualities and, in more detail, it is induced by the following functors:

- $\mathcal{M}\colon \mathbf{ssFrm} \to \mathbf{ssRMV}_\sigma$, defined

  on objects by $\mathcal{M}(L) = C(\mathtt{pt}(L))$, where $\mathtt{pt}$ is the usual *point functor* that sends each L in the set of frame homomorphisms from L to $\{0, 1\}$;

  on morphisms by precomposition.

- $\mathcal{F}\colon \mathbf{ssRMV}_\sigma \to \mathbf{ssFrm}$, defined

  on objects by $\mathcal{F}(A) = \Omega(\mathsf{Max}\,(A))$;

  on morphisms by taking preimages twice: if $f\colon A \to B$, then

  $$\mathcal{F}(f)\colon \Omega(\mathsf{Max}\,(A)) \to \Omega(\mathsf{Max}\,(B))$$

  is defined by

  $$\mathcal{F}(f)(O) = \{\mathfrak{m} \in \mathsf{Max}\,(B) \mid f^{-1}(\mathfrak{m}) \in O\}.$$

Theorem 17 is the intended pointfree view on measurability. Indeed, we had already established that Borel measurable functions are characterized as elements of the pre-variety $ISP([0, 1])$. The previous theorem adds another piece of information, giving a more concrete characterization, that is completely algebraic and, additionally, pointfree.

We now take a little step further to describe how the point of view taken in this section can be also used to obtain a frame-theoretical translation of observables. As we have already mentioned, in [8] the authors defined observables as the algebraic counterpart of random variables and formally, for any countable κ and any $A \in \mathbf{ssRMV}_\sigma$, a κ-dimensional observable posed in A is a σ-homomorphism $\mathcal{X}\colon \mathsf{Borel}([0, 1]^\kappa) \to A$.

Proposition 18 *For any* $A \in \mathbf{ssRMV}_\sigma$ *and any observable* $\mathcal{X}$ *posed in* A *there exists a unique frame homomorphism* $\mathcal{X}_\mathcal{F}$ *and functions* ι_κ *and* ι_A *such that the following diagram is commutative, for any* κ.

$$\begin{tikzcd}
\mathsf{Borel}([0,1]^\kappa) \arrow[r, "\mathcal{X}"] \arrow[d, "\iota_\kappa"'] & A \arrow[d, "\iota_A"] \\
\mathcal{F}(\mathsf{Borel}([0,1]^\kappa)) \arrow[r, "\mathcal{X}_\mathcal{F}"] & \mathcal{F}(A)
\end{tikzcd}$$

Proof The existence of a unique $\mathcal{X}_\mathcal{F}$ that corresponds to $\mathcal{X}$ is a consequence of Theorem 17. Let us prove that the diagram is commutative. For any B the functions $\iota\colon B \to \Omega(\mathsf{Max}\,(B))$, depicted vertically, are defined by

$$\iota(a) = \{\mathfrak{m} \in \mathsf{Max}\,(B) \mid a \notin \mathfrak{m}\},\ a \in B.$$

For any $f \in \mathsf{Borel}([0, 1]^\kappa)$ we have that $\iota_A(\mathcal{X}(f)) = \{\mathfrak{m} \in \mathsf{Max}\,(A) \mid \mathcal{X}(f) \notin \mathfrak{m}\}$, while

$$\begin{aligned}
\mathcal{X}_\mathcal{F}(\iota_\kappa(f)) =& \mathcal{X}_\mathcal{F}(\{\mathfrak{n} \in \mathsf{Max}\,(\mathsf{Borel}([0, 1]^\kappa)) \mid f \notin \mathfrak{n}\}) \\
=& \{\mathfrak{m} \in \mathsf{Max}\,(A) \mid f \notin \mathcal{X}^{-1}(\mathfrak{m})\} \\
=& \{\mathfrak{m} \in \mathsf{Max}\,(A) \mid \mathcal{X}(f) \notin \mathfrak{m}\}.
\end{aligned}$$

■

In this section we described how the setting of σ-complete Riesz MV-algebras can provide an adequate setting to investigate the algebraic nature of algebras of random variables. It follows from the representation of σ-semisimple algebras given in Theorem 1 that this approach is limited to dealing with Borel measurable functions. Nonetheless, it provides a fruitful starting point for a *pointfree probability theory* that takes the one of measurable function as a primitive notion.

5. σ-semisimple Algebras and Statistical Models

In this section we take a little step in a complementary direction. In [12] the authors define statistical models in a logico-algebraic framework. Formally, for a $\kappa \leq \omega$ a statistical model is a function $\eta = (\eta_i)_{i\in\kappa}\colon P \to \Delta_\kappa$, where $P \subseteq [0, 1]^d$ is an intersection of Borel measurable sets and Δ_κ is the standard κ-dimensional simplex. When $\kappa = \omega$, we take Δ_ω to be $\{x \in [0, 1]^\omega \mid \sum_{i=1}^\infty x_i \leq 1\}$, which is known to be closed (and convex) and therefore it is a Borel subset of $[0, 1]^\omega$. In [12] the authors also give a list of examples to test the applicability of this approach to classical models. This definition was inspired by the theory of algebraic statistics, whose main reference is [20]. The intuition behind this definition is the following:

- $[0, 1]^\kappa$ is the set of observations on the real world and $\mathsf{Borel}([0, 1]^\kappa)$ is the algebra of many-valued events;
- the set $P \subseteq [0, 1]^d$ is the set of states of the world, or parameters, we allow d to be any countable cardinal;
- the tuple of functions $\eta := (\eta_i)_{i\in\kappa}\colon P \to [0, 1]^\kappa$ is our statistical model: to each parameter $\mathbf{x} \in P$ it associates the tuple $(\eta_i(\mathbf{x}))_{i\in\kappa}$. Each $\eta_i\colon [0, 1]^d \to [0, 1]$ is a Borel measurable function.

We call κ-dimensional any statistical model whose codomain is Δ_κ.

Here, we want to read this definition from the point of view of algebra and category theory. We start with the following definition.

Definition 19 *Let κ be a countable cardinal. Let* **Par** *be the category defined as follows:*

- *objects are intersections of Borel measurable subsets of some $[0,1]^\mu$, for a countable μ;*
- *arrows are tuples of Borel measurable functions $\eta = (\eta_i)_{i\in\alpha}\colon P \subseteq [0,1]^\mu \to Q \subseteq [0,1]^\alpha$, with α, μ countable and $\eta_i \in \mathsf{Borel}([0,1]^\mu)$.*

We call presheaf of κ-dimensional models *the functor $\mathcal{SM}_\kappa\colon$ **Par** $\to$ **Set** given by $\mathcal{SM}_\kappa(P) = \mathsf{Hom}_{\mathsf{Par}}(P, \Delta_\kappa)$, where Δ_κ is fixed.*

Proposition 20 *Let κ be a countable cardinal. Let $A \in$ $\mathbf{ssRMV}_\sigma$ be a σ-semisimple algebra with at most a countable number of generators. There exists a set of parameters P_A such that the set $\mathsf{Hom}_{\mathsf{ssRMV}_\sigma}(\mathsf{Borel}(\Delta_\kappa), A)$ is isomorphic to the set of all κ-dimensional statistical models defined on P_A.*

Proof Preliminarily notice that statistical models are exactly morphisms in the above-defined category **Par**. Suppose that A is μ-generated. By Theorem 1 there exists $P_A \subseteq [0,1]^\mu$ such that P_A is an intersection of Borel sets and $A \simeq \mathsf{Borel}(P_A)$. Moreover, this isomorphism can be lifted to the corresponding categories: [9, Proposition 4.10] yields that **Par** is a dual to a full subcategory of $\mathbf{ssRMV}_\sigma$. Consequently, we infer that $\mathsf{Hom}_{\mathsf{ssRMV}_\sigma}(\mathsf{Borel}(\Delta_\kappa), A)$ is isomorphic with $\mathsf{Hom}_{\mathsf{Par}}(P_A, \Delta_\kappa)$, the set of κ-dimensional statistical models with set of parameters P_A. ■

Denote by $\mathbf{RMV}^{\mathbf{c}}_\sigma$ the full subcategory of $\mathbf{ssRMV}_\sigma$ whose objects are algebras with a countable number of generators. We can define the dual functor $\mathcal{SM}^d\colon \mathbf{RMV}^{\mathbf{c}}_\sigma \to \mathbf{Set}$ obtained by composing the functor $\mathcal{SM}$ with the one that is implicitly given in Theorem 20, via [9, Corollary 4.16]. More precisely, $\mathcal{SM}^d$ is defined as follows:

- $\mathcal{SM}^d_\kappa(A) = \mathsf{Hom}_{\mathsf{ssRMV}_\sigma}(\mathsf{Borel}(\Delta_\kappa), A)$;
- For any $f\colon A \to B$, $\mathcal{SM}^d_\kappa(f)\colon \mathcal{SM}^d_\kappa(A) \to \mathcal{SM}^d_\kappa(B)$, $\mathcal{SM}^d_\kappa(f)(g) = f \circ g$.

To close the circle, consider the map $A \mapsto \mathsf{St}(A) = \{s\colon A \to [0,1] \mid s \textit{ is a } \sigma\textit{-state}\}$. The assignment is clearly functorial and can be seen as a contravariant functor $\mathcal{S}\colon \mathbf{ssRMV}_\sigma \to \mathbf{Set}$, given on objects by $\mathcal{S}(A) = \mathsf{St}(A)$. Thus, the functor $\mathcal{S}$ is a presheaf on $\mathbf{ssRMV}_\sigma$.

An application of Yoneda'a lemma, yields the following.

$$\begin{tikzcd}
\mathsf{Borel}(\Delta_\kappa) \arrow[d, dash, "g"] & \\
A \arrow[d, "f"] & \mathsf{Hom}_{\mathsf{ssRMV}_\sigma}(\mathsf{Borel}(\Delta_\kappa), A) \arrow[d, "f \circ g"] \\
B & \mathsf{Hom}_{\mathsf{ssRMV}_\sigma}(\mathsf{Borel}(\Delta_\kappa), B)
\end{tikzcd}$$

Figure 1: The functor $\mathcal{SM}^d_\kappa$.

Theorem 21 *Let A be a countably generated σ-semisimple algebra. For any $s \in \mathcal{S}(A)$ there exists a unique family of homomorphisms $\{\eta_i\}_{(i\leq\omega)}$ such that the following diagram commutes for any $\mu, \kappa \leq \omega$ and any $f\colon \mathsf{Borel}(\Delta_\mu) \to \mathsf{Borel}(\Delta_\kappa)$. Conversely, if $A := \mathsf{Borel}(\Delta_\kappa)$ for some κ, any family $\{\eta_i\}_{(i\leq\omega)}$ gives a unique σ-state on A.*

$$\begin{tikzcd}
\mathcal{SM}^d_\kappa(A) \arrow[r, "\eta_\kappa"] \arrow[d, "- \circ f"'] & \mathcal{S}(\mathsf{Borel}(\Delta_\kappa)) \arrow[d, "- \circ f"] \\
\mathcal{SM}^d_\mu(A) \arrow[r, "\eta_\mu"] & \mathcal{S}(\mathsf{Borel}(\Delta_\mu))
\end{tikzcd}$$

Proof It is an application of Yoneda's lemma to the contravariant functor $\mathcal{S}$ since, for any $i \leq \omega$, $\mathcal{SM}^d_i(A) = \mathsf{Hom}_{\mathsf{ssRMV}_\sigma}(\mathsf{Borel}(\Delta_i), A)$ is a hom-functor defined on the same category as $\mathcal{S}$, where the A is varying.

The correspondences are defined as follows. For any $s \in \mathsf{St}(A)$, and any $i \leq \omega$, define

$$\eta^s_i\colon \mathsf{Hom}_{\mathsf{ssRMV}_\sigma}(\mathsf{Borel}(\Delta_i), A) \to \mathsf{St}(\mathsf{Borel}(\Delta_i))$$

by $\eta^s_i(h) := \mathcal{S}(h)(s) = s \circ h$.

One can easily see that the diagram is commutative for any $\kappa, \mu \leq \omega$, that is,

$$(\mathcal{S}(f) \circ \eta^s_\kappa)(h) = s \circ (h \circ f) = (\eta_\mu \circ \mathcal{SM}^d_\mu(f))(h),$$

for any $h \in \mathcal{SM}^d_\mu(A)$.

Conversely, if $A = \mathsf{Borel}(\Delta_\kappa)$ for some countable κ, for any family of arrows $\{\eta_i\}_{(i\leq\omega)}\colon \mathcal{SM}^d_i(\mathsf{Borel}(\Delta_\kappa)) \to \mathcal{S}(\mathsf{Borel}(\Delta_i))$, we define s to be the image through η_κ of the identity $id\colon \mathsf{Borel}(\Delta_\kappa) \to \mathsf{Borel}(\Delta_\kappa)$. ■

Traditionally, a parametrized statistical model can be seen as a function $P\colon \Theta \to \mathcal{P}(S)$ where Θ is a set of parameters and $\mathcal{P}(S)$ is the set of probability measures on a sample space S. Theorem 21 gives a bridge between our approach on stochastic models and the traditional point of view.

With a slight abuse of notation, let us denote by $\mathsf{Nat}(\mathcal{SM}(A), \mathcal{S})$ the set of all families of maps $\{\eta_i\}_{(i\leq\omega)}$ given in Theorem 21. Then, the function $\Phi_A\colon \mathcal{S}(A) \to \mathsf{Nat}(\mathcal{SM}(A), \mathcal{S})$ is our *dual parametrization* for any countably generated and σ-semisimple algebra A. If we restrict to algebras of type $A = \mathsf{Borel}(\Delta_\kappa)$ for some $\kappa \leq \omega$, the map Φ_A can be also be inverted. Via Theorem 2, each probability on Δ_κ can be seen as a function that maps a μ-dimensional statistical model on A to the set of all probabilities on Δ_μ, which is turn can be seen as the set of all probabilities on μ points. In a sense, this approach describes a way to obtain a sort of redistribution of the probability from κ points to μ points. Future work will explore the condition under which the pre-sheaf of statistical models $\mathcal{SM}_\kappa$ can become a sheaf and make a comparison with [22].

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