Supplementary material: proofs

These proofs are, due to page constraints, not included in the paper. We provide them here so as to allow the interested reader to check the results.

Proof of Lemma 2

We will abbreviate $F \coloneqq \bigcup_{B\in\mathcal{B}} \mathbb{I}_B(D\rfloor B)$, and show that $\mathcal{L}(\Omega)_{>0} \subseteq \operatorname{posi}(F)$. This will imply the desired result that $\operatorname{posi}(\breve{D} \cup F \cup \mathcal{L}(\Omega)_{>0}) = \operatorname{posi}(\breve{D} \cup F)$: indeed, since posi is a closure operator, we infer that $\operatorname{posi}(\breve{D} \cup F \cup \mathcal{L}_{>0}) \subseteq \operatorname{posi}(\breve{D} \cup \operatorname{posi}(F) \cup \mathcal{L}_{>0}) = \operatorname{posi}(\breve{D} \cup \operatorname{posi}(F)) \subseteq \operatorname{posi}(\operatorname{posi}(\breve{D} \cup F)) = \operatorname{posi}(\breve{D} \cup F) \subseteq \operatorname{posi}(\breve{D} \cup F \cup \mathcal{L}_{>0})$, where the first equality follows once we establish that $\mathcal{L}_{>0} \subseteq \operatorname{posi}(F)$. So consider any f in $\mathcal{L}(\Omega)_{>0}$. For any B in $\mathcal{B}$, let $f_B \colon B \to \mathbb{R} \colon x \mapsto f(x)$ be f's restriction to B, so that $f = \sum_{B\in\mathcal{B}} \mathbb{I}_B f_B$. Collect in $\mathcal{E} \coloneqq \{B \in \mathcal{B} \colon f_B \in \mathcal{L}(B)_{>0}\} = \{B \in \mathcal{B} \colon f(x) > 0 \text{ for some } x \text{ in } B\} \subseteq \mathcal{B}$ the events in $\mathcal{B}$ on which f attains a positive value. That f belongs to $\mathcal{L}(\Omega)_{>0}$ implies that $\mathcal{E}$ is non-empty. For every B in $\mathcal{B} \setminus \mathcal{E}$ it follows that $f_B = 0$, and hence $f = \sum_{B\in\mathcal{E}} \mathbb{I}_B f_B$. Note that, for every B in $\mathcal{E}$, the gamble $f_B > 0$ belongs to $D\rfloor B$ by its coherence whence $\mathbb{I}_B f \in \mathbb{I}_B D\rfloor B$, and therefore, indeed, $f = \sum_{B\in\mathcal{B}} \mathbb{I}_B f_{\mathcal{B}} \in \operatorname{posi}(\bigcup_{B\in\mathcal{B}} \mathbb{I}_B(D\rfloor B)) = \operatorname{posi}(F)$.

Lemma 16 *For any $F^\star \subseteq \mathcal{L}$ such that $\operatorname{posi}(F^\star) \cap \mathcal{L}_{<0} = \emptyset$, we have*

$$K_{\operatorname{posi}(F^\star)} = \operatorname{Rs}(\operatorname{Posi}(K_{F^\star})).$$

Proof We will show that (i) $K_{\operatorname{posi}(F^\star)} \subseteq \operatorname{Rs}(\operatorname{Posi}(K_{F^\star}))$ and (ii) $K_{\operatorname{posi}(F^\star)} \supseteq \operatorname{Rs}(\operatorname{Posi}(K_{F^\star}))$.

For (i), consider any F in $K_{\operatorname{posi}(F^\star)}$, implying that there are n in $\mathbb{N}$, real coefficients $\lambda_{1:n} > 0$ and $g_1, \dots, g_n$ in $F^\star$ such that $g \coloneqq \sum_{k=1}^n \lambda_k g_k \in F$. Note that the requirement $\operatorname{posi}(F^\star) \cap \mathcal{L}_{<0} = \emptyset$ implies that $g \notin \mathcal{L}_{<0}$. By letting $F_1 \coloneqq \{g_1\} \in K_{F^\star}$, $\dots$, $F_n \coloneqq \{g_n\} \in K_{F^\star}$, and $\lambda_{1:n}^{g_{1:n}} \coloneqq \lambda_{1:n} > 0$, we find that $\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n F_k\} = \{\sum_{k=1}^n \lambda_k g_k\} = \{g\}$ belongs to $\operatorname{Posi}(K_{F^\star})$, whence, indeed, $F \in \operatorname{Rs}(\operatorname{Posi}(K_{F^\star}))$ since $g \notin \mathcal{L}_{<0}$.

Conversely, for (ii), consider any F in $\operatorname{Rs}(\operatorname{Posi}(K_{F^\star}))$, so $F \supseteq F' \setminus \mathcal{L}_{>0}$ for some F' in $\operatorname{Posi}(K_{F^\star})$. This implies that $F' = \{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n F_k\}$ for some n in $\mathbb{N}$, $F_1, \dots, F_n$ in $K_{F^\star}$ and real coefficients $\lambda_{1:n}^{f_{1:n}} > 0$ for every $f_{1:n}$ in $\times_{k=1}^n F_k$. That all of $F_1, \dots, F_n$ belong to $K_{F^\star}$ means that $F_1 \cap F^\star \neq \emptyset, \dots, F_n \cap F^\star \neq \emptyset$, so there are $g_1 \in F_1 \cap F^\star, \dots$, $g_n \in F_n \cap F^\star$. Then the specific $g \coloneqq \sum_{k=1}^n \lambda_k^{g_{1:n}} g_k \in F'$ belongs to $\operatorname{posi}(F^\star)$, which tells us that $g \notin \mathcal{L}_{<0}$, and hence g belongs to F, whence $F \cap \operatorname{posi}(F^\star) \neq \emptyset$. Therefore indeed $F \in K_{\operatorname{posi}(F^\star)}$. ■

Lemma 17 *For any $\mathcal{F} \subseteq \mathcal{P}(\mathcal{L})$ we have*

$$K_{\bigcup \mathcal{F}} = \bigcup_{F\in\mathcal{F}} K_F.$$

Proof Consider any $F^\star$ in $\mathcal{Q}$, and infer that, indeed,

$$F^\star \in K_{\bigcup \mathcal{F}} \Leftrightarrow F^\star \cap \bigcup \mathcal{F} \neq \emptyset \Leftrightarrow (\exists F \in \mathcal{F}) F^\star \cap F \neq \emptyset \Leftrightarrow (\exists F \in \mathcal{F}) F^\star \in K_F \Leftrightarrow F^\star \in \bigcup_{F\in\mathcal{F}} K_F,$$

which establishes the desired equality. ■

Proof of Theorem 5

We start with the first statement. We will first show that $\mathcal{L}^{\mathrm{s}}(\Omega)_{>0} \subseteq \operatorname{Posi}(\bigcup_{B\in\mathcal{B}} \mathbb{I}_B(K\rfloor B))$, which will establish the second equality: indeed, abbreviating $\mathcal{F} \coloneqq \bigcup_{B\in\mathcal{B}} \mathbb{I}_B(K\rfloor B)$, since K and Posi are closure operators, we infer that $\operatorname{Rs}(\operatorname{Posi}(\breve{K} \cup \mathcal{F} \cup \mathcal{L}^{\mathrm{s}}_{>0})) \subseteq \operatorname{Rs}(\operatorname{Posi}(\breve{K} \cup \operatorname{Posi}(\mathcal{F}) \cup \mathcal{L}^{\mathrm{s}}_{>0})) = \operatorname{Rs}(\operatorname{Posi}(\breve{K} \cup \operatorname{Posi}(\mathcal{F}))) \subseteq \operatorname{Rs}(\operatorname{Posi}(\operatorname{Posi}(\breve{K} \cup \mathcal{F}))) = \operatorname{Rs}(\operatorname{Posi}(\breve{K} \cup \mathcal{F})) \subseteq \operatorname{Rs}(\operatorname{Posi}(\breve{K} \cup \mathcal{F} \cup \mathcal{L}^{\mathrm{s}}_{>0}))$, where the first equality follows once we establish that $\mathcal{L}^{\mathrm{s}}_{>0} \subseteq \operatorname{Posi}(\mathcal{F})$. So consider any $\{f\}$ in $\mathcal{L}^{\mathrm{s}}(\Omega)_{>0}$—which implies that $f \in \mathcal{L}(\Omega)_{>0}$—and any D in $\mathbf{D}(K)$. Then using the same argument as in the proof of Lemma 2 we infer that $f \in \operatorname{posi}(\bigcup_{B\in\mathcal{B}} \mathbb{I}_B(D\rfloor B))$, or, in other words, that $\{f\} \in K_{\operatorname{posi}(\bigcup_{B\in\mathcal{B}} \mathbb{I}_B(D\rfloor B))}$. Now use Lemma 16 to infer that then $\{f\} \in \operatorname{Rs}(\operatorname{Posi}(K_{\bigcup_{B\in\mathcal{B}} \mathbb{I}_B(D\rfloor B)}))$ and hence $\{f\} \in \operatorname{Posi}(K_{\bigcup_{B\in\mathcal{B}} \mathbb{I}_B(D\rfloor B)})$ since

$f \in \mathcal{L}_{>0}$ and therefore $f \notin \mathcal{L}_{<0}$, and use subsequently Lemma 17 to infer that then $\{f\} \in \operatorname{Posi}(\bigcup_{B\in\mathcal{B}} K_{\mathbb{I}_B D\rfloor B}) = \operatorname{Posi}(\bigcup_{B\in\mathcal{B}} \mathbb{I}_B K_{D\rfloor B}) = \operatorname{Posi}(\bigcup_{B\in\mathcal{B}} \mathbb{I}_B K_D\rfloor\mathcal{B})$.

Next we show that $\widehat{K}$ satisfies "agreeing on $\mathcal{B}$" and "rigidity". To this end, note that $\widehat{K}$ satisfies "agreeing on $\mathcal{B}$" by its definition and the fact that Rs and Posi are closure operators. Moreover, for any B in $\mathcal{B}$, we have that $\widehat{K}\rfloor B = \{F \in \mathcal{Q}(B) \colon \mathbb{I}_B F \in \widehat{K}\} \supseteq \{F \in \mathcal{Q}(B) \colon \mathbb{I}_B F \in \mathbb{I}_B(K\rfloor B)\} = K\rfloor B$ again using that Rs and Posi are closure operators, so $\widehat{K}$ satisfies "rigidity" also.

We now turn to showing that $\widehat{K}$ is coherent. To this end, we infer from [6, Thm. 10] that if $\breve{K} \cup \bigcup_{B\in\mathcal{B}} \mathbb{I}_B(K\rfloor B)$ is consistent, then $\widehat{K}$ is the expression for its natural extension, which then is guaranteed to be coherent. We verify that $\breve{K} \cup \bigcup_{B\in\mathcal{B}} \mathbb{I}_B(K\rfloor B)$ is consistent by considering any $\widehat{D}$ in the non-empty $\widehat{\mathbf{D}} \subseteq \overline{\mathbf{D}}$,[8] and showing that $\breve{K} \cup \bigcup_{B\in\mathcal{B}} \mathbb{I}_B(K\rfloor B)$ is a subset of $K_{\widehat{D}}$, which is a coherent set of desirable gamble sets by [6, Lem. 12]. This will prove in one fell swoop that $\widehat{K} \subseteq \bigcap_{\widehat{D}\in\widehat{\mathbf{D}}} K_{\widehat{D}}$, a useful property that we will use later on in this proof when establishing the second statement .

In order to do so, note that $\widehat{D} = \operatorname{posi}(\breve{D} \cup \bigcup_{B\in\mathcal{B}} \mathbb{I}_B(D\rfloor B))$ for some $\breve{D}$ in $\mathbf{D}(\breve{K})$ and D in $\mathbf{D}(K)$. Consider any F in $\widehat{K}$, meaning that $F \supseteq F' \setminus \mathcal{L}_{<0}$ for some n in $\mathbb{N}$, $F_1, \dots, F_n$ in $\breve{K} \cup \bigcup_{B\in\mathcal{B}} \mathbb{I}_B(K\rfloor B)$, and, for every $f_{1:n}$ in $\bigtimes_{k=1}^n F_k$, real coefficients $\lambda_{1:n}^{f_{1:n}} > 0$ such that $F' = \{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \bigtimes_{k=1}^n F_k\}$. So any F_k belongs to $\breve{K}$—in which case it also belongs to $K_{\breve{D}}$ as $\breve{D} \in \mathbf{D}(\breve{K})$, and hence F_k contains a gamble $g_k \in \breve{D}$—or F_k belongs to $\mathbb{I}_B(K\rfloor B)$ for some B in $\mathcal{B}$—in which case it also belongs to $\mathbb{I}_B(K_D\rfloor B) = \mathbb{I}_B(K_{D\rfloor B})$ as $D \in \mathbf{D}(K)$, and hence F contains a gamble $\mathbb{I}_B g_k$ where $g_k \in D\rfloor B$. In any case, we find that $\sum_{k=1}^n \lambda_k^{g_{1:n}} g_k \in F'$ belongs to $\operatorname{posi}(\breve{D} \cup \bigcup_{B\in\mathcal{B}} \mathbb{I}_B(D\rfloor B)) = \widehat{D}$, and hence $F' \in K_{\widehat{D}}$. This implies that, indeed, $F \in K_{\widehat{D}}$.

So we have established that $\widehat{K}$ satisfies "agreeing on $\mathcal{B}$", "rigidity" and "coherence". To complete the proof for the first statement, we show that $\widehat{K}$ is the smallest such set of desirable gamble sets. To this end, consider any set of desirable gamble sets $K^\star$ satisfying "agreeing on $\mathcal{B}$", "rigidity" and "coherence". Note that $K^\star$ must include $\breve{K}$ by "agreeing on $\mathcal{B}$" and $\bigcup_{B\in\mathcal{B}} \mathbb{I}_B K\rfloor B$ by "rigidity". By "coherence" it must therefore include $\operatorname{Rs}\big(\operatorname{Posi}\big(\breve{D} \cup \bigcup_{B\in\mathcal{B}} \mathbb{I}_B(D\rfloor B)\big)\big) = \widehat{K}$, whence $K^\star \supseteq \widehat{K}$, showing that, indeed, $\widehat{K}$ is the smallest set of desirable gambles that satisfies "agreeing on $\mathcal{B}$", "rigidity" and "coherence". This also establishes that the smallest set of desirable gamble sets that satisfies "agreeing on $\mathcal{B}$", "rigidity" and "coherence" is necessarily unique.

Now we turn to the second statement. We need to show that $\widehat{K} = \bigcap_{\widehat{D}\in\widehat{\mathbf{D}}} K_{\widehat{D}}$. Recall from the proof of the first statement that $\widehat{K} \subseteq \bigcap_{\widehat{D}\in\widehat{\mathbf{D}}} K_{\widehat{D}}$, so it suffices to prove the converse set inclusion $\bigcap_{\widehat{D}\in\widehat{\mathbf{D}}} K_{\widehat{D}} \subseteq \widehat{K}$. To this end, we use a theorem privately communicated to us by Jasper De Bock and Gert de Cooman that follows from their [7, Thm. 9], in the form of [38, Thm. 6]: 'The natural extension of a consistent assessment $\mathcal{F}$ is given by $\bigcap\{K_D \colon D \in \overline{\mathbf{D}}, \mathcal{F} \subseteq K_D\}$'. Applied to the current case, as the assessment $\breve{K} \cup \bigcup_{B\in\mathcal{B}} \mathbb{I}_B(K\rfloor B)$ is already known to be consistent from the proof for the first statement, we infer that

$$\widehat{K} = \bigcap\Big\{K_D \colon D \in \overline{\mathbf{D}}, \breve{K} \cup \bigcup_{B\in\mathcal{B}} \mathbb{I}_B(K\rfloor B) \subseteq K_D\Big\},$$

and hence to establish that $\bigcap_{\widehat{D}\in\widehat{\mathbf{D}}} K_{\widehat{D}} \subseteq \widehat{K}$ it suffices to show that any $D^\star$ in $\overline{\mathbf{D}}$ such that $\breve{K} \cup \bigcup_{B\in\mathcal{B}} \mathbb{I}_B(K\rfloor B) \subseteq K_{D^\star}$ belongs to $\widehat{\mathbf{D}}$. So consider such a $D^\star$, which implies that $\widehat{K} \subseteq K_{D^\star}$—meaning that $D^\star \in \mathbf{D}(\widehat{K})$—and $\bigcup_{B\in\mathcal{B}} \mathbb{I}_B(K\rfloor B) \subseteq K_{D^\star}$—meaning that $\mathbb{I}_B(K\rfloor B) \subseteq K_{D^\star}$ and hence $K\rfloor B \subseteq K_{D^\star}\rfloor B = K_{D^\star\rfloor B}$ whence $D^\star\rfloor B \in \mathbf{D}(K\rfloor B)$ for all B in $\mathcal{B}$. As $\bigcup_{B\in\mathcal{B}} \mathbb{I}_B(D^\star\rfloor B) = \bigcup_{B\in\mathcal{B}}\{\mathbb{I}_B f \colon f \in D^\star\rfloor B\} = \bigcup_{B\in\mathcal{B}}\{\mathbb{I}_B f \colon \mathbb{I}_B f \in D^\star\} \subseteq D^\star$, we find, taking into account its coherence, that $D^\star = \operatorname{posi}(D^\star \cup \bigcup_{B\in\mathcal{B}} \mathbb{I}_B(D^\star\rfloor B))$, whence indeed $D^\star \in \widehat{\mathbf{D}}$.

Proof of Proposition 6

We will show (i) that $\mathcal{M}(\widehat{\underline{E}}) \subseteq \{P \colon (\forall A \in \mathcal{P}(\mathcal{B}))P(\mathbb{I}_A) \geq \underline{P}(A) \text{ and } (\forall A' \subseteq \Omega, B \in \mathcal{B})P(A'|B) \geq \underline{P}(A'|B)\}$ and (ii) that $\mathcal{M}(\widehat{\underline{E}}) \supseteq \{P \colon (\forall A \in \mathcal{P}(\mathcal{B}))P(\mathbb{I}_A) \geq \underline{P}(A) \text{ and } (\forall A' \subseteq \Omega, B \in \mathcal{B})P(A'|B) \geq \underline{P}(A'|B)\}$. For (i), consider any P in $\mathcal{M}(\widehat{\underline{E}})$, which implies that $P \geq \widehat{\underline{E}}$. Since $\widehat{\underline{E}} \geq \breve{\underline{P}}$ as $\widehat{\underline{E}}$ extends $\breve{\underline{E}}$ [and $\breve{\underline{E}}$ is $\breve{\underline{P}}$'s natural extension] and $\breve{\underline{P}}(B) > 0$ for every B in $\mathcal{B}$, we find that also $P(B) > 0$ for all $B \in \mathcal{B}$, whence $P = P(P(\bullet|\mathcal{B}))$, so $P(\bullet|B)$ is determined uniquely by Bayes' rule. This implies that $P(f) \geq \breve{\underline{E}}(f)$ for every $f \in \mathcal{L}(\mathcal{B})$, and in particular that $P(\mathbb{I}_A) \geq \underline{P}(A)$ for every $A \in \mathcal{P}(\mathcal{B})$. Moreover, this also implies that $P(f|B) \geq \underline{E}(f|B)$ for every $B \in \mathcal{B}$ and every $f \in \mathcal{L}$, whence $P(A'|B) \geq \underline{P}(A'|B)$ for all $A' \subseteq \Omega$ and $B \in \mathcal{B}$.

[8] It is a consequence of Thm. 1, which is a direct consequence of [10, Thm. 3], that every element of $\widehat{\mathbf{D}}$ is a coherent set of desirable gambles.

To show (ii), the inverse inclusion, consider any P such that $P(\mathbb{I}_A) \geq \underline{P}(A)$ for all $A \in \mathcal{P}(\mathcal{B})$ and $P(A'|B) \geq \underline{P}(A'|B)$ for all $A' \subseteq \Omega$ and $B \in \mathcal{B}$. It follows by the natural extension that $P(f|B) \geq \underline{E}(f|B)$ for every f in $\mathcal{L}(B)$ and B in $\mathcal{B}$, and similarly, that $P(g) \geq \check{\underline{E}}(g)$ for every g in $\mathcal{L}(\mathcal{B})$. consequence,

$$P(f) = P(P(f|\mathcal{B})) \geq \check{\underline{E}}(\underline{E}(f|\mathcal{B})) = \widehat{\underline{E}}(f)$$

for every $f \in \mathcal{L}(\Omega)$, which completes the proof.

Proof of Proposition 7

Assume first of all that $\widehat{\underline{P}} \geq \widehat{\underline{E}} = \check{\underline{E}}(\underline{E}(\bullet|\mathcal{B}))$. Then for any gamble $f \in \mathcal{L}(\mathcal{B})$ we infer $\widehat{\underline{P}}(f) \geq \widehat{\underline{E}}(f) = \check{\underline{E}}(f)$, whence (a) holds. With respect to (b), for any $B \in \mathcal{B}$ such that $\widehat{\underline{P}}(B) > 0$, the conditional $\widehat{\underline{P}}(\bullet|B)$ coincides with the model $\widehat{\underline{P}}$ induces applying regular extension; since $\mathcal{M}(\widehat{\underline{P}}) \supseteq \mathcal{M}(\widehat{\underline{E}})$, this in turn dominates the conditional induced by $\widehat{\underline{E}}$ from regular extension, which must then dominate $\underline{E}(\bullet|B)$, that satisfies GBR with respect to $\widehat{\underline{E}}$, using [23, Lem. 2]. Thus (c) holds.

Conversely, if (a) and (b) holds but there is some gamble such that $\widehat{\underline{P}}(f) < \widehat{\underline{E}}(f)$, then it cannot be $f \in \mathcal{L}(\mathcal{B})$ by (a); consider then the conditional lower prevision $\widehat{\underline{P}}(\bullet|\mathcal{B})$ where $\widehat{\underline{P}}(\bullet|B)$ is defined by regular extension if $\widehat{\underline{P}}(B) > 0$ and $\widehat{\underline{P}}(\bullet|B) = \underline{P}(\bullet|B)$ if $\widehat{\underline{P}}(B) = 0$. Then $\widehat{\underline{P}}$ is coherent with $\widehat{\underline{P}}(\bullet|\mathcal{B})$, whence $\widehat{\underline{P}}(f) \geq \widehat{\underline{P}}(\widehat{\underline{P}}(f|\mathcal{B}))$. As a consequence, there must be some $B \in \mathcal{B}$ such that $\widehat{\underline{P}}(f|B) < \underline{P}(f|B)$. But then can neither be $\widehat{\underline{P}}(B) > 0$ (by (b)) nor $\widehat{\underline{P}}(B) = 0$ (by definition), which leads to a contradiction.

Proof of Proposition 8

1. Consider two gambles f_1, f_2 on Ω. Then $\widehat{\underline{E}}(f_1 \wedge f_2) = \check{\underline{E}}(\underline{E}(f_1 \wedge f_2|\mathcal{B})) = \check{\underline{E}}(g_1 \wedge g_2) = \min\{\check{\underline{E}}(g_1), \check{\underline{E}}(g_2)\}$, where $g_1 = \underline{E}(f_1|\mathcal{B}), g_2 = \underline{E}(f_2|\mathcal{B})$. Thus, $\widehat{\underline{E}}$ is minitive.

2. Assume first of all that $\check{\underline{E}}$ is minitive on gambles. Given two events A_1, A_2, infer that $\widehat{\underline{E}}(A_1 \cap A_2) = \check{\underline{E}}(\underline{E}(A_1 \cap A_2|\mathcal{B})) = \check{\underline{E}}(g_1 \wedge g_2) = \min\{\check{\underline{E}}(g_1), \check{\underline{E}}(g_2)\} = \min\{\widehat{\underline{E}}(A_1), \widehat{\underline{E}}(A_2)\}$, where $g_1 = \underline{E}(A_1|\mathcal{B}), g_2 = \underline{E}(A_2|\mathcal{B})$.

   Next, if $\underline{E}(\bullet|B)$ is minitive on gambles, it is $\{0, 1\}$-valued on events by [12, Prop. 7]. As a consequence, there exists a filter $\mathcal{F}_B$ such that

$$\underline{P}(A_1 \cap A_2|B) = \begin{cases} 1 & \text{if } A_1 \cap A_2 \in \mathcal{F}_B \\ 0 & \text{otherwise.} \end{cases}$$

   This implies that $\widehat{\underline{E}}(A_1 \cap A_2) = \check{\underline{E}}(H)$ for $H := \bigcup\{B \colon A_1 \cap A_2 \in \mathcal{F}_B\}$. But since $H = H_1 \cap H_2$ for $H_1 := \bigcup\{B \colon A_1 \in \mathcal{F}_B\}$ and $H_2 := \bigcup\{B \colon A_2 \in \mathcal{F}_B\}$ since filters are closed under finite intersections, we deduce that $\widehat{\underline{E}}(A_1) = \check{\underline{E}}(H_1)$ and $\widehat{\underline{E}}(A_2) = \check{\underline{E}}(H_2)$, and therefore $\widehat{\underline{E}}(A_1 \cap A_2) = \min\{\widehat{\underline{E}}(A_1), \widehat{\underline{E}}(A_2)\}$.

3. To see this, we need to find some $B \in \mathcal{B}$ such that $\check{\underline{P}}(B) \in (0, 1)$, which always exists because $\check{\underline{E}}$ is not minitive on gambles. Similarly, there is some $A_1 \subset B$ such that $\underline{P}(A_1|B) \in (0, 1)$. By defining the events $H_1 := A_1 \cup B^c$ and $H_2 := B$, we infer that:

$$\begin{aligned} \widehat{\underline{E}}(H_1 \cap H_2) &= \widehat{\underline{E}}(A_1) = \check{\underline{P}}(B) \cdot \underline{P}(A_1|B) \\ \widehat{\underline{E}}(H_2) &= \check{\underline{P}}(B) \\ \widehat{\underline{E}}(H_1) &= \underline{P}(A_1|B) + (1 - \underline{P}(A_1|B)) \cdot \underline{P}(B^c), \end{aligned}$$

   whence $\widehat{\underline{E}}(H_1 \cap H_2) < \min\{\widehat{\underline{E}}(H_1), \widehat{\underline{E}}(H_2)\}$.

Proof of Proposition 11

That the first statement implies the second is trivial. To see the converse, let us establish first of all the implication

$$(\forall \omega \in B)\underline{P}(\mathbb{I}_B(\mathbb{I}_{\{\omega\}} - \underline{E}(\{\omega\}|B))) = 0 \Rightarrow (\forall f \in \mathcal{L})\underline{P}(\mathbb{I}_B(f - \underline{E}(f|B))) = 0 \tag{6}$$

To this end, consider first of all any event $A \subseteq B$, and we will show that $\underline{P}(B(\mathbb{I}_A - \underline{E}(A|B))) = 0$. If $A = B$ then we have $\underline{P}(B(\mathbb{I}_B - \underline{E}(B|B))) = \underline{P}(\mathbb{I}_B - \mathbb{I}_B\underline{E}(B|B)) = \underline{P}(\mathbb{I}_B - \mathbb{I}_B 1) = 0$, so we assume that $A \subset B$. Then indeed

$$\underline{P}(B(\mathbb{I}_A - \underline{E}(A|B))) = \underline{P}\left(\sum_{\omega\in A} B(\mathbb{I}_{\{\omega\}} - \underline{E}(\{\omega\}|B))\right) = \sum_{\omega\in A} \underline{P}(\mathbb{I}_B(\mathbb{I}_{\{\omega\}} - \underline{E}(\{\omega\}|B))) = 0,$$

where the first equality follows from applying Eq. (3) twice, taking into account that $\underline{E}(A|B)$ is a constant, and that coherent lower previsions satisfy constant additivity. The second equality follows once we realise that $\min \mathbb{I}_B(\mathbb{I}_A - \underline{E}(A|B)) = -\underline{E}(A|B) = -\sum_{\omega\in A}\underline{E}(\{\omega\}|B) = \sum_{\omega\in A}\min \mathbb{I}_B(\mathbb{I}_{\{\omega\}} - \underline{E}(\{\omega\}|B))$, using that $A \subset B$, and the third one by the assumption in Eq. (6).

Next, consider a gamble f on Ω such that $f = Bf$, and let us express it as $f = \sum_{i=1}^n x_i \mathbb{I}_{A_i}$, for $x_1 > x_2 > \cdots > x_n$ and a partition $\{A_1, \ldots, A_n\}$ of B. Since a coherent lower prevision always satisfies constant additivity, we can assume without loss of generality that $x_n = 0$. Then

$$\begin{aligned}\underline{P}\left(\sum_{i=1}^n x_i\mathbb{I}_{A_i} - \underline{E}(f|B)\right) &= \underline{P}\left(\sum_{i=1}^n x_i\mathbb{I}_{A_i} - \sum_{i=1}^n x_i\underline{E}(A_i|B)\right)\\ &= \underline{P}\left(\sum_{i=1}^n x_i(\mathbb{I}_B(\mathbb{I}_{A_i} - \underline{E}(A_i|B)))\right) = \sum_{i=1}^n x_i\underline{P}(\mathbb{I}_B(\mathbb{I}_{A_i} - \underline{E}(A_i|B))) = 0.\end{aligned}$$

Here the first equality follows from

$$\underline{E}(f|B) = (1-\delta_B)P_B\left(\sum_{i=1}^n x_i\mathbb{I}_{A_i}\right) + \delta_B \min\left(\sum_{i=1}^n x_i\mathbb{I}_{A_i}\right) = (1-\delta_B)P_B\left(\sum_{i=1}^n x_i\mathbb{I}_{A_i}\right) = \sum_{i=1}^n x_i\underline{E}(A_i|B),$$

the third from

$$\begin{aligned}\underline{P}\left(\sum_{i=1}^n x_i\mathbb{I}_B(\mathbb{I}_{A_i} - \underline{E}(A_i|B))\right) &= (1-\delta)P\left(\sum_{i=1}^n x_i\mathbb{I}_B(\mathbb{I}_{A_i} - \underline{E}(A_i|B))\right) + \delta\min\left(\sum_{i=1}^n x_i\mathbb{I}_B(\mathbb{I}_{A_i} - \underline{E}(A_i|B))\right)\\ &= (1-\delta)\sum_{i=1}^n x_iP(\mathbb{I}_B(\mathbb{I}_{A_i} - \underline{E}(A_i|B))) - \delta\sum_{i=1}^n x_i\underline{E}(A_i|B)\\ &= \sum_{i=1}^n x_i\left((1-\delta)P(\mathbb{I}_B(\mathbb{I}_{A_i} - \underline{E}(A_i|B))) + \delta\min\mathbb{I}_B(\mathbb{I}_{A_i} - \underline{E}(A_i|B))\right)\\ &= \sum_{i=1}^n x_i\underline{P}(\mathbb{I}_B(\mathbb{I}_{A_i} - \underline{E}(A_i|B))),\end{aligned}$$

and the fourth one by the assumption in Eq. (6). This establishes Eq. (6). Since $\underline{P}$ is coherent with $\check{\underline{E}}, \underline{E}(\bullet|\mathcal{B})$ if and only if $\underline{P}(f) = \check{\underline{E}}(f)$ and $\underline{P}(\mathbb{I}_B(f - \underline{E}(f|B)) = 0$ for every $f \in \mathcal{L}$ and every $B \in \mathcal{B}$, we deduce (a) from (b).

Let us now prove the equivalence between the second and third statements. To this end, note already that two LV models determined by (P_1, δ_1) and (P_2, δ_2) are equal if and only if $P_1 = P_2$ and $\delta_1 = \delta_2$. Therefore, we see that $\underline{P} = \check{\underline{E}}$ is equivalent to $(B \in \mathcal{B})P(B) = P_{\mathcal{B}}(B)$ and $\delta = 1 - \sum_{B\in\mathcal{B}}\check{\underline{E}}(B) = 1 - \sum_{B\in\mathcal{B}}\underline{P}_{\mathcal{B}}(B) = \delta_{\mathcal{B}}$:

Next, given $B \in \mathcal{B}$ and $\omega \in B$,

$$\begin{aligned}\underline{P}(B(\mathbb{I}_{\{\omega\}} - \underline{E}(\{\omega\}|B))) &= (1-\delta)P(B(\mathbb{I}_{\{\omega\}} - \underline{E}(\{\omega\}|B))) + \delta\min(B(\mathbb{I}_{\{\omega\}} - \underline{E}(\{\omega\}|B)))\\ &= (1-\delta)(P(\omega) - P(B)\underline{E}(\{\omega\}|B)) + \delta(-\underline{E}(\{\omega\}|B)),\end{aligned}$$

whence $\underline{P}(B(\mathbb{I}_{\{\omega\}} - \underline{E}(\{\omega\}|B))) = 0$ if and only if $P(\{\omega\}) = \frac{\underline{E}(\{\omega\}|B)(\delta+P(B))}{1-\delta}$. Moreover, since $P(B) = \sum_{\omega\in B}P(\{\omega\})$, we infer that

$$\sum_{\omega\in B}\frac{\underline{E}(\{\omega\}|B)(\delta + P(B))}{1-\delta} = P(B).$$

The left hand side is equal to

$$\sum_{\omega\in B}\frac{(1-\delta_B)P_B(\{\omega\}|B)(\delta+P(B))}{1-\delta}=\frac{(1-\delta_B)(\delta+P(B))}{1-\delta},$$

so this is equal to $P(B)$ if and only if

$$\delta_B=\frac{\delta+\delta P(B)}{\delta+P(B)}.$$

This completes the proof.

Proof of Proposition 12

Consider the conditional probability measure $P(\bullet|\mathcal{B})$ given by $P(A|B)=P_B(A)$ for every B in $\mathcal{B}$ and $A\subseteq B$, and let P_0 denote the probability measure on Ω determined by $P_{\mathcal{B}}, P(\bullet|\mathcal{B})$. Then it holds that, for any $B\in\mathcal{B}$ and $\omega\in B$,

$$\underline{\widehat{E}}(\{\omega\})=\underline{\check{E}}((1-\delta_B)P(\{\omega\}|B)\mathbb{I}_B)=(1-\delta_{\mathcal{B}})(1-\delta_B)P_0(\{\omega\}),$$

whence

$$\sum_{\omega\in B}\underline{\widehat{E}}(\{\omega\})=(1-\delta_{\mathcal{B}})(1-\delta_B)\sum_{\omega\in B}P_0(\{\omega\})=(1-\delta_{\mathcal{B}})(1-\delta_B)P_0(B),$$

while

$$\underline{\widehat{E}}(B)=\underline{\check{E}}(\mathbb{I}_B)=(1-\delta_{\mathcal{B}})P_{\mathcal{B}}(B).$$

Thus, Eq. (3) is not satisfied and therefore $\underline{\widehat{E}}\notin\mathcal{C}_{\text{LV}}$, since δ_B and $\delta_{\mathcal{B}}$ belong to the open interval $(0,1)$.

Proof of Proposition 14

Assume *ex absurdo* that there is some such PMM $\overline{P}$, and let (P,δ) be its associated parameters. Consider a gamble f on B given by $f=\sum_{i=1}^n x_i\mathbb{I}_{A_i}$ for $x_1=1>x_2>\cdots>x_n=0$ and for a partition $\{A_1,\ldots,A_n\}$ of B, and let us characterise under which conditions we have that $\overline{P}(\overline{P}(f|B)-Bf)=0$. Also, by coherence we get that

$$\underline{P}(A)\geq\underline{P}_{\mathcal{B}}(\underline{P}(A|\mathcal{B}))>0 \tag{7}$$

for any event A.

First of all, taking into account that for any $x>x_n$ it holds that

$$P(\{f\leq x\})\geq P(A_n)>\frac{\delta}{1+\delta},$$

since $(1+\delta)P(A_n)-\delta=\underline{P}(A_n|B)>0$ by assumption, we deduce that if we apply Eq. (4) to compute $\overline{P}(f|B)$ we obtain

$$\overline{P}(f|B)=x_n+(1+\delta)P_B((f-x_n)^+)=(1+\delta)P_B(f),$$

whence $\overline{P}(f|B)-Bf=(1+\delta)P_B(f)-Bf$.

On the other hand, it follows from Eq. (7) that $\underline{P}(\{\omega\})>0$ for every $\omega\in B$. As a consequence, defining $g:=\overline{P}(f|B)-Bf$, for any value $x>\min g=\overline{P}(f|B)-x_1=\overline{P}(f|B)-1$ it holds that

$$P(\{g\leq x\})\geq P(A_1)>\frac{\delta}{1+\delta},$$

since $(1+\delta)P(A_1)-\delta=\underline{P}(A_1)>0$ by Eq. (7). Thus, Eq. (4) gives

$$\overline{E}(g)=\overline{P}(f|B)-1+(1+\delta)P((G(f|B)-\min G(f|B))^+)=(1+\delta)P_B(f)-1+(1+\delta)P(B(1-f)).$$

Therefore,

$$\overline{E}(g)=0\Leftrightarrow(1+\delta)P_B(f)-1+(1+\delta)P(B(1-f))=0\Leftrightarrow P(f)=P_B(f)+P_0(B)-\frac{1}{1+\delta}.$$

Applying this to $f = \mathbb{I}_{\{\omega\}}$ for some $\omega \in B$, we obtain that P should satisfy

$$P(\{\omega\}) = P_B(\{\omega\}) + P_0(B) - \frac{1}{1+\delta}. \tag{8}$$

This means that

$$\sum_{\omega \in B} P(\{\omega\}) = 1 + |B| P_0(B) - \frac{|B|}{1+\delta} = P_0(B)$$
$$\Leftrightarrow P_0(B) = \frac{|B| - 1 - \delta}{(|B|-1)(1+\delta)},$$

and this for every $B \in \mathcal{B}$. If we consider B with more than two elements and take both $\omega_1, \omega_2 \in B$ with $\omega_1 \neq \omega_2$, then $\overline{E}(\overline{P}(\mathbb{I}_{\{\omega_1,\omega_2\}}|B) - B\mathbb{I}_{\{\omega_1,\omega_2\}}) = 0$ if and only if

$$P(\{\omega_1, \omega_2\}) = P_B(\{\omega_1, \omega_2\}) + P_0(B) - \frac{1}{1+\delta};$$

but by Eq. (8) it is

$$P(\{\omega_1, \omega_2\}) = P_B(\{\omega_1, \omega_2\}) + 2P_0(B) - 2\frac{1}{1+\delta};$$

and this can only be if $P_0(B) = \frac{1}{1+\delta}$. Since $\frac{1}{1+\delta} \neq \frac{|B|-1-\delta}{(|B|-1)(1+\delta)}$, we obtain a contradiction.

Finally, if $|B| = 2$ for all B then it must be $|\mathcal{B}| = \frac{n}{2}$. We get on the one hand $P_0(B) = \frac{1-\delta}{1+\delta}$ for all B, and the equality $1 = \sum_B P_0(B) = \frac{n}{2} P_0(B)$ implies that $\delta = \frac{n-2}{n+2}$; but on the other hand for $\underline{P}(B) > 0$ we should have then $n < 4$; this means that $n = 2$ and that $\mathcal{B}$ has only one element.