Supplementary Material

Our proof for Theorem 20 builds on the following intermediary lemmata. In order not to unnecessarily repeat ourselves in this section, we fix some upper rate operator $\overline{Q}$ for the remainder. Furthermore, we let

$$D := \{\delta \in \mathbb{R}_{>0} \colon \delta\|\overline{Q}\| \leq 2\}$$

and for all $\delta \in D$, let $\overline{T}(\delta) := I + \delta\overline{Q}$; due to Lemma 12, $\overline{T}(\delta)$ is an upper transition operator whenever $\delta \in D$, and henceforth we will use this fact implicitly.

Lemma 23 *For all $\delta \in D$, $f, h \in \mathcal{L}$ and $n \in \mathbb{N}$,*

$$\|(I + \delta\overline{Q}_f)^n[h]\| \leq n\delta\|f\| + \|h\|. \tag{13}$$

Proof Let us prove the result by induction. For the base case $n = 1$, it follows from the definition of $\overline{Q}_f$ and (T7) that

$$\|(I + \delta\overline{Q}_f)[h]\| \leq \|\delta f\| + \|\overline{T}(\delta)[h]\| \leq \delta\|f\| + \|h\|,$$

as required. For the inductive step, we assume that (13) holds for $n = k$ with $k \in \mathbb{N}$, and set out to show that it then also holds for $n = k + 1$. From the definition of $\overline{Q}_f$, (T7) and the induction hypothesis, it follows immediately that

$$\begin{aligned}\|(I + \delta\overline{Q}_f)^{k+1}[h]\| &\leq \delta\|f\| + \|\overline{T}(\delta)(I + \delta\overline{Q}_f)^k[h]\| \\ &\leq \delta\|f\| + \|(I + \delta\overline{Q}_f)^k[h]\| \\ &\leq \delta(k + 1)\|f\| + \|h\|,\end{aligned}$$

as required. ■

The second intermediary lemma builds on Lemma 23.

Lemma 24 *Fix some $\delta \in D$ and $f, h \in \mathcal{L}$. Then for all $n \in \mathbb{N}$,*

$$\|(I + \delta\overline{Q}_f)^n[h] - h\| \leq n\delta c_1 + n^2\delta^2 c_2, \tag{14}$$

with $c_1 := \|f\| + \|\overline{Q}\|\|h\|$ and $c_2 := \|\overline{Q}\|\|f\|$.

Proof We again give a proof by induction. For the base case $n = 1$, note that

$$\begin{aligned}\|(I + \delta\overline{Q}_f)[h] - h\| &= \|\delta f + h + \delta\overline{Q}[h] - h\| \\ &\leq \delta\|f\| + \delta\|\overline{Q}\|\|h\| = \delta c_1,\end{aligned}$$

which implies the inequality in the statement for $n = 1$.

For the inductive step, we assume that (14) holds for $n = k$ with $k \in \mathbb{N}$, and set out to verify that it holds for $n = k + 1$ as well. Observe that

$$\begin{aligned}(I + \delta\overline{Q}_f)^{k+1}[h] - h \\ = \delta f + (I + \delta\overline{Q}_f)^k[h] - h + \delta\overline{Q}(I + \delta\overline{Q}_f)^k[h].\end{aligned}$$

Recall from (R5) that

$$\|\delta\overline{Q}(I + \delta\overline{Q}_f)^k[h]\| \leq \delta\|\overline{Q}\|\|(I + \delta\overline{Q}_f)^k[h]\|.$$

We infer from these two observations, the induction hypothesis and Lemma 23 that

$$\begin{aligned}&\|(I + \delta\overline{Q}_f)^{k+1}[h] - h\| \\ &\quad\leq \delta\|f\| + (k\delta c_1 + k^2\delta^2 c_2) + \delta\|\overline{Q}\|(k\delta\|f\| + \|h\|) \\ &\quad= (k + 1)\delta c_1 + k^2\delta^2 c_2 + k\delta^2 c_2.\end{aligned}$$

Since $k^2 + k \leq (k + 1)^2$, we infer from this that

$$\|(I + \delta\overline{Q}_f)^{k+1}[h] - h\| \leq (k + 1)\delta c_1 + (k + 1)^2\delta^2 c_2,$$

which is the inequality we were after. ■

Our next step is to use Lemma 24 to prove a 'generalisation' of Lemma E.5 in [14]. In this result, we need the fact that $\overline{Q}$ is Lipschitz:

R7. $\|\overline{Q}[f] - \overline{Q}[g]\| \leq \|\overline{Q}\|\|f - g\|$ for all $f, g \in \mathcal{L}$;

this is trivial if $\|\overline{Q}\| = 0$ and follows from Lemma 12 (with $\Delta = 2/\|\overline{Q}\|$) and (T8) (for $I + \Delta\overline{Q}$) otherwise, see also [4, R11] or [7, LR8].

Lemma 25 *Fix some $\delta \in D$ and $f, h \in \mathcal{L}$. Then for all $n \in \mathbb{N}$,*

$$\|(I + \delta\overline{Q}_f)^n[h] - (I + n\delta\overline{Q}_f)[h]\| \leq n^2\delta^2 c_3 + n^3\delta^3 c_4,$$

with $c_3 := \|\overline{Q}\|\|f\| + \|\overline{Q}\|^2\|h\|$ and $c_4 := \|\overline{Q}\|^2\|f\|$.

Proof Our proof will be one by induction. The base case $n = 1$ is trivially satisfied. For the inductive step, we assume that the inequality in the statement holds for $n = k$ with $k \in \mathbb{N}$. To prove that the inequality in the statement holds for $n = k + 1$, we observe that

$$\begin{aligned}&(I + \delta\overline{Q}_f)^{k+1}[h] - (I + (k + 1)\delta\overline{Q}_f)[h] \\ &\quad= \delta f + (I + \delta\overline{Q}_f)^k[h] - (I + k\delta\overline{Q}_f)[h] \\ &\qquad\quad - \delta f - \delta\overline{Q}[h] + \delta\overline{Q}(I + \delta\overline{Q}_f)^k[h].\end{aligned}$$

It follows from this, the induction hypothesis, (R7) and Lemma 24 that

$$\begin{aligned}&\|(I + \delta\overline{Q}_f)^{k+1}[h] - (I + (k + 1)\delta\overline{Q}_f)[h]\| \\ &\quad\leq (k^2\delta^2 c_3 + k^3\delta^3 c_4) \\ &\qquad\quad + \delta\|\overline{Q}\|\|(I + \delta\overline{Q}_f)^k[h] - h\| \\ &\quad\leq (k^2\delta^2 c_3 + k^3\delta^3 c_4) + \delta\|\overline{Q}\|(k\delta c_1 + k^2\delta^2 c_2) \\ &\quad= (k^2 + k)\delta^2 c_3 + (k^3 + k^2)\delta^3 c_4\end{aligned}$$

$$\leq (k+1)^2\delta^2 c_3 + (k+1)^3\delta^3 c_4,$$

which is the inequality we were after. ■

As a final intermediary step, we generalise Lemma 25; this result is to Lemma 25 what Lemma E.6 is to Lemma E.5 in [14].

Lemma 26 *Fix some $\delta \in D$, $f, h \in \mathcal{L}$ and $k \in \mathbb{N}$. Then for all $n \in \mathbb{N}$,*

$$\left\| \left(I + \frac{\delta}{k}\overline{Q}_f\right)^{nk}[h] - (I + \delta\overline{Q}_f)^n[h] \right\| \leq n\delta^2 c_3 + n^2\delta^3 c_4,$$

with c_3 and c_4 as in Lemma 25.

Proof Let us prove the result by induction. For the base case $n = 1$, we apply Lemma 25 (with $\delta/k \in D$ here as δ there and k here as n there) to find that

$$\|(I + \frac{\delta}{k}\overline{Q}_f)^k[h] - (I + k\frac{\delta}{k}\overline{Q}_f)[h]\| \leq k^2\left(\frac{\delta}{k}\right)^2 c_3 + k^3\left(\frac{\delta}{k}\right)^3 c_4 = \delta^2 c_3 + \delta^3 c_4.$$

For the inductive step, we assume that the inequality in the statement holds for $n = \ell$ with $\ell \in \mathbb{N}$, and set out to establish the inequality in the statement for $n = \ell + 1$. Observe that

$$\begin{aligned}
&\left(I + \frac{\delta}{k}\overline{Q}_f\right)^{(\ell+1)k}[h] - (I + \delta\overline{Q}_f)^{\ell+1}[h] \\
&= \left(I + \frac{\delta}{k}\overline{Q}_f\right)^k \left(I + \frac{\delta}{k}\overline{Q}_f\right)^{\ell k}[h] \\
&\quad - \left(I + \frac{\delta}{k}\overline{Q}_f\right)^k (I + \delta\overline{Q}_f)^\ell[h] \\
&\quad + \left(I + \frac{\delta}{k}\overline{Q}_f\right)^k (I + \delta\overline{Q}_f)^\ell[h] \\
&\quad - (I + \delta\overline{Q}_f)(I + \delta\overline{Q}_f)^\ell[h].
\end{aligned}$$

Let us denote the norm of the first two terms on the right hand side by $\eta_{1:2}$ and that of the last two terms by $\eta_{3:4}$, such that

$$\left\| \left(I + \frac{\delta}{k}\overline{Q}_f\right)^{(\ell+1)k}[h] - (I + \delta\overline{Q}_f)^{\ell+1}[h] \right\| \leq \eta_{1:2} + \eta_{3:4}.$$

Since $\overline{T}(\delta/k)$ satisfies (T8) because $\delta/k \in D$, the same is true for $\overline{T}(\delta/k)_{\delta f/k}$ – we leave this for the reader to check – and therefore also for $\overline{T}(\delta/k)^k_{\delta f/k} = \left(I + \frac{\delta}{k}\overline{Q}_f\right)^k$; consequently,

$$\begin{aligned}
\eta_{1:2} &\leq \left\| \left(I + \frac{\delta}{k}\overline{Q}_f\right)^{\ell k}[h] - (I + \delta\overline{Q}_f)^\ell[h] \right\| \\
&\leq \ell\delta^2 c_3 + \ell^2\delta^3 c_4,
\end{aligned}$$

where the second inequality is exactly the induction hypothesis. Moreover, it follows from Lemma 25 (with $(I + \delta\overline{Q}_f)^\ell[h]$ here as h there, k here as n there and $\delta/k \in D$ here as δ there) and Lemma 23 (with ℓ here as n there) that

$$\begin{aligned}
\eta_{3:4} &\leq \delta^2\left(\|\overline{Q}\|\|f\| + \|\overline{Q}\|^2\|(I + \delta\overline{Q}_f)^\ell[h]\|\right) + \delta^3 c_4 \\
&\leq \delta^2\left(\|\overline{Q}\|\|f\| + \|\overline{Q}\|^2\ell\delta\|f\| + \|\overline{Q}\|^2\|h\|\right) + \delta^3 c_4 \\
&= \delta^2 c_3 + \ell\delta^3 c_4 + \delta^3 c_4.
\end{aligned}$$

Combining our observations, we find that

$$\begin{aligned}
&\left\| \left(I + \frac{\delta}{k}\overline{Q}_f\right)^{(\ell+1)k}[h] - (I + \delta\overline{Q}_f)^{\ell+1}[h] \right\| \\
&\leq \ell\delta^2 c_3 + \ell^2\delta^3 c_4 + \delta^2 c_3 + \ell\delta^3 c_4 + \delta^3 c_4 \\
&= (\ell+1)\delta^2 c_3 + (\ell^2 + \ell + 1)\delta^3 c_4 \\
&\leq (\ell+1)\delta^2 c_3 + (\ell+1)^2\delta^3 c_4,
\end{aligned}$$

which is the inequality we were after. ■

Proving Theorem 20 is now simply a matter of combining (6) and Lemma 26.

Proof of Theorem 20 Fix some $n \in \mathbb{N}$. Then for all $k \in \mathbb{N}$

$$\begin{aligned}
&e^{n\Delta\overline{Q}_f}[h] - \left(I + \frac{\Delta}{n^2}\overline{Q}_f\right)^{n^3}[h] \\
&= e^{n\Delta\overline{Q}_f}[h] - \left(I + \frac{n\Delta}{kn^3}\overline{Q}_f\right)^{kn^3}[h] \\
&\quad + \left(I + \frac{n\Delta}{kn^3}\overline{Q}_f\right)^{kn^3}[h] - \left(I + \frac{\Delta}{n^2}\overline{Q}_f\right)^{n^3}[h]
\end{aligned}$$

From (6) with $t = \Delta n$, we know that

$$\begin{aligned}
e^{n\Delta\overline{Q}_f}[h] &= \lim_{k\to+\infty}\left(I + \frac{n\Delta}{k}\overline{Q}_f\right)^k[h] \\
&= \lim_{k\to+\infty}\left(I + \frac{n\Delta}{kn^3}\overline{Q}_f\right)^{kn^3}[h].
\end{aligned}$$

Furthermore, if $\Delta\|\overline{Q}\| \leq 2n^2$, it follows from Lemma 26 (with $\delta = \Delta/n^2$ and n^3 here as n there) that for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\left\| \left(I + \frac{n\Delta}{kn^3}\overline{Q}_f\right)^{kn^3}[h] - \left(I + \frac{\Delta}{n^2}\overline{Q}_f\right)^{n^3}[h] \right\| \\
&\leq n^3\left(\frac{\Delta}{n^2}\right)^2 c_3 + n^6\left(\frac{\Delta}{n^2}\right)^3 c_4 \\
&= \frac{1}{n}\Delta^2 c_3 + \Delta^3 c_4.
\end{aligned}$$

Combining the preceding and taking the limit for $k \to +\infty$ gives that, for all $n \in \mathbb{N}$ such that $\Delta\|\overline{Q}\| \leq 2n^2$,

$$\frac{1}{n\Delta}\left\| e^{n\Delta\overline{Q}_f}[h] - \left(I + \frac{\Delta}{n^2}\overline{Q}_f\right)^{n^3}[h] \right\| \leq \frac{1}{n^2}\Delta c_3 + \frac{1}{n}\Delta^2 c_4.$$

The right hand side of this inequality vanishes as $n \to +\infty$, which implies the statement. ■