Appendix A. Technical Lemmas and Proofs

Lemma 18 For any non-degenerate prequential situation \( v = (i, w) \in (\mathcal{F} \times \mathcal{X})^* \) and any non-negative superfarthingale \( F \in \mathbb{F} \), \( F(v) \leq \prod_{i=1}^{w} \frac{1}{(\max I_r)^{1/(1-\min I_r)}} F(\omega) \).

Proof Consider any non-degenerate prequential situation \( vI_r x \in (\mathcal{F} \times \mathcal{X})^* \). If \( x = 1 \) then \( 0 < \max I_r \leq 1 \), and

\[
F(vI_r x) \leq \frac{1}{\max I_r} \left[ \max I_r F(vI_r 1) + (1 - \max I_r) F(vI_r 0) \right] \\
\leq \frac{1}{\max I_r} E_{I_r}(F(vI_r \cdot)) \leq \frac{1}{\max I_r} F(v).
\]

If \( x = 0 \) then \( 0 \leq \min I_r < 1 \), and

\[
F(vI_r x) \leq \frac{1}{1 - \min I_r} \left[ \min I_r F(vI_r 1) + (1 - \min I_r) F(vI_r 0) \right] \\
\leq \frac{1}{1 - \min I_r} E_{I_r}(F(vI_r \cdot)) \leq \frac{1}{1 - \min I_r} F(v).
\]

Above, the first, second and third inequalities follow from the non-negativity of \( F \), Equation (1) and the supermartingale property, respectively. Hence, \( F(vI_r x) \leq \frac{1}{(\max I_r)^{1/(1-\min I_r)}} F(v) \). A simple induction argument now leads to the desired result. ■

Lemma 19 For every non-degenerate supercomputable forecasting system \( \varphi \in \Phi \) there’s a recursive natural map \( C : \mathcal{X}^* \rightarrow \mathbb{N} \) such that for every test supermartingale \( T \in \overline{\mathbb{T}}(\varphi) \) it holds that \( T(w) \leq C(w) \) for all \( w \in \mathcal{X}^* \).

Proof Define the map \( C' : \mathcal{X}^* \rightarrow \mathbb{N} \) by \( C'(w) := \prod_{i=1}^{w} \frac{1}{\varphi(i,w \cdot)} \) for all \( w \in \mathcal{X}^* \). This map is real-valued, since \( \emptyset < \varphi \) and \( \varphi \leq 1 \) by the non-degeneracy of \( \varphi \). Since \( \varphi \) is computable, \( C' \) is computable as well. Let’s now prove that \( T(w) \leq C'(w) \) for all \( w \in \mathcal{X}^* \). Fix any situation \( w \in \mathcal{X}^* \) and any \( x \in \mathcal{F} \). If \( x = 1 \), then

\[
T(wx) \leq \frac{1}{\varphi(w)} \left[ \varphi(w) T(w1) + (1 - \varphi(w)) T(w0) \right] \\
\leq \frac{1}{\varphi(w)} E_{\varphi(w)}(T(w \cdot)) \leq \frac{1}{\varphi(w)} T(w).
\]

If \( x = 0 \), then

\[
T(wx) \leq \frac{1}{1 - \varphi(w)} \left[ \varphi(w) T(w1) + (1 - \varphi(w)) T(w0) \right] \\
\leq \frac{1}{1 - \varphi(w)} E_{\varphi(w)}(T(w \cdot)) \leq \frac{1}{1 - \varphi(w)} T(w).
\]

Above, the first, second and third inequalities follow from the non-negativity of \( T \), Equation (1) and the supermartingale property, respectively. Hence,

\[
T(wx) \leq \frac{1}{\varphi(w)G(w)(1 - \varphi(w))} T(w).
\]

A simple induction argument now shows that indeed \( T(w) \leq C'(w) \) for all \( w \in \mathcal{X}^* \).

Since \( C' \) is a computable real map, there’s a recursive rational map \( q : \mathcal{X}^* \times \mathbb{N}_0 \rightarrow \mathbb{Q} \) such that \( |C'(w) - q(w,n)| \leq \varepsilon_n \) for all \( w \in \mathcal{X}^* \) and \( n \in \mathbb{N}_0 \). Let \( C : \mathcal{X}^* \rightarrow \mathbb{N} \) be defined as \( C(w) := \max \{ 1, \lfloor q(w,1) + 1 \rfloor \} \) for all \( w \in \mathcal{X}^* \), with \( \lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z} \) the ceiling function and \( \mathbb{Z} \) the set of integer numbers. It’s easy to see that \( C \) is natural-valued, positive and recursive. Furthermore, we have that \( T(w) \leq C'(w) \leq q(w,1) + \frac{1}{2} \leq C(w) \) for all \( w \in \mathcal{X}^* \).

Lemma 20 There’s a single algorithm that, upon the input of a code for a lower semicomputable superfarthingale \( F' \in \mathbb{F} \),

(i) \( F'(v) = 0 \) for all degenerate prequential situations \( v \in (\mathcal{F} \times \mathcal{X})^* \);

(ii) for any rational forecasting system \( \varphi \in \Phi_r \), \( F'(\varphi, [w], w) = F(\varphi, [w], w) \) for all \( w \in \mathcal{X}^* \) for which \( (\varphi, [w], w) \) is non-degenerate, provided that the map \( F(\varphi, [\cdot], \cdot) : \mathcal{X}^* \rightarrow \mathbb{R} \) is a positive test supermartingale for \( \varphi \).

Proof Start from a code for the map \( F : (\mathcal{F} \times \mathcal{X})^* \rightarrow [0, +\infty] \) that is lower semicomputable. By Corollary 6, we can invoke the single algorithm that outputs a code \( q : (\mathcal{F} \times \mathcal{X})^* \times \mathbb{N}_0 \rightarrow \mathbb{Q} \) for such that \( q(v, \cdot) \not\in F(v) \) and \( q(v, n) < q(v, n + 1) \) for all \( v \in (\mathcal{F} \times \mathcal{X})^* \) and \( n \in \mathbb{N}_0 \).

We’ll now use the code \( q \) to construct a code \( q' \) for a lower semicomputable test superfarthingale \( F' \in \mathbb{F} \) that satisfies the requirements of the lemma.

Let \( q' : (\mathcal{F} \times \mathcal{X})^* \times \mathbb{N}_0 \rightarrow \mathbb{Q} \) be defined by \( q'(\emptyset, n) := 1 \) and

\[
q'(vI_r x, n) := \\
\begin{cases} 
\max \{ A(v, I_r, x, n) \cup \{ 0 \} \} & \text{if } vI_r x \text{ is non-degenerate} \\
0 & \text{if } vI_r x \text{ is degenerate},
\end{cases}
\]

for all \( v = (i, w) \in (\mathcal{F} \times \mathcal{X})^* \), \( I_r \in \mathcal{F} \), \( x \in \mathcal{X} \) and \( n \in \mathbb{N}_0 \), where \( A : (\mathcal{F} \times \mathcal{X})^* \times \mathcal{F} \times \mathcal{X} \times \mathbb{N}_0 \rightarrow \{ \emptyset \subseteq \mathcal{Q} : (\mathcal{Q} \subset \mathbb{Q}) < \infty \} \) is defined by

\[
A(v, I_r, x, n) := \{ q(vI_r x, m) \in \mathbb{Q} : 0 \leq m \leq n, q(vI_r x, m) \leq q(vI_r x, n) \}.
\]
By construction, since the map $A$ outputs finite sequences of rationals, the map $q'$ is well-defined, non-negative and rational. It’s not too difficult to see that the map $A$, and therefore also the map $q'$, is recursive.

The map $q'$ is non-decreasing in its second argument, as we now show by induction on its first argument. We start by observing that $q'(\mathbb{Z}, n) \leq q'(\mathbb{Z}, n+1)$ for all $n \in \mathbb{N}$. For the induction step, fix any $v = (i, w) \in (\mathcal{I}_r \times X)^*$, $I_r \in \mathcal{I}_r$, $x \in X$ and $n \in \mathbb{N}_0$, and assume that $q'(v, n) \leq q'(v, n + 1)$. We then have to show that also $q'(vlx, n) \leq q'(vlx, n + 1)$. This is trivial when $vlx$ is degenerate; when $vlx$ is non-degenerate, it follows readily from the inequality $A(v, I_r, x, n) \subseteq A(v, I_r, x, n + 1)$, which is itself immediate from Equation (6).

For any $n \in \mathbb{N}_0$, the map $q'(\ast, n) : (\mathcal{I}_r \times X)^* \to \mathbb{R}$ is a test superfarthingale. To prove this, we may clearly concentrate on the superfarthingale condition. Fix any $v = (i, w) \in (\mathcal{I}_r \times X)^*$, $I_r \in \mathcal{I}_r$ and $n \in \mathbb{N}_0$, and infer from Equation (6) that $A(v, I_r, 1, n) = \emptyset \Rightarrow A(v, I_r, 0, n) = \emptyset$, so we only need to consider two cases. If $A(v, I_r, 1, n) = A(v, I_r, 0, n) = \emptyset$, then $q'(vlx, n) = 0$, and therefore trivially $\overline{E}_L(q'(v, n, \cdot)) = \overline{E}_L(0) = 0 \leq q'(v, n)$, where the second equality follows from C1. Otherwise, because the map $q$ is increasing in its second argument, there’s an $m \in \{0, \ldots, n\}$ such that $\overline{E}_L(q'(v, \cdot, m)) \leq q'(v, n)$, with $q'(vlx, \cdot, m) = \max(A(v, I_r, \cdot, n) \cup \{0\}) \geq q'(vlx, \cdot, n)$, where the last inequality takes into account that there may be some $x \in X$ such that $vlx$ is degenerate. Hence, indeed, in this case also

\[ \overline{E}_L(q'(vlx, \cdot, n)) \leq \overline{E}_L(q'(v, \cdot, m)) \leq q'(v, n). \]

As a final preliminary step, we infer from Lemma 18 that for every (non-degenerate) prequivalent situation $v \in (\mathcal{I}_r \times X)^*$ there’s some real $B_v \in \mathbb{R}$ such that $q'(v, n) \leq B_v$ for all $n \in \mathbb{N}_0$.

With this set-up phase completed, let $F'$ be defined as $q'(v, \ast) \mapsto F'(v)$ for all $v \in (\mathcal{I}_r \times X)^*$; note that $F'(\mathbb{Z}) = 1$. This map is well-defined, real-valued, non-negative and lower semicomputable due to the non-decreasingness, boundedness, non-negativity and recursiveness of $q'$ respectively, so we only need to check the superfarthingale property explicitly in order to conclude that $F'$ is a lower semicomputable test superfarthingale. To this end, fix any $v \in (\mathcal{I}_r \times X)^*$ and $I_r \in \mathcal{I}_r$. If we recall that the map $q'(\ast, n) : (\mathcal{I}_r \times X)^* \to \mathbb{R}$ is a test superfarthingale for every $n \in \mathbb{N}_0$, we immediately infer from C6 and the real-valuedness of $F'$ that $\overline{E}_L(F'(v, \cdot, n)) = \lim_{n \to \infty} \overline{E}_L(q'(vlx, \cdot, n)) \leq \lim_{n \to \infty} q'(v, n) = F'(v)$.

We are done if we can show that $F'$ satisfies the conditions (i) and (ii). For (i), fix any degenerate prequivalent situation $v \in (\mathcal{I}_r \times X)^*$ and note that then $q'(v, n) = 0$ for all $n \in \mathbb{N}_0$ by construction. Hence, indeed, $F'(v) = 0$.

For (ii), fix any rational forecasting system $\varphi_j \in \mathcal{Q}_r$, consider the map $T : X^* \to \mathbb{R}$ defined by $T(w) := F(\varphi_j[w], w)$ for all $w \in X^*$, and assume that $T$ is a positive test supermartingale. We must now show that $F'(\varphi_j[w], w) = T(w)$ for all $w \in X^*$ for which the pre-quivalent situation $(\varphi_j[w], w)$ is non-degenerate.

By construction, $F'(\varphi_j[w], w) \leq F(\varphi_j[w], w) = T(w)$ for all $w \in X^*$. Assume towards contradiction that there’s some $\overline{w} \in X^*$ for which $(\varphi_j[\overline{w}], \overline{w})$ is non-degenerate and $F'(\varphi_j[\overline{w}], \overline{w}) < T(\overline{w})$, implying that there’s some $c > 0$ such that $q'((\varphi_j[\overline{w}], \overline{w}), n) + c < T(\overline{w})$ for all $n \in \mathbb{N}_0$. We’ll use an induction argument to show that this is impossible.

Since by assumption $q((\varphi_j[\overline{w}], \overline{w}), \cdot) \not\geq T(\overline{w}) > 0$ and $q((\varphi_j[\overline{w}], \overline{w}), n) < q((\varphi_j[\overline{w}], \overline{w}), n + 1)$ for all $n \in \mathbb{N}_0$, there are $\epsilon_0, \epsilon_1, \ldots, \epsilon_{|\overline{w}|} \in \mathbb{R}$ and $n_0, n_1, \ldots, n_{|\overline{w}|} \in \mathbb{N}_0$ such that

\begin{align*}
0 &< \epsilon_0 < \epsilon_1 < \cdots < \epsilon_{|\overline{w}|} < \epsilon & (7) \\
T(\overline{w}_{1, \ell}) &< q((\varphi_j[\overline{w}_{1, \ell}], \overline{w}_{1, \ell}), n_\ell) + \epsilon & (8) \\
0 &\leq q((\varphi_j[\overline{w}_{1, k}], \varphi_j(\overline{w}_{1, k}), \overline{w}_{1, k}.), n_{k+1}) & (9) \\
q((\varphi_j[\overline{w}_{1, k}], \varphi_j(\overline{w}_{1, k}), \overline{w}_{1, k}.), n_{k+1}) + \epsilon_k &< T(\overline{w}_{1, k} -) & (10)
\end{align*}

for all $k \in \{0, 1, \ldots, |\overline{w}|-1\}$ and $\ell \in \{0, 1, \ldots, |\overline{w}|\}$. The argument starts with $\ell := |\overline{w}|$ and $k := |\overline{w}|-1$, finding $\epsilon_k$ such that (7) is satisfied, and finding $n_{k+1}$ such that (8) and (9) are satisfied. We then move to $\ell := |\overline{w}|-1$ and $k := |\overline{w}|-2$, find $\epsilon_k$ such that (7) and (10) are satisfied, and find $n_{k+1}$ such that (8) and (9) are satisfied. And so on . . . ; these conditions are depicted below for a situation $\overline{w} \in X^*$ for which $|\overline{w}| = 5$.

\[ T(\overline{w}_{1, \ell}) \]

\[ q((\varphi_j[\overline{w}_{1, \ell}], \overline{w}_{1, \ell}), n_\ell) \]

\[ 0 \]

\[ 1 \]

\[ 2 \]

\[ 3 \]

\[ 4 \]

\[ 5 \]

Now, let $N := \max\{n_0, n_1, \ldots, n_{|\overline{w}|}\}$. To start the induction argument, observe that, trivially, $q'(\mathbb{Z}, N) = 1 > T(\mathbb{Z}) - \epsilon_0$. For the induction step, we fix any $k \in \{0, 1, \ldots, |\overline{w}|-1\}$ and assume that $q'(\varphi_j[\overline{w}_{1, k}], \overline{w}_{1, k}), N) > T(\overline{w}_{1, k}) - \epsilon_k$. It then follows that

\[ \overline{E}_{\varphi_j[\overline{w}_{1, k}]} q((\varphi_j[\overline{w}_{1, k}], \varphi_j(\overline{w}_{1, k}), \overline{w}_{1, k}.), n_{k+1}) \]
We'll now show in a number of steps that if \( T \) is a supermartingale, the last inequality from the induction hypothesis. Hence, by Equations (6) and (9),

\[
q((\varphi_r[\overline{w}_{1,k+1}],\overline{w}_{1,k+1}),n_{k+1}) \in A((\varphi_r[\overline{w}_1],\overline{w}_1),\varphi_r(\overline{w}_{1,k+1}), N),
\]

which implies that

\[
q'(((\varphi_r[\overline{w}_{1,k+1}],\overline{w}_{1,k+1}),N) \geq \max A((\varphi_r[\overline{w}_1],\overline{w}_1),\varphi_r(\overline{w}_{1,k+1}),N) \geq q((\varphi_r[\overline{w}_{1,k+1}],\overline{w}_{1,k+1}),n_{k+1}) \geq T(\overline{w}_{1,k+1}) - \epsilon_{k+1}.
\]

Repeating this argument until we reach \( k = |\overline{w}| - 1 \), we eventually find that \( q'(((\varphi_r[\overline{w}],\overline{w}),N) > T(\overline{w}) - \epsilon|\overline{w}| > T(\overline{w}) - \epsilon \), which is the desired contradiction. \( \square \)

The following result is now immediate.

**Corollary 21** There's a single algorithm that, upon the input of a code for a lower semicomputable map \( F' : (\mathcal{J} \times \mathcal{X})^* \rightarrow [0, +\infty] \), outputs a code for a lower semicomputable test superfarthingale \( F' \in \mathcal{F} \) such that, for all prequential situations \( v \in (\mathcal{J} \times \mathcal{X})^* \),

(i) \( F'(v) = 0 \) if \( v \) is degenerate;

(ii) \( F'(v) = F(v) \) if \( v \) is non-degenerate and \( F \) is a positive test superfarthingale.

**Proof of Theorem 17.** We'll give a proof for the first inequality, the proof for the second one is similar. Assume towards contradiction that there's some real number \( \epsilon \), with \( 0 < \epsilon < 1 \), such that

\[
\liminf_{n \to \infty} \frac{\sum_{k=0}^{n-1} S(v_{1,k}, t_{k+1})[\omega_{k+1} - \min t_{k+1}]}{\sum_{k=0}^{n-1} S(v_{1,k}, t_{k+1})} < -\epsilon.
\]

Let the map \( F : (\mathcal{J} \times \mathcal{X})^* \rightarrow \mathbb{R} \) be defined by

\[
F(v) := \prod_{k=0}^{n-1} \left[ 1 - \frac{\epsilon}{3} S(v_{1,k}, t_{k+1})[w_{k+1} - \min t_{k+1}] \right]
\]

for all \( v = (i,w) \in (\mathcal{J} \times \mathcal{X})^* \).

We'll now show in a number of steps that \( F \) is a lower semicomputable test superfarthingale for which

\[
\limsup_{n \to \infty} F(v_{1,n}) = \infty,
\]

implying that \( \nu \) can't be game-random.

Trivially, \( F(\emptyset) = 1 \), and also \( F \geq 0 \), since \( \epsilon < 1 \), \( |S| \leq 1 \) and \( |x - \min I_\nu| \leq 1 \) for all \( x \in \mathcal{X} \) and \( I_\nu \in \mathcal{J}_\nu \). Moreover, for any \( v \in (\mathcal{J} \times \mathcal{X})^* \) and \( I_\nu \in \mathcal{J}_\nu \), we have that

\[
E_{I_\nu}(F(vI_\nu)) \leq F(v)E_{I_\nu}\left(1 + \frac{\epsilon}{3} S(v, I_\nu)[\min I_\nu - X]\right)
\]

so we find that \( F \) is a test superfarthingale. From the recursiveness of \( S \) and the rational-valuedness of the forecasts \( I_\nu \in \mathcal{J}_\nu \) and outcomes \( x \in \mathcal{X} \) it follows that \( F \) is recursive, and therefore lower semicomputable as well. We conclude that \( F \) is a lower semicomputable test superfarthingale.

By assumption, for any \( m, M \in \mathbb{N}_0 \), there's some \( N > m \) such that \( \sum_{k=0}^{N-1} S(v_{1,k}, t_{k+1}) \geq M \) and

\[
\sum_{k=0}^{N-1} S(v_{1,k}, t_{k+1})[\omega_{k+1} - \min t_{k+1}] > -\epsilon. \quad (11)
\]

This will allow us to obtain a lower bound for \( F(v_{1,N}) \).

Since \( 1 - \frac{\epsilon}{3} S(v, I_\nu)[x - \min I_\nu] > \frac{1}{2} \) for all \( v \in (\mathcal{J} \times \mathcal{X})^* \), \( I_\nu \in \mathcal{J}_\nu \) and \( x \in \mathcal{X} \), it holds that \( F(v_{1,N}) = \exp(K) \), with

\[
K := \ln \left( 1 - \frac{\epsilon}{3} S(v_{1,k}, t_{k+1})[\omega_{k+1} - \min t_{k+1}] \right).
\]

Since \( \ln(1 + x) \geq x - x^2 \) for all \( x > -1/2 \), we infer that

\[
K \geq \frac{\epsilon}{3} \sum_{k=0}^{N-1} S(v_{1,k}, t_{k+1})[\omega_{k+1} - \min t_{k+1}]
- \frac{\epsilon^2}{9} \sum_{k=0}^{N-1} S(v_{1,k}, t_{k+1})^2[\omega_{k+1} - \min t_{k+1}]^2
\]

and, also taking into account Equation (11), \( S^2 = S \) and \( [\omega_{k+1} - \min t_{k+1}]^2 \leq 1 \),

\[
K \geq \frac{\epsilon^2}{9} \sum_{k=0}^{N-1} S(v_{1,k}, t_{k+1}) - \frac{\epsilon^2}{9} \sum_{k=0}^{N-1} S(v_{1,k}, t_{k+1})
= \frac{2\epsilon^3}{9} \sum_{k=0}^{N-1} S(v_{1,k}, t_{k+1}).
\]

Hence,

\[
F(v_{1,N}) \geq \exp \left( \frac{2\epsilon^2}{9} \sum_{k=0}^{N-1} S(v_{1,k}, t_{k+1}) \right) \geq \exp \left( \frac{2\epsilon^2}{9} M \right).
\]
After recalling that the inequality above holds for any 
\( M \in \mathbb{N}_0 \) and for arbitrarily large well-chosen \( N \in \mathbb{N}_0 \), we 
conclude that \( \limsup_{n \to \infty} F(u_{1,n}) = \infty. \)