Supplementary Material for ”Open World Dempster-Shafer Using Complementary Sets”

Theorem 4 *Let m_1 and m_2 be CBPAs with frame of discernments Ω_1 and Ω_2 respectively. For a new frame of discernment Ω, the evaluation distributes through the conjunctive join,*

$$m_1|_\Omega \oplus m_2|_\Omega = (m_1 \oplus m_2)|_\Omega \ .$$

Proof Proven directly.
For $w \in 2^\Omega$:

$$(m_1|_\Omega \oplus m_2|_\Omega)(w) = \sum_{\substack{u\in 2^\Omega \\ v\in 2^\Omega \\ u\cap v=w}} m_1|_\Omega(u)\cdot m_2|_\Omega(v) \qquad \text{(by conjunctive join definition)}$$

$$= \sum_{\substack{u\in 2^\Omega \\ v\in 2^\Omega \\ u\cap v=w}} \left(\sum_{\substack{x\in 2^{\Omega_1}\times\mathbb{B} \\ x|_\Omega=u}} m_1(x)\right)\cdot\left(\sum_{\substack{y\in 2^{\Omega_2}\times\mathbb{B} \\ y|_\Omega=v}} m_2(y)\right) \qquad \text{(by evaluation of CBPAs Equation (4))}$$

$$= \sum_{\substack{u\in 2^\Omega \\ v\in 2^\Omega \\ u\cap v=w}} \left(\sum_{\substack{q\in 2^{\Omega_1} \\ (q,T)|_\Omega=u}} m_1((q,T)) + \sum_{\substack{r\in 2^{\Omega_1} \\ (r,F)|_\Omega=u}} m_1((r,F))\right)\cdot\left(\sum_{\substack{s\in 2^{\Omega_2} \\ (s,T)|_\Omega=v}} m_2((s,T)) + \sum_{\substack{t\in 2^{\Omega_2} \\ (t,F)|_\Omega=v}} m_2((t,F))\right)$$

(by splitting complementary focal elements into T and F components)

$$= \sum_{\substack{u\in 2^\Omega \\ v\in 2^\Omega \\ u\cap v=w}} \left(\sum_{\substack{q\in 2^{\Omega_1} \\ \Omega\cap q=u}} m_1((q,T)) + \sum_{\substack{r\in 2^{\Omega_1} \\ \Omega-r=u}} m_1((r,F))\right)\cdot\left(\sum_{\substack{s\in 2^{\Omega_2} \\ \Omega\cap s=v}} m_2((s,T)) + \sum_{\substack{t\in 2^{\Omega_2} \\ \Omega-t=v}} m_2((t,F))\right)$$

(by evaluation of complementary focal elements Equation (3))

$$= \sum_{\substack{u\in 2^\Omega \\ v\in 2^\Omega \\ u\cap v=w}} \Bigg(\sum_{\substack{q\in 2^{\Omega_1} \\ \Omega\cap q=u}} m_1((q,T))\cdot\sum_{\substack{s\in 2^{\Omega_2} \\ \Omega\cap s=v}} m_2((s,T)) + \sum_{\substack{q\in 2^{\Omega_1} \\ \Omega\cap q=u}} m_1((q,T))\cdot\sum_{\substack{t\in 2^{\Omega_2} \\ \Omega-t=v}} m_2((t,F))$$

$$+ \sum_{\substack{r\in 2^{\Omega_1} \\ \Omega-r=u}} m_1((r,F))\cdot\sum_{\substack{s\in 2^{\Omega_2} \\ \Omega\cap s=v}} m_2((s,T)) + \sum_{\substack{r\in 2^{\Omega_1} \\ \Omega-r=u}} m_1((r,F))\cdot\sum_{\substack{t\in 2^{\Omega_2} \\ \Omega-t=v}} m_2((t,F))\Bigg)$$

(by distributive expansion of a product of sums)

$$= \sum_{\substack{q\in 2^{\Omega_1} \\ s\in 2^{\Omega_2} \\ (\Omega\cap q)\cap(\Omega\cap s)=w}} m_1((q,T))\cdot m_2((s,T)) + \sum_{\substack{q\in 2^{\Omega_1} \\ t\in 2^{\Omega_2} \\ (\Omega\cap q)\cap(\Omega-t)=w}} m_1((q,T))\cdot m_2((t,F))$$

$$+ \sum_{\substack{r\in 2^{\Omega_1} \\ s\in 2^{\Omega_2} \\ (\Omega-r)\cap(\Omega\cap s)=w}} m_1((r,F))\cdot m_2((s,T)) + \sum_{\substack{r\in 2^{\Omega_1} \\ t\in 2^{\Omega_2} \\ (\Omega-r)\cap(\Omega-t)=w}} m_1((r,F))\cdot m_2((t,F))$$

(by substitution)

$$
\begin{aligned}
&= \sum_{\substack{q\in 2^{\Omega_1}\\ s\in 2^{\Omega_2}\\ \Omega\cap(q\cap s)=w}} m_1((q,T))\cdot m_2((s,T)) + \sum_{\substack{q\in 2^{\Omega_1}\\ t\in 2^{\Omega_2}\\ \Omega\cap(q-t)=w}} m_1((q,T))\cdot m_2((t,F))\\
&\quad + \sum_{\substack{r\in 2^{\Omega_1}\\ s\in 2^{\Omega_2}\\ \Omega\cap(s-r)=w}} m_1((r,F))\cdot m_2((s,T)) + \sum_{\substack{r\in 2^{\Omega_1}\\ t\in 2^{\Omega_2}\\ \Omega-(r\cup t)=w}} m_1((r,F))\cdot m_2((t,F)) && \text{(by set theory)}\\
&= \sum_{\substack{q\in 2^{\Omega_1}\\ s\in 2^{\Omega_2}\\ (q\cap s,T)|_\Omega=w}} m_1((q,T))\cdot m_2((s,T)) + \sum_{\substack{q\in 2^{\Omega_1}\\ t\in 2^{\Omega_2}\\ (q-t,T)|_\Omega=w}} m_1((q,T))\cdot m_2((t,F))\\
&\quad + \sum_{\substack{r\in 2^{\Omega_1}\\ s\in 2^{\Omega_2}\\ (s-r,T)|_\Omega=w}} m_1((r,F))\cdot m_2((s,T)) + \sum_{\substack{r\in 2^{\Omega_1}\\ t\in 2^{\Omega_2}\\ (r\cup t,F)|_\Omega=w}} m_1((r,F))\cdot m_2((t,F))\\
&&& \text{(by evaluation of complementary focal elements Equation (3))}\\
&= \sum_{\substack{q\in 2^{\Omega_1}\\ s\in 2^{\Omega_2}\\ ((q,T)\cap(s,T))|_\Omega=w}} m_1((q,T))\cdot m_2((s,T)) + \sum_{\substack{q\in 2^{\Omega_1}\\ t\in 2^{\Omega_2}\\ ((q,T)\cap(t,F))|_\Omega=w}} m_1((q,T))\cdot m_2((t,F))\\
&\quad + \sum_{\substack{r\in 2^{\Omega_1}\\ s\in 2^{\Omega_2}\\ ((s,T)\cap(r,F))|_\Omega=w}} m_1((r,F))\cdot m_2((s,T)) + \sum_{\substack{r\in 2^{\Omega_1}\\ t\in 2^{\Omega_2}\\ ((r,F)\cap(t,F))|_\Omega=w}} m_1((r,F))\cdot m_2((t,F))\\
&&& \text{(by intersection identities Equation (6))}\\
&= \sum_{\substack{x\in 2^{\Omega_1}\times\mathbb{B}\\ y\in 2^{\Omega_2}\times\mathbb{B}\\ (x\cap y)|_\Omega=w}} m_1(x)\cdot m_2(y)\\
&&& \text{(by contracting } T \text{ and } F \text{ components into complementary focal elements)}\\
&= \sum_{\substack{z\in 2^{\Omega_1\cup\Omega_2}\times\mathbb{B}\\ z|_\Omega=w}} \sum_{\substack{x\in 2^{\Omega_1}\times\mathbb{B}\\ y\in 2^{\Omega_2}\times\mathbb{B}\\ (x\cap y)=z}} m_1(x)\cdot m_2(y) && \text{(by substitution)}\\
&= \sum_{\substack{z\in 2^{\Omega_1\cup\Omega_2}\times\mathbb{B}\\ z|_\Omega=w}} (m_1\oplus m_2)(z) && \text{(by conjunctive join Definition 3)}\\
&= (m_1\oplus m_2)|_\Omega(w) && \text{(by evaluation of CBPAs Equation (4))}
\end{aligned}
$$

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Theorem 6 *For a CBPA m with a frame of discernment Ω and for $u \in 2^\Omega$,*

$$
\mathrm{bel}_m((u,T)) \le \mathrm{bel}_{m|_\Omega}(u) \le \mathrm{bel}_m((\Omega-u,F)) \ .
$$

Proof Done directly as two inequalities.

$$
\begin{aligned}
\mathrm{bel}_m((u,T)) &= \sum_{\substack{x\in 2^\Omega\times\mathbb{B}\\ x\subseteq(u,T)\\ x\neq(\emptyset,T)}} m(x) && \text{(by belief Definition 5)}\\
&= \sum_{\substack{v\in 2^\Omega\\ (v,T)\subseteq(u,T)\\ (v,T)\neq(\emptyset,T)}} m((v,T)) + \sum_{\substack{v\in 2^\Omega\\ (v,F)\subseteq(u,T)\\ (v,F)\neq(\emptyset,T)}} m((v,F))\\
&&& \text{(by splitting complementary focal elements into } T \text{ and } F \text{ components)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{v\in 2^{\Omega}\\ v\subseteq u\\ v\neq\emptyset}} m((v,T)) && \text{(by subset identities Equation (10))}\\
&\leq \sum_{\substack{v\in 2^{\Omega}\\ v\subseteq u\\ v\neq\emptyset}} \left(m((v,T)) + m((\Omega - v, F))\right) && \text{(by adding additional components)}\\
&= \sum_{\substack{v\in 2^{\Omega}\\ v\subseteq u\\ v\neq\emptyset}} m|_{\Omega}(v) && \text{(by evaluation identity Equation (5))}\\
&= \mathrm{bel}_{m|_{\Omega}}(u) && \text{(by belief definition)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{bel}_{m|_{\Omega}}(u) &= \sum_{\substack{v\in 2^{\Omega}\\ v\subseteq u\\ v\neq\emptyset}} m|_{\Omega}(v) && \text{(by belief definition)}\\
&= \sum_{\substack{v\in 2^{\Omega}\\ v\subseteq u\\ v\neq\emptyset}} \left(m((v,T)) + m((\Omega - v, F))\right) && \text{(by evaluation identity Equation (5))}\\
&= \sum_{\substack{v\in 2^{\Omega}\\ v\subseteq u\\ v\neq\emptyset}} m((v,T)) + \sum_{\substack{v\in 2^{\Omega}\\ v\subseteq u\\ v\neq\emptyset}} m((\Omega - v, F)) && \text{(by distributive property)}\\
&= \sum_{\substack{v\in 2^{\Omega}\\ v\cap(\Omega-u)=\emptyset\\ v\neq\emptyset}} m((v,T)) + \sum_{\substack{v\in 2^{\Omega}\\ \Omega-u\subseteq\Omega-v\\ v\neq\emptyset}} m((\Omega - v, F)) && \text{(by set theory)}\\
&\leq \sum_{\substack{v\in 2^{\Omega}\\ v\cap(\Omega-u)=\emptyset\\ v\neq\emptyset}} m((v,T)) + \sum_{\substack{v\in 2^{\Omega}\\ \Omega-u\subseteq\Omega-v}} m((\Omega - v, F)) && \text{(by adding additional components)}\\
&= \sum_{\substack{v\in 2^{\Omega}\\ v\cap(\Omega-u)=\emptyset\\ v\neq\emptyset}} m((v,T)) + \sum_{\substack{y\in 2^{\Omega}\\ \Omega-u\subseteq y}} m((y, F)) && \text{(change of variables, } y = \Omega - v\text{)}\\
&= \sum_{\substack{x\in 2^{\Omega}\times\mathbb{B}\\ x\subseteq(\Omega-u,F)\\ x\neq(\emptyset,T)}} m(x) && \text{(by contracting } T \text{ and } F \text{ components into complementary focal elements)}\\
&= \mathrm{bel}_m((\Omega - u, F)) && \text{(by belief Definition 5)}
\end{aligned}
$$

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Theorem 8 *For a CBPA m with a frame of discernment Ω and for* $u \in 2^{\Omega}$,

$$\mathrm{pl}_m((u,T)) = \mathrm{pl}_{m|_{\Omega}}(u) \leq \mathrm{pl}_m((\Omega - u, F)) \ .$$

Proof Done directly in two steps.

$$
\begin{aligned}
\mathrm{pl}_m((u,T)) &= \sum_{\substack{x\in 2^{\Omega}\times\mathbb{B}\\ x\cap(u,T)\neq(\emptyset,T)}} m(x) && \text{(by plausibility Definition 7)}\\
&= \sum_{\substack{v\in 2^{\Omega}\\ (v,T)\cap(u,T)\neq(\emptyset,T)}} m((v,T)) + \sum_{\substack{v\in 2^{\Omega}\\ (v,F)\cap(u,T)\neq(\emptyset,T)}} m((v,F))
\end{aligned}
$$

(by splitting complementary focal elements into T and F components)

$$= \sum_{\substack{v \in 2^{\Omega} \\ v \cap u \neq \emptyset}} m((v,T)) + \sum_{\substack{v \in 2^{\Omega} \\ u - v \neq \emptyset}} m((v,F)) \quad \text{(by intersection identities Equation (6))}$$

$$= \sum_{\substack{v \in 2^{\Omega} \\ \Omega \cap v \cap u \neq \emptyset}} m((v,T)) + \sum_{\substack{v \in 2^{\Omega} \\ (\Omega - v) \cap u \neq \emptyset}} m((v,F)) \quad \text{(by set theory)}$$

$$= \sum_{\substack{w \in 2^{\Omega} \\ w \cap u \neq \emptyset}} \left(\sum_{\substack{v \in 2^{\Omega} \\ \Omega \cap v = w}} m((v,T)) + \sum_{\substack{v \in 2^{\Omega} \\ \Omega - v = w}} m((v,F)) \right) \quad \text{(by algebra)}$$

$$= \sum_{\substack{w \in 2^{\Omega} \\ w \cap u \neq \emptyset}} \left(\sum_{\substack{v \in 2^{\Omega} \\ (v,T)|_{\Omega} = w}} m((v,T)) + \sum_{\substack{v \in 2^{\Omega} \\ (v,F)|_{\Omega} = w}} m((v,F)) \right)$$

(by evaluation of complementary focal elements Equation (3))

$$= \sum_{\substack{w \in 2^{\Omega} \\ w \cap u \neq \emptyset}} \sum_{\substack{x \in 2^{\Omega} \times \mathbb{B} \\ x|_{\Omega} = w}} m(x) \quad \text{(by contracting } T \text{ and } F \text{ components into complementary focal elements)}$$

$$= \sum_{\substack{w \in 2^{\Omega} \\ w \cap u \neq \emptyset}} m|_{\Omega}(w) \quad \text{(by evaluation of CBPAs Equation (4))}$$

$$= \mathrm{pl}_{m|_{\Omega}}(u) \quad \text{(by plausibility definition)}$$

With the first equality $\mathrm{pl}_m((u,T)) = \mathrm{pl}_{m|_{\Omega}}(u)$ shown to be true, we now prove $\mathrm{pl}_m((u,T)) \leq \mathrm{pl}_m((\Omega - u, F))$.

$$\mathrm{pl}_m((u,T)) = \sum_{\substack{x \in 2^{\Omega} \times \mathbb{B} \\ x \cap (u,T) \neq (\emptyset, T)}} m(x) \quad \text{(by plausibility Definition 7)}$$

$$= \sum_{\substack{v \in 2^{\Omega} \\ (v,T) \cap (u,T) \neq (\emptyset,T)}} m((v,T)) + \sum_{\substack{v \in 2^{\Omega} \\ (v,F) \cap (u,T) \neq (\emptyset,T)}} m((v,F))$$

(by splitting complementary focal elements into T and F components)

$$= \sum_{\substack{v \in 2^{\Omega} \\ v \cap u \neq \emptyset}} m((v,T)) + \sum_{\substack{v \in 2^{\Omega} \\ u - v \neq \emptyset}} m((v,F)) \quad \text{(by intersection identities Equation (6))}$$

$$\leq \sum_{\substack{v \in 2^{\Omega} \\ v \cap u \neq \emptyset}} m((v,T)) + \sum_{v \in 2^{\Omega}} m((v,F)) \quad \text{(by adding additional components)}$$

$$= \sum_{\substack{v \in 2^{\Omega} \\ v - (\Omega - u) \neq \emptyset}} m((v,T)) + \sum_{v \in 2^{\Omega}} m((v,F)) \quad \text{(by set theory)}$$

$$= \sum_{\substack{v \in 2^{\Omega} \\ (v,T) \cap (\Omega - u, F) \neq (\emptyset,T)}} m((v,T)) + \sum_{\substack{v \in 2^{\Omega} \\ (v,F) \cap (\Omega - u, F) \neq (\emptyset,T)}} m((v,F)) \quad \text{(by intersection identities Equation (6))}$$

$$= \sum_{\substack{x \in 2^{\Omega} \times \mathbb{B} \\ x \cap (\Omega - u, F) \neq (\emptyset,T)}} m(x) \quad \text{(by contracting } T \text{ and } F \text{ components into complementary focal elements)}$$

$$= \mathrm{pl}_m((\Omega - u, F)) \quad \text{(by plausibility Definition 7)}$$

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Theorem 9 *Belief and plausibility are related through the following equation:*

$$m((\emptyset, T)) + \mathrm{bel}((u, \neg a) + \mathrm{pl}((u,a)) = 1 \tag{1}$$

Proof We prove this by breaking it up into two cases and expanding.
Case 1: $a = T$

$$
\begin{aligned}
& m((\emptyset,T)) + \mathrm{bel}((u,F) + \mathrm{pl}((u,T)) \\
&= m((\emptyset,T)) + \sum_{\substack{x\in 2^{\Omega}\times\mathbb{B} \\ x\subseteq(u,F) \\ x\neq(\emptyset,T)}} m(x) + \sum_{\substack{x\in 2^{\Omega}\times\mathbb{B} \\ x\cap(u,T)\neq(\emptyset,T)}} m(x) && \text{(by belief and plausibility Definitions 5 and 7)} \\
&= \sum_{\substack{x\in 2^{\Omega}\times\mathbb{B} \\ x\subseteq(u,F)}} m(x) + \sum_{\substack{x\in 2^{\Omega}\times\mathbb{B} \\ x\cap(u,T)\neq(\emptyset,T)}} m(x) && \text{(by grouping the first two terms together)} \\
&= \sum_{\substack{v\in 2^{\Omega} \\ (v,T)\subseteq(u,F)}} m((v,T)) + \sum_{\substack{v\in 2^{\Omega} \\ (v,F)\subseteq(u,F)}} m((v,F)) + \sum_{\substack{v\in 2^{\Omega} \\ (v,T)\cap(u,T)\neq(\emptyset,T)}} m((v,T)) + \sum_{\substack{v\in 2^{\Omega} \\ (v,F)\cap(u,T)\neq(\emptyset,T)}} m((v,F)) \\
& && \text{(by splitting complementary focal elements into } T \text{ and } F \text{ components)} \\
&= \sum_{\substack{v\in 2^{\Omega} \\ v\cap u=\emptyset}} m((v,T)) + \sum_{\substack{v\in 2^{\Omega} \\ u\subseteq v}} m((v,F)) + \sum_{\substack{v\in 2^{\Omega} \\ u\cap v\neq\emptyset}} m((v,T)) + \sum_{\substack{v\in 2^{\Omega} \\ u-v\neq\emptyset}} m((v,F)) \\
& && \text{(by applying intersection and subset Equations (6) and (10))} \\
&= \sum_{v\in 2^{\Omega}} m((v,T)) + \sum_{v\in 2^{\Omega}} m((v,F)) && \text{(by grouping like terms, } m((v,T)) \text{ together and } m((v,F)) \text{ together)} \\
&= \sum_{x\in 2^{\Omega}\times\mathbb{B}} m(x) && \text{(by contracting } T \text{ and } F \text{ components into complementary focal elements)} \\
&= 1 && \text{(by CBPA Definition 2)}
\end{aligned}
$$

Case 2: $a = F$

$$
\begin{aligned}
& m((\emptyset,T)) + \mathrm{bel}((u,T) + \mathrm{pl}((u,F)) \\
&= m((\emptyset,T)) + \sum_{\substack{x\in 2^{\Omega}\times\mathbb{B} \\ x\subseteq(u,T) \\ x\neq(\emptyset,T)}} m(x) + \sum_{\substack{x\in 2^{\Omega}\times\mathbb{B} \\ x\cap(u,F)\neq(\emptyset,T)}} m(x) && \text{(by belief and plausibility Definitions 5 and 7)} \\
&= \sum_{\substack{x\in 2^{\Omega}\times\mathbb{B} \\ x\subseteq(u,T)}} m(x) + \sum_{\substack{x\in 2^{\Omega}\times\mathbb{B} \\ x\cap(u,F)\neq(\emptyset,T)}} m(x) && \text{(by grouping the first two terms together)} \\
&= \sum_{\substack{v\in 2^{\Omega} \\ (v,T)\subseteq(u,T)}} m((v,T)) + \sum_{\substack{v\in 2^{\Omega} \\ (v,F)\subseteq(u,T)}} m((v,F)) + \sum_{\substack{v\in 2^{\Omega} \\ (v,T)\cap(u,F)\neq(\emptyset,T)}} m((v,T)) + \sum_{\substack{v\in 2^{\Omega} \\ (v,F)\cap(u,F)\neq(\emptyset,T)}} m((v,F)) \\
& && \text{(by splitting complementary focal elements into } T \text{ and } F \text{ components)} \\
&= \sum_{\substack{v\in 2^{\Omega} \\ v\subseteq u}} m((v,T)) + \sum_{\substack{v\in 2^{\Omega} \\ v-u\neq\emptyset}} m((v,T)) + \sum_{v\in 2^{\Omega}} m((v,F)) \\
& && \text{(by applying intersection and subset Equations (6) and (10))} \\
&= \sum_{v\in 2^{\Omega}} m((v,T)) + \sum_{v\in 2^{\Omega}} m((v,F)) && \text{(by grouping like terms, } m((v,T)) \text{ together)} \\
&= \sum_{x\in 2^{\Omega}\times\mathbb{B}} m(x) && \text{(by contracting } T \text{ and } F \text{ components into complementary focal elements)} \\
&= 1 && \text{(by CBPA Definition 2)}
\end{aligned}
$$

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