Before presenting the proof of the propositions, we need the following definition on Tensor Connect Product (TCP), which computes the tensor core merging.

**Definition 1** (Tensor Connect Product (TCP) [Wang et al., 2017]). Let $Z_k \in \mathbb{R}^{r_k \times I_k \times r_{k+1}}$, $k = 1, \ldots, N$ be $N$ 3-order tensors. The tensor connect product (TCP) between $Z_k$ and $Z_{k+1}$ is defined as,

\[
Z^{(k,k+1)} = \text{fold}(L(Z_k) \times R(Z_{k+1}))
\]

where $\text{fold}(X)$ denotes the operation of reshaping the unfolding matrix $X$ back to tensor $X$ and

\[
L(X) = (X^{(3)})^T \in \mathbb{R}^{(r_k I_k) \times r_{k+1}}
\]

\[
R(X) = X^{(1)} \in \mathbb{R}^{r_k \times (I_k r_{k+1})}.
\]

First, we consider the computation of the Gram matrix using only two core tensors. According to the tensor core merging of two core tensors $Z_k$ and $Z_{k+1}$, we establish the following lemma.

**Lemma 1.** Let $Z_k \in \mathbb{R}^{r_k \times I_k \times r_{k+1}}$, $k = 1, \ldots, N$, be 3rd order tensors. The Gram matrix of $Z^{(k,k+1)}$ can be computed as

\[
G_{Z^{(k,k+1)}} = Z^{(k,k+1),T}_{[2]} Z^{(k,k+1)}_{[2]} = \Phi(Q_k Q_{k+1})
\]

where $Q_k(:; i \times r_{k+1} + j) = \text{vec}\{(Z_k(:,; i)) Z_k(:,; j)^T\}$, with $\text{vec}\{\cdot\}$ denoting the vectorization operation, and $\Phi(X)$ is a reshape operation by which $X \in \mathbb{R}^{m^2 \times n^2}$ is first divided into $m \times n$ blocks $\{X_{ij}\}_{i,j=1}^{m,n} \in \mathbb{R}^{m \times n}$, then reshaped as

\[
\Phi(X) = \text{vec}\{X_{11}^T\} \text{ vec}\{X_{21}^T\} \ldots \text{ vec}\{X_{mn}^T\}^T.
\]

1 **PROOF OF LEMMA 1**

**Proof.** From TCP in Definition 1, we can express the fiber-wise relation between mode-2 fibers of $Z_k$, $Z_{k+1}$ and $Z^{(k,k+1)}$ as

\[
Z^{(k,k+1)}(i,:; j) = \sum_{m=1}^{r_{k+1}} Z_{k+1}(m,:; j) \otimes Z_k(i,:; m),
\]

with $i \in [1, r_k], j \in [1, r_{k+2}]$ and $\otimes$ denotes the kronecker product. Then, the $(i,j)$-th entry of the Gram matrix of $Z^{(k,k+1)}_{[2]}$ can be computed as

\[
\left[Z^{(k,k+1),T}_{[2]} Z^{(k,k+1)}_{[2]}\right]_{i,j} = \left(Z^{(k,k+1)}(p_i,:; q_i)\right)^T Z^{(k,k+1)}(p_j,:; q_j),
\]

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where \( p_i = \lfloor i/r_{k+1} \rfloor \), \( q_i = \mod(i - 1, r_{k+1}) + 1 \). Substituting (2) in (3) and using the property
\[
(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}),
\]
we have that
\[
\mathbf{v}_1 = \text{vec}\{\mathbf{Z}_k(p_i, :, :)^T \mathbf{Z}_k(p_j, :, :)}\}
\]
\[
\mathbf{v}_2 = \text{vec}\{\mathbf{Z}_{k+1}(::, q_i) \mathbf{Z}_{k+1}(::, q_j)^T\}.
\]
Therefore, by defining \( \mathbf{Q}_k(:, i \times r_{k+1} + j) = \text{vec}\{(\mathbf{Z}_k(:, :, i))^T \mathbf{Z}_k(:, :, j)^T\} \), the Gram matrix of \( \mathbf{Z}_{k+1}^{(k,k+1)} \) can be computed as
\[
\mathbf{G}_{\mathbf{Z}_{k+1}^{(k,k+1)}} = \mathbf{Z}_{k+1}^{(k,k+1)\top} \mathbf{Z}_{k+1}^{(k,k+1)} = \Phi(\mathbf{Q}_k \mathbf{Q}_k^+) \]
where \( \Phi(\mathbf{X}) \) is a reshape operation by which \( \mathbf{X} \in \mathbb{R}^{m^2 \times n^2} \) is first divided into \( m \times n \) blocks \( \{\mathbf{X}_{ij}\}_{i,1}^{m,n} \in \mathbb{R}^{m \times n} \), then reshaped as
\[
\Phi(\mathbf{X}) = [\text{vec}\{\mathbf{X}_{i1}^T\} \text{ vec}\{\mathbf{X}_{21}^T\} \ldots \text{ vec}\{\mathbf{X}_{mn}^T\}]^T.
\]

2 PROOF OF PROPOSITION 1

Following (1) in the proof of Lemma 1, for \( \mathbf{Z}^{\leq c} \in \mathbb{R}^{t_1 \times \prod_{k=1}^c j_k \times r_{c+1}} \) which is a subchain obtained by merging \( c \) cores \( \{\mathbf{Z}_k\}_{k=1}^c \), according to TCP, we can express the fiber-wise relation between mode-2 fibers of \( \mathbf{Z}_c \) and \( \mathbf{Z}^{\leq c-1} \) as
\[
\mathbf{Z}^{\leq c}(i, :, j) = \sum_{m=1}^{r_c} \mathbf{Z}_c(m, :, j) \otimes \mathbf{Z}^{\leq c-1}(i, :, m).
\]

With the above recursion equation we have
\[
\mathbf{Z}^{\leq c}(i, :, j)
= \sum_{m=1}^{r_c} \mathbf{Z}_c(m, :, j) \otimes \left( \sum_{m=1}^{r_{c-1}} \mathbf{Z}_{c-1}(m, :, j) \otimes \left( \sum_{m=1}^{r_2} \mathbf{Z}_2(m, :, j) \otimes \mathbf{Z}_1(i, :, m) \right) \right).
\]

Again, using the property in (4), we can obtain that
\[
\mathbf{v}_1 = \text{vec}\{\mathbf{Z}_1(p_i, :, :)^T \mathbf{Z}_1(p_j, :, :)}\}
\]
\[
\mathbf{v}_2 = \text{vec}\{\mathbf{Z}_{c}(::, q_i) \mathbf{Z}_{c}(::, q_j)^T\}
\]
\[
\mathbf{Q}_k(:, i \times r_{k+1} + j) = \text{vec}\{(\mathbf{Z}_k(:, :, i))^T \mathbf{Z}_k(:, :, j)^T\}, k = 2, \ldots, c - 1
\]
\[
p_i = \lfloor i/r_{c+1} \rfloor, q_i = \mod(i - 1, r_{c+1}) + 1.
\]

Then, the Gram matrix of \( \mathbf{Z}_{k+1}^{\leq c} \) can be computed as
\[
\mathbf{G}_{\mathbf{Z}^{\leq c}} = \mathbf{Z}_{k+1}^{\leq c\top} \mathbf{Z}_{k+1}^{\leq c} = \Phi\left(\prod_{k=1}^c \mathbf{Q}_k\right),
\]
where \( \mathbf{Q}_k(:, i \times r_{k+1} + j) = \text{vec}\{(\mathbf{Z}_k(:, :, i))^T \mathbf{Z}_k(:, :, j)^T\} \) for \( k > 1 \) and
\[
\mathbf{Q}_1(:, i \times r_2 + j) = \begin{cases} 
\text{vec}\{(\mathbf{Z}_1(:, :, i))^T \mathbf{Z}_1(:, :, j)^T\}, & c \text{ is even} \\
\text{vec}\{(\mathbf{Z}_1(:, :, j))^T \mathbf{Z}_1(:, :, i)^T\}, & c \text{ is odd}
\end{cases}
\]

References