Incentivising Diffusion while Preserving Differential Privacy (Supplementary Material)

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A IDM

Here, we introduce the first diffusion auction for selling single item, IDM [13]. A key concept of IDM is diffusion critical sequence. Given a profile digraph $G_{\theta'}$, for any buyers $i, j \in V_{\theta'}$, i is *θ'-critical* to j, denoted by $i \preceq_{\theta'} j$, if all paths from s to j in $G_{\theta'}$ go through i. A *diffusion critical sequence* of i, denoted by C_i, is a sequence of all diffusion critical nodes of i and i itself ordered by θ'-critical relation. That is, $C_i = (x_1, x_2 \dots, x_k, i)$, where $x_1 \preceq_{\theta'} x_2 \preceq_{\theta'} \dots \preceq_{\theta'} x_k \preceq_{\theta'} i$. Based on this concept, IDM works as follows. IDM first locates the buyer m with the highest valuation among all buyers. Then it allocates the item to the buyer w, who has the highest valuation when the buyers after w are not considered. The winner w pays the highest bid without her participation, and each diffusion critical node is rewarded by the increased payment due to her participation.

B PROOF OF LEMMA 4.6

Lemma 4.6. *Given a reported global profile θ', recursive DPDM* REC *is $\epsilon d_{\max}\Delta\sigma$-differentially private, where ϵ is the DP parameter of* REC.

Proof. Let θ and θ' be two profiles where a buyer i's reports i reports v_i in θ and v_i' in θ' such that $v_i \neq v_i'$. Consider the probabilities that $\mathrm{REC}(\theta)$ and $\mathrm{REC}(\theta')$ return a winner w. In a critical diffusion tree T_θ, let d_w denote the depth of w, a_w^ℓ be an ancestor of w with distance ℓ. Also, let $\mathrm{Exp}^\theta(T(a_w^1) - T(w))$ and $\mathrm{Exp}^{\theta'}(T(a_w^1) - T(w))$ denote the value derived from θ and θ', respectively. Then by Equation (3), we have

$$\frac{\Pr[\mathrm{REC}(\theta) = o_w]}{\Pr[\mathrm{REC}(\theta') = o_w]} = \frac{\frac{\mathrm{Exp}(w)}{\mathrm{Exp}^\theta(T(a_w^1) - T(w))}}{\frac{\mathrm{Exp}^{\theta'}(w)}{\mathrm{Exp}^{\theta'}(T(a_w^1) - T(w))}} \times \frac{\Pr^\theta_{T[a_w^1]} - \Pr^\theta_{a_w^1}}{\Pr^{\theta'}_{T[a_w^1]} - \Pr^{\theta'}_{a_w^1}}$$

We repeatedly replace $\Pr^\theta_{T[a_w^\ell]}$, $\Pr^\theta_{a_w^\ell}$, $\Pr^{\theta'}_{T[a_w^\ell]}$, $\Pr^{\theta'}_{a_w^\ell}$ by expressions of $a_w^{\ell+1}$ until we get an expression of s. For each distance $0 \le \ell < d_w$, we denote $\frac{\mathrm{Exp}(T[a_w^\ell])}{\mathrm{Exp}(T(a_w^{\ell+1}))}$ as A_ℓ^θ, $\frac{\mathrm{Exp}(a_w^\ell)}{\mathrm{Exp}(T(a_w^{\ell+1}) \setminus T(a_w^\ell))}$ as B_ℓ^θ. For θ', we have similar notations as $A_\ell^{\theta'}$ and $B_\ell^{\theta'}$. Then the above ratio can be written as

$$\frac{\Pr[\mathrm{REC}(\theta) = o_w]}{\Pr[\mathrm{REC}(\theta') = o_w]} = \frac{B_0^\theta}{B_0^{\theta'}} \times \prod_{\ell=1}^{d_w - 1} \frac{A_\ell^\theta - B_\ell^\theta}{A_\ell^{\theta'} - B_\ell^{\theta'}}$$

Next we show for each $0 \le \ell < d_w$, $\frac{A_\ell^\theta - B_\ell^\theta}{A_\ell^{\theta'} - B_\ell^{\theta'}}$ is bounded by $\exp(\epsilon\Delta\sigma)$. To prove it, we first show for for each ℓ, $(A_\ell^\theta - A_\ell^{\theta'}) \times (B_\ell^\theta - B_\ell^{\theta'}) \ge 0$ by cases.
(1) When $i \in T[a_w^\ell]$, we have $A_\ell^\theta - A_\ell^{\theta'} \le 0, B_\ell^\theta - B_\ell^{\theta'} \le 0$ or $A_\ell^\theta - A_\ell^{\theta'} \ge 0, B_\ell^\theta - B_\ell^{\theta'} \ge 0$
(2) When $i \in T[a_w^{\ell+1}] \setminus T[a_w^\ell]$, then $A_\ell^\theta - A_\ell^{\theta'} \le 0, B_\ell^\theta - B_\ell^{\theta'} \le 0$ or $A_\ell^\theta - A_\ell^{\theta'} \ge 0, B_\ell^\theta - B_\ell^{\theta'} \ge 0$
(3) When $i \notin T[a_w^{\ell+1}]$, then $A_\ell^\theta - A_\ell^{\theta'} = 0, B_\ell^\theta - B_\ell^{\theta'} = 0$.

Without loss of generality, we assume that $A_\ell^{\theta'} = \alpha_1 A_\ell^\theta, B_\ell^{\theta'} = \alpha_2 B_\ell^\theta, \alpha_1, \alpha_2 \in \mathbb{R}^+$. Plug in these two equations, and we get

$$\frac{A_\ell^\theta - B_\ell^\theta}{A_\ell^{\theta'} - B_\ell^{\theta'}} = \frac{A_\ell^\theta - B_\ell^\theta}{\alpha_1 A_\ell^\theta - \alpha_2 B_\ell^\theta}.$$

Then we consider two cases:
(1) When $\alpha_1 \ge \alpha_2$, we have $\frac{A_\ell^\theta - B_\ell^\theta}{\alpha_1 A_\ell^\theta - \alpha_2 B_\ell^\theta} \le \frac{A_\ell^\theta - B_\ell^\theta}{\alpha_1 A_\ell^\theta - \alpha_1 B_\ell^\theta} \le \frac{1}{\alpha_1}$.

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(2) When $\alpha_2 \geq \alpha_1$, we have $\frac{A_\ell^\theta - B_\ell^\theta}{\alpha_1 A_\ell^\theta - \alpha_2 B_\ell^\theta} \leq \frac{A_\ell^\theta - B_\ell^\theta}{\alpha_2 A_\ell^\theta - \alpha_2 B_\ell^\theta} \leq \frac{1}{\alpha_2}$.

After that, we show that both $\frac{1}{\alpha_1}$ and $\frac{1}{\alpha_2}$ are bounded by $\exp(\epsilon\Delta\sigma)$ as follows. By definition of α_1, we have $\frac{1}{\alpha_1} = \frac{A_\ell^\theta}{A_\ell^{\theta'}} = \frac{\mathrm{Exp}^\theta(T[a_w^\ell])}{\mathrm{Exp}^{\theta'}(T[a_w^\ell])} \times \frac{\mathrm{Exp}^{\theta'}(T(a_w^{\ell+1}))}{\mathrm{Exp}^\theta(T(a_w^{\ell+1}))}$.
(1) When valuation $v_i' \leq v_i$, the second ratio is at most 1. Then we have

$$\frac{1}{\alpha_1} = \frac{A_\ell^\theta}{A_\ell^{\theta'}} \leq \frac{\mathrm{Exp}^\theta(T[a_w^\ell])}{\mathrm{Exp}^{\theta'}(T[a_w^\ell])}$$
$$\leq \frac{\sum_{k\in T[a_w^\ell]} \exp(\epsilon\sigma(\theta, o_k))}{\sum_{k\in T[a_w^\ell]} \exp(\epsilon(\sigma(\theta, o_k) - \Delta\sigma))} \leq \exp(\epsilon\Delta\sigma)$$

(2) When valuation $v_i' \geq v_i$, the first ratio is at most 1. We have

$$\frac{1}{\alpha_1} = \frac{A_\ell^\theta}{A_\ell^{\theta'}} \leq \frac{\mathrm{Exp}^{\theta'}(T(a_w^{\ell+1}))}{\mathrm{Exp}^\theta(T(a_w^{\ell+1}))}$$
$$\leq \frac{\sum_{k\in T(a_w^{\ell+1})} \exp(\epsilon(\sigma(\theta, o_k) + \Delta\sigma))}{\sum_{k\in T(a_w^{\ell+1})} \exp(\epsilon\sigma(\theta, o_k))} \leq \exp(\epsilon\Delta\sigma)$$

In a similar way, we can show that $\frac{1}{\alpha_2} \leq \exp(\epsilon\Delta\sigma)$.

Therefore we have

$$\frac{\Pr[\mathrm{REC}(\theta) = o_w]}{\Pr[\mathrm{REC}(\theta') = o_w]} \leq \exp(\epsilon\Delta\sigma) \times \prod_{1\leq \ell < d_w} \exp(\epsilon\Delta\sigma)$$
$$\leq \exp(\epsilon d_w \Delta\sigma) \leq \exp(\epsilon d_{\max}\Delta\sigma)$$

□

C PROOF OF LEMMA 5.2

Lemma 5.2. *Given a reported global profile θ', layered DPDM* LAY *is $\epsilon\Delta\sigma$-differential private, where ϵ is the privacy parameter of* LAY.

Proof. Given a global profile θ, for each buyer i with (v_i, r_i), we have

$$\mathbf{E}_{\mathrm{LAY}}[u_i(\theta)] = (v_i - p_i(\theta))\Pr_i(\theta_i)$$
$$= \int_0^{v_i} \Pr_i^{\mathrm{LAY}}((x, r_i))dx \geq 0.$$

Therefore, the lemma holds. □

D PROOF OF LEMMA 5.4

Lemma 5.4. *Given a reported global profile θ', layered DPDM* LAY *is $\epsilon\Delta\sigma$-differential private, where ϵ is the privacy parameter of* LAY.

Proof. Given two reported global profiles θ and θ' that differ in an arbitrary buyer i's reported valuation such that i reports v_i in θ and v_i' in θ', we consider the probabilities that $\mathrm{LAY}(\theta)$ and $\mathrm{LAY}(\theta')$ return a winner w.

Without loss of generality, we assume that w is in L_ℓ, then we have

$$\frac{\Pr[\mathrm{LAY}(\theta) = o_w]}{\Pr[\mathrm{LAY}(\theta') = o_w]} = \frac{\Pr_{L_\ell} \times \frac{\mathrm{Exp}^\theta(w)}{\mathrm{Exp}^\theta(L_\ell)}}{\Pr_{L_\ell} \times \frac{\mathrm{Exp}^{\theta'}(w)}{\mathrm{Exp}^{\theta'}(L_\ell)}}$$
$$= \frac{\mathrm{Exp}^\theta(w)}{\mathrm{Exp}^{\theta'}(w)} \frac{\mathrm{Exp}^{\theta'}(L_\ell)}{\mathrm{Exp}^\theta(L_\ell)}$$

When i is not on layer L_ℓ, $\frac{\Pr[\mathrm{LAY}(\theta)=o_w]}{\Pr[\mathrm{LAY}(\theta')=o_w]} = 1 \leq \exp(\epsilon\Delta\sigma)$. Otherwise, when i is on layer L_ℓ, we consider two cases.
(1) $v_i < v_i'$. As $\sigma(\cdot)$ is non-decreasing in v_i, the first ratio is at most 1. Then we have

$$\frac{\Pr[\mathrm{LAY}(\theta) = o_w]}{\Pr[\mathrm{LAY}(\theta') = o_w]} \leq \frac{\mathrm{Exp}^{\theta'}(L_\ell)}{\mathrm{Exp}^\theta(L_\ell)}$$
$$\leq \frac{\sum_{j\in L_\ell} \exp(\epsilon(\sigma(\theta, o_j) + \Delta\sigma))}{\sum_{j\in L_\ell} \exp(\epsilon\sigma(\theta, o_j))}$$
$$\leq \exp(\epsilon\Delta\sigma)$$

(2) $v_i > v_i'$. In this case, the second ratio is at most 1. Then we have

$$\frac{\Pr[\mathrm{LAY}(\theta) = o_w]}{\Pr[\mathrm{LAY}(\theta') = o_w]} \leq \frac{\mathrm{Exp}^\theta(w)}{\mathrm{Exp}^{\theta'}(w)} \leq \frac{\exp(\epsilon\sigma(\theta, o_w))}{\exp(\epsilon(\sigma(\theta, o_w) - \Delta\sigma))}$$
$$\leq \exp(\epsilon\Delta\sigma)$$

□

E PROOF OF THEOREM 5.6

Theorem 5.6 *Given a global profile θ, layered DPDM* LAY *has $\mathbf{E}_{\mathrm{LAY}}[sw_{\mathrm{LAY}}(\theta)] \geq \gamma_{d_{\max}} \mathbf{E}_{\mathrm{EMD}}[sw_{\mathrm{EMD}}(\theta)]$. .*

Proof. Given a global profile θ, the expected social welfare of LAY is

$$\mathbf{E}_{\mathrm{LAY}}[sw_{\mathrm{LAY}}(\theta)] = \sum_{i\in V} \left(v_i \times \Pr_i^{\mathrm{LAY}}(\theta_i)\right)$$
$$= \sum_{i\in V} v_i \frac{\exp(\epsilon, \sigma(\theta, o_i))}{\sum_{j\in L_{d_i}} \frac{1}{\gamma_{d_i}} \exp(\epsilon, \sigma(\theta, o_j))}$$
$$\geq \gamma_{d_{\max}} \sum_{i\in N} v_i \frac{\exp(\epsilon, \sigma(\theta, o_i))}{\sum_{j\in L_{d_i}} \exp(\epsilon, \sigma(\theta, o_j))}$$
$$\geq \gamma_{d_{\max}} \sum_{i\in N} v_i \frac{\exp(\epsilon, \sigma(\theta, o_i))}{\sum_{j\in V} \exp(\epsilon, \sigma(\theta, o_j))}$$
$$= \gamma_{d_{\max}} \mathbf{E}_{\mathrm{EMD}}[sw_{\mathrm{EMD}}(\theta)]$$

□