Mixture of Normalizing Flows for European Option Pricing
(Supplementary Material)

1 CONSTRAINTS TRANSLATION

Given our NF model for option pricing,

\[ f_\theta(K; z) = e^{-r\tau} F_\tau \int_{-\infty}^{+\infty} \left[ \mathbb{I}(z) \left( \frac{e^x}{\text{SG}(\mu_\theta)} - \frac{K}{F_\tau} \right) \right] + q_\theta(x) \, dx \]

(1)

where \( \mu_\theta = \int_{-\infty}^{+\infty} e^x q_\theta(x) \, dx \). We are going to prove that, as long as \( q_\theta \) is a valid density function, i.e.,

\[ q_\theta \geq 0, \quad \int_{-\infty}^{+\infty} q_\theta(x) \, dx = 1 \]

(2)

The constraints (C1) – (C4) for call option pricing and the constraints (P1) – (P4) for put option pricing will be met.

For the ease of notation, we drop some subscripts and write the pricing functions for call and put option separately.

\[ C(K) = e^{-r\tau} F \int_{\log(K\mu / \mu)}^{+\infty} \left( \frac{e^x}{\mu} - \frac{K}{F} \right) q(x) \, dx \]

(C)

\[ P(K) = e^{-r\tau} F \int_{-\infty}^{\log(K\mu / \mu)} \left( \frac{K}{F} - \frac{e^x}{\mu} \right) q(x) \, dx \]

(P)

The following constraints must be met for a non-arbitrage pricing model.

\[ \frac{\partial C(K)}{\partial K} \leq 0 \]  \hspace{1cm} (C1)

\[ \frac{\partial^2 C(K)}{\partial K^2} \geq 0 \]  \hspace{1cm} (C2)

\[ C(\infty) = 0 \]  \hspace{1cm} (C3)

\[ \max(0, e^{-r\tau} (F - K)) \leq C(K) \leq e^{-r\tau} F \]  \hspace{1cm} (C4)

\[ \frac{\partial P(K)}{\partial K} \geq 0 \]  \hspace{1cm} (P1)

\[ \frac{\partial^2 P(K)}{\partial K^2} \geq 0 \]  \hspace{1cm} (P2)

\[ P(0) = 0 \]  \hspace{1cm} (P3)

\[ \max(0, e^{-r\tau} (K - F)) \leq P(K) \leq e^{-r\tau} K \]  \hspace{1cm} (P4)
For (C1),
\[
\frac{\partial C(K)}{\partial K} = -e^{-r\tau} \int_{\log(\frac{K_0}{F})}^{\infty} q(x)dx \leq 0
\]
(3)

For (C2),
\[
\frac{\partial^2 C(K)}{\partial K^2} = \frac{e^{-r\tau}}{K} q(\log(\frac{K}{F})) \geq 0
\]
(4)

For (C3),
\[
C(\infty) = e^{-r\tau} F \int_{-\infty}^{\infty} \left( e^{x} \mu - \frac{K}{F} \right) q(x)dx = 0
\]
(5)

For (C4), the upper bound is achieved when \( K = 0 \) (C1), and
\[
C(0) = e^{-r\tau} F \int_{-\infty}^{\infty} \frac{e^{x}}{\mu} q(x)dx = e^{-r\tau} F
\]
(6)

Note that we have \( C'(0) = \frac{\partial C(K)}{\partial K} |_{K=0} = -e^{-r\tau} \), and \( C'(\cdot) \) is a non-decreasing function (C2), which means \( C'(K) \geq C'(0) \). Then,
\[
C(K) = C(0) + \int_{0}^{K} C'(y)dy 
\]
(7)
\[
\geq C(0) + \int_{0}^{K} C'(0)dy 
\]
(8)
\[
= C(0) + KC'(0) 
\]
(9)
\[
= e^{-r\tau} F - Ke^{-r\tau} 
\]
(10)
\[
= e^{-r\tau} (F - K) 
\]
(11)

Besides, \( C(K) \geq C(\infty) = 0 \), thus the tighter lower bound should be \( \max(0, e^{-r\tau} (F - K)) \).

For (P1),
\[
\frac{\partial P(K)}{\partial K} = e^{-r\tau} \int_{-\infty}^{\log(\frac{K}{F})} q(x)dx \geq 0
\]
(12)

For (P2),
\[
\frac{\partial^2 P(K)}{\partial K^2} = \frac{e^{-r\tau}}{K} q(\log(\frac{K}{F})) \geq 0
\]
(13)

For (P3),
\[
P(0) = e^{-r\tau} F \int_{-\infty}^{\infty} \left( \frac{K}{F} - \frac{e^{x}}{\mu} \right) q(x)dx = 0
\]
(14)

For (P4), we can follow the similar procedure as we prove (C4). However, since C(4) has already been proved, we can simply apply put-call parity,
\[
P(K) = C(K) - e^{-r\tau} F + e^{-r\tau} K
\]
(15)

Thus we have
\[
\max(0, e^{-r\tau} (K - F)) \leq P(K) \leq e^{-r\tau} K
\]
(16)