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Wasserstein Distributionally Robust Regret-Optimal Control over Infinite-Horizon

Taylan Kargin* TKARGIN@CALTECH.EDU
Joudi Hajar* JHAJAR@CALTECH.EDU
Vikrant Malik* VMALIK@CALTECH.EDU
Babak Hassibi HASSIBI@CALTECH.EDU
**Equal contribution*
Department of Electrical Engineering, Caltech, Pasadena, CA 91125

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Abstract

We investigate the Distributionally Robust Regret-Optimal (DR-RO) control of discrete-time linear dynamical systems with quadratic cost over an infinite horizon. Regret is the difference in cost obtained by a causal controller and a clairvoyant controller with access to future disturbances. We focus on the infinite-horizon framework, which results in stability guarantees. In this DR setting, the probability distribution of the disturbances resides within a Wasserstein-2 ambiguity set centered at a specified nominal distribution. Our objective is to identify a control policy that minimizes the worst-case expected regret over an infinite horizon, considering all potential disturbance distributions within the ambiguity set. In contrast to prior works, which assume time-independent disturbances, we relax this constraint to allow for time-correlated disturbances, thus *actual* distributional robustness. While we show that the resulting optimal controller is non-rational and lacks a finite-dimensional state-space realization, we demonstrate that it can still be uniquely characterized by a finite dimensional parameter. Exploiting this fact, we introduce an efficient numerical method to compute the controller in the frequency domain using fixed-point iterations. This method circumvents the computational bottleneck associated with the finite-horizon problem, where the semi-definite programming (SDP) solution dimension scales with the time horizon. Numerical experiments demonstrate the effectiveness and performance of our framework.

Keywords: Distributionally Robust Control, Regret-Optimal Control, Wasserstein distance, Infinite-Horizon Control.

1. Introduction

Ensuring reliable and effective operation in the face of uncertainty is a fundamental challenge in decision-making and control. Control systems are inherently subject to diverse uncertainties, including exogenous disturbances, measurement inaccuracies, modeling discrepancies, and temporal variations in the underlying dynamics (van der Grinten, 1968; Doyle, 1985). Disregarding these uncertainties during controller design can lead to significant performance degradation and even unsafe and unintended behavior (Samuelson and Yang, 2017).

Traditionally, stochastic and robust control frameworks have addressed this issue primarily through the lens of exogenous disturbances (Kalman, 1960; Zames, 1981; Doyle et al., 1988). Stochastic control, exemplified by Linear–quadratic–Gaussian (LQG), or H_2, control, aims to minimize the expected cost, assuming disturbances are generated randomly from a known probability distribution (Hassibi et al., 1999). However, in practice, the true distribution is often estimated from sampled data, rendering this approach vulnerable to inaccurate models. Robust control, on the other hand, seeks to minimize the worst-case cost across a set of potential disturbance realizations, like those with bounded energy or power (H_∞ control) (Zhou et al., 1996). While this guarantees robustness against adversarial disturbances, it can be overly conservative, discarding potentially valuable statistical information. To address this issue, two recent approaches have emerged.

Regret-Optimal (RO) Control. Introduced by Sabag et al. (2021); Goel and Hassibi (2023), this framework provides a promising approach to address both stochastic and adversarial uncertainties. It defines regret as the performance gap between a causal control policy and a clairvoyant, non-causal policy with perfect knowledge of the system and future disturbances. In the full-information Linear-Quadratic Regulator (LQR) setting, RO controllers minimize the worst-case regret across all bounded energy disturbances (Sabag et al., 2021; Goel and Hassibi, 2023). Furthermore, the infinite-horizon RO controller admits a state-space form, rendering this approach amenable to efficient real-time computation (Sabag et al., 2021).

Extensions of this framework have been explored for the measurement-feedback setting (Goel and Hassibi, 2021a; Hajar et al., 2023b), the dynamic environment setting (Goel and Hassibi, 2021b), safety critical control (Martin et al., 2022; Didier et al., 2022), filtering (Sabag and Hassibi, 2022; Goel and Hassibi, 2023), and distributed control (Martinelli et al., 2023). While these controllers closely track the performance of the non-causal controller in the worst-case disturbance setting, they can, however, become overly conservative when dealing with stochastic disturbances.

Distributionally Robust (DR) Control. This framework, on the other hand, addresses uncertainty in system dynamics and disturbances by considering ambiguity sets, *i.e.*, a set of plausible probability distributions, rather than considering a single distribution as in H_2 or worst-case realization of disturbances as in H_∞ and RO control (Yang, 2020; Kim and Yang, 2021; Hakobyan and Yang, 2022; Taskesen et al., 2023; Aolaritei et al., 2023a,b). This approach seeks to design controllers that perform well across all probability distributions of disturbances within an ambiguity set. The size of the ambiguity set allows one to control the amount of desired robustness against distributional uncertainty so that, unlike H_∞ and RO controllers, the resulting controller is not overly conservative.

While various distributional mismatch measures like total variation (Tzortzis et al., 2014, 2016) and KL divergence (Liu et al., 2023) are considered in DR control, ambiguity sets are commonly chosen as Wasserstein-2 balls around a nominal distribution due to computational tractability (Mohajerin Esfahani and Kuhn, 2018; Gao and Kleywegt, 2022). Therefore, this approach provides a tractable means to bridge the gap between the realms of stochastic and adversarial uncertainties.

1.1. Contributions

In this work, we consider the Wasserstein-2 distributionally robust regret-optimal (DR-RO) control framework introduced by Taha et al. (2023) for the full-information LQR setting and extended by Hajar et al. (2023a) to the partial-observability one. DR-RO control aims to design controllers that minimize the worst-case expected regret across all distributions chosen adversarially within a Wasserstein-2 ambiguity set. We summarize our contributions as follows.

Stabilizing Infinite-Horizon Controller. Rather than the finite-horizon setting prevalent in the DR control literature (Hakobyan and Yang, 2022; Taskesen et al., 2023; Aolaritei et al., 2023b; Taha et al., 2023; Hajar et al., 2023a), we focus on the infinite-horizon DR-RO control in the full-information LQR setting. Thus, we are able to provide long-term stability and robustness guarantees.

Robustness to Arbitrarily Correlated Disturbances. Unlike several prior works which assume time-independence of the disturbances (Yang, 2020; Kim and Yang, 2021; Hakobyan and Yang, 2022; Taskesen et al., 2023; Zhong and Zhu, 2023; Aolaritei et al., 2023a,b), we do not impose such assumptions so that the resulting controllers are robust against time-correlated and non-Gaussian disturbances, thus better capturing distributional robustness.

Computationally Efficient Controller Synthesis. Leveraging a strong duality result, we obtain the exact Karush-Khun-Tucker (KKT) conditions for the worst-case distribution and the optimal causal controller. We show that, although the resulting controller is non-rational, *i.e.*, it does not admit a finite state-space form, it does admit a non-linear finite-dimensional parametric form. We exploit this parametric structure and provide a computationally efficient numerical method to compute the optimal DR-RO controller in the frequency domain via fixed-point iterations. Prior works focus on finite horizon problems (see Taha et al. (2023); Hajar et al. (2023a); Taskesen et al. (2023)) and are hampered by the fact that they require solving a semi-definite program (SDP) whose size scales with the time horizon. This prohibits their applicability when the time horizon is large. Our approach enables efficient implementation of the infinite-horizon DR-RO controller.

A concurrent study by Brouillon et al. (2023) addresses the constrained infinite-horizon DR control problem with time-correlated disturbances, stability guarantees, and reduction to a finite convex program. Unlike our approach, it assumes order T stationarity, formulates a stationary control problem, and uses ambiguity sets centered on nominal empirical distributions, similar to Yang (2020) and Kim and Yang (2021).

2. Preliminaries and Problem Setup

Notations: From hereon, calligraphic letters ($\mathcal{K}$, $\mathcal{M}$, $\mathcal{L}$, etc.) denote operators. $\mathcal{I}$ is the identity operator. Sans serif type letters (x, u, w, etc.) denote infinite sequences. Boldface letters ($\mathbf{K}$, $\mathbf{C}$, $\mathbf{w}$, etc.) denote matrices with finite-horizon. Asterisk $\mathcal{M}^*$ denotes the adjoint of $\mathcal{M}$ and $\succ$ denotes the positive-definite ordering. $\mathscr{P}(\cdot)$ denotes the space of probability measures over a domain. $\mathscr{K}$ stands for (Hardy-2) space of causal block Toeplitz operators. Tr is the normalized trace function over block Toeplitz operators such that $\operatorname{Tr}(\mathcal{I}) = p$, and tr is the trace of matrices. $\|\cdot\|_{\text{op}}$ and $\|\cdot\|_F$ are the operator (H_∞) and Frobenius (H_2) norms for operators, respectively. $\|\cdot\|$ is the Euclidean norm of vectors. $\{\mathcal{M}\}_+$ and $\{\mathcal{M}\}_-$ denote the causal and strictly anti-causal parts of an operator $\mathcal{M}$.

2.1. A Linear Dynamical System

We consider a discrete-time linear time-invariant (LTI) dynamical system expressed in its state-space representation as follows:

$$x_{t+1} = Ax_t + B_u u_t + B_w w_{t+1}, \tag{1}$$

Here, $x_t \in \mathbb{R}^n$ denotes the state, $u_t \in \mathbb{R}^d$ the control input, and $w_t \in \mathbb{R}^p$ the exogenous disturbance process. We assume that (A, B_u) and (A, B_w) are stabilizable.

In the rest of this paper, we adopt an operator-theoretic representation of system dynamics (1). To this end, we denote by $\mathsf{x} \coloneqq \{x_t\}_{t\in\mathbb{Z}}$, $\mathsf{u} \coloneqq \{u_t\}_{t\in\mathbb{Z}}$, and $\mathsf{w} \coloneqq \{w_t\}_{t\in\mathbb{Z}}$ the bi-infinite state, control input and disturbance sequences, respectively. For a finite-horizon index set $\mathbb{I}_T \coloneqq \{-T, -T+1, \dots T-1, T\}$ with $T > 0$, we adopt the notation $\mathbf{x}_T \coloneqq \{x_t\}_{t\in\mathbb{I}_T}$, $\mathbf{u}_T \coloneqq \{u_t\}_{t\in\mathbb{I}_T}$, and $\mathbf{w}_T \coloneqq \{w_t\}_{t\in\mathbb{I}_T}$ to denote the finite-horizon counterparts. Using these definitions, we can represent the infinite-horizon system dynamics (1) equivalently in operator notation as

$$\mathsf{x} = \mathcal{F}\mathsf{u} + \mathcal{G}\mathsf{w}, \tag{2}$$

where $\mathcal{F}$ and $\mathcal{G}$ are bi-infinite strictly causal (*i.e.*, strictly lower triangular) and causal (*i.e.*, lower triangular) time-invariant block Toeplitz operators, respectively, corresponding to the dynamics (1). We use $\mathbf{F}_T$ and $\mathbf{G}_T$ to denote the finite-horizon counterparts of $\mathcal{F}$ and $\mathcal{G}$ for the interval $\mathbb{I}_T$.

Controller. In this paper, we consider linear time-invariant (LTI) disturbance feedback control (DFC) policies $\mathcal{K} : \mathsf{w} \to \mathsf{u}$ in the form

$$\mathsf{u} = \mathcal{K}\mathsf{w}. \tag{3}$$

Here, $\mathcal{K} \in \mathscr{K}$ stands for the controller, a causal and time-invariant block Toeplitz operator mapping past disturbance realizations to control inputs. We define the closed-loop transfer operator as

$$\mathcal{T}_{\mathcal{K}} : \mathsf{w} \mapsto \begin{bmatrix} \mathsf{x} \\ \mathsf{u} \end{bmatrix} := \begin{bmatrix} \mathcal{F}\mathcal{K} + \mathcal{G} \\ \mathcal{K} \end{bmatrix} \mathsf{w}, \tag{4}$$

which maps the disturbances to the regulated output and the control input of the system (1) under a fixed control policy $\mathcal{K}$. We similarly adopt the notations $\mathbf{K}_T$ and $\mathbf{T}_{\mathbf{K},T}$ to respectively denote the finite-horizon controller and closed-loop transfer matrix for the interval $\mathbb{I}_T$.

Cost. We assume that the cumulative cost incurred by a control policy $\mathbf{K}_T$ within the time interval $\mathbb{I}_T$ for the disturbance realization $\mathbf{w}_T$ is given by:

$$\mathrm{cost}_T(\mathbf{K}_T, \mathbf{w}_T) := \sum\nolimits_{t \in \mathbb{I}_T} x_t^\intercal Q x_t + u_t^\intercal R u_t, \tag{5}$$

where $Q, R \succ 0$. By redefining $x_t \leftarrow Q^{\frac{1}{2}} x_t$ and $u_t \leftarrow R^{\frac{1}{2}} u_t$, we can rewrite the cumulative cost (5) in terms of the closed-loop transfer operator (4) as $\mathrm{cost}_T(\mathbf{K}_T, \mathbf{w}_T) = \mathbf{w}_T^* \mathbf{T}_{\mathbf{K},T}^* \mathbf{T}_{\mathbf{K},T} \mathbf{w}_T$.

2.2. The Regret Measure

In the full-information setting, there exists a unique optimal non-causal policy, $\mathcal{K}_\circ$, defined as

$$\mathcal{K}_\circ := -(\mathcal{I} + \mathcal{F}^*\mathcal{F})^{-1}\mathcal{F}^*\mathcal{G}, \tag{6}$$

that minimizes the infinite-horizon cost, $\lim_{T\to\infty} \frac{1}{|\mathbb{I}_T|} \mathrm{cost}_T(\mathbf{K}_T, \mathbf{w}_T)$, and a unique optimal non-causal policy, $\mathbf{K}_{\circ,T}$, defined as

$$\mathbf{K}_{\circ,T} := -(I + \mathbf{F}_T^*\mathbf{F}_T)^{-1}\mathbf{F}_T^*\mathbf{G}_T, \tag{7}$$

that minimizes the finite-horizon cost (5) for all bounded power disturbance realizations (Hassibi et al., 1999; Sabag et al., 2021). Since a non-causal controller is physically unrealizable, we aim to design a causal control policy that performs as best as the optimal non-causal policy $\mathcal{K}_\circ$, which has access to the entire disturbance trajectory at the outset. To quantify the disparity in accumulated costs between a causal controller and the optimal non-causal controller , $\mathbf{K}_{\circ,T}$, we define the regret as

$$\begin{aligned} \mathrm{Regret}_T(\mathbf{K}_T, \mathbf{w}_T) &:= \mathrm{cost}_T(\mathbf{K}_T, \mathbf{w}_T) - \mathrm{cost}_T(\mathbf{K}_{\circ,T}, \mathbf{w}_T) \\ &= \mathbf{w}_T^* \left(\mathbf{T}_{\mathbf{K},T}^* \mathbf{T}_{\mathbf{K},T} - \mathbf{T}_{\mathbf{K}_\circ,T}^* \mathbf{T}_{\mathbf{K}_\circ,T} \right) \mathbf{w}_T. \end{aligned} \tag{8}$$

Put differently, regret measures the excess cost that a causal controller suffers as a result of not foreseeing the realization of future disturbances. In the regret-optimal control framework, the objective is to design a causal controller minimizing the worst-case regret among all bounded energy disturbances, formulated as follows:

Problem 1 (Regret-Optimal Control (Sabag et al., 2021)) *Find a causal control policy, $\mathcal{K}$, that minimizes the time-averaged worst-case regret as $T \to \infty$, i.e.,*

$$\inf_{\mathcal{K}\in\mathscr{K}} \lim_{T\to\infty} \frac{1}{|\mathbb{I}_T|} \sup_{\|\mathbf{w}_T\|^2\leq 1} \mathrm{Regret}_T(\mathbf{K}_T, \mathbf{w}_T). \tag{9}$$

By leveraging the time-invariance of the dynamics (1) and the controller (3), problem (9) can be reframed as $\inf_{\mathcal{K}\in\mathscr{K}} \|\mathcal{T}_{\mathcal{K}}^{*}\mathcal{T}_{\mathcal{K}} - \mathcal{T}_{\mathcal{K}_\circ}^{*}\mathcal{T}_{\mathcal{K}_\circ}\|_{\mathrm{op}}$, which can be solved by reducing it to a Nehari problem (Sabag et al., 2021). The resulting controller closely mirrors the non-causal controller's performance under worst-case disturbance but may be overly conservative in stochastic disturbance scenarios.

2.3. Distributionally Robust Regret-Optimal Control

This paper explores the distributionally robust regret-optimal control approach, aiming to design a causal controller minimizing the worst-case expected regret within an *ambiguity set* of probability distributions of disturbances. The ambiguity set during the time interval $\mathbb{I}_T$ is described as a Wasserstein-2 ball of radius $r\sqrt{|\mathbb{I}_T|}$ centered around a nominal probability distribution $\mathbb{P}_\circ \in \mathscr{P}(\mathbb{R}^{p|\mathbb{I}_T|})$, *i.e.*,

$$\mathfrak{W}_T := \left\{\mathbb{P} \in \mathscr{P}(\mathbb{R}^{p|\mathbb{I}_T|}) \mid \mathrm{W}_2(\mathbb{P}, \mathbb{P}_\circ) \leq r\sqrt{|\mathbb{I}_T|}\right\}. \tag{10}$$

Here, the Wasserstein-2 distance is defined as

$$\mathrm{W}_2(\mathbb{P}_1, \mathbb{P}_2)^2 := \inf_{\pi\in\Pi(\mathbb{P}_1,\mathbb{P}_2)} \int \|w_1 - w_2\|^2 \, \pi(dw_1, dw_2), \tag{11}$$

where the set $\Pi(\mathbb{P}_1, \mathbb{P}_2)$ consists of all joint distributions with marginals $\mathbb{P}_1$ and $\mathbb{P}_2$ (Villani, 2009; Santambrogio, 2015). The growth rate of the radius with the horizon is justified, as the total squared energy of a random vector of iid variables scales linearly with its dimension.

In Taha et al. (2023); Hajar et al. (2023a), the worst-case expected regret incurred by a causal controller $\mathbf{K}_T$ during the time interval $\mathbb{I}_T$ is given by

$$\sup_{\mathbb{P}\in\mathfrak{W}_T} \mathbb{E}_{\mathbb{P}}\left[\mathrm{Regret}_T(\mathbf{K}_T, \mathbf{w}_T)\right] \tag{12}$$

where $\mathbb{E}_{\mathbb{P}}$ denotes the expectation under the distribution $\mathbb{P}$ such that $\mathbf{w}_T \sim \mathbb{P}$. Using this formulation in the finite-horizon, we define the worst-case expected regret in infinite-horizon as follows:

Definition 2 (Worst-Case Expected Regret) *The time-averaged worst-case expected regret suffered by a causal control policy, $\mathcal{K} \in \mathscr{K}$, over an infinite horizon is given by*

$$R(\mathcal{K}, \mathsf{w}) := \lim_{T\to\infty} \frac{1}{|\mathbb{I}_T|} \sup_{\mathbb{P}\in\mathfrak{W}_T} \mathbb{E}_{\mathbb{P}}\left[\mathrm{Regret}_T(\mathbf{K}_T, \mathbf{w}_T)\right]. \tag{13}$$

Using this definition, we formally cast the infinite-horizon DR-RO control problem as follows:

Problem 3 (Distributionally Robust Regret-Optimal (DR-RO) Control) *Find a causal control policy, $\mathcal{K}$, that minimizes the time-averaged worst-case expected regret (13) as $T \to \infty$, i.e.,*

$$\inf_{\mathcal{K}\in\mathscr{K}} \lim_{T\to\infty} \frac{1}{|\mathbb{I}_T|} \sup_{\mathbb{P}\in\mathfrak{W}_T} \mathbb{E}_{\mathbb{P}}\left[\mathrm{Regret}_T(\mathbf{K}_T, \mathbf{w}_T)\right]. \tag{14}$$

In Section 3, we provide an equivalent formulation to Problem 3 in terms of the closed-loop transfer operator by appealing to strong duality.

3. Main Theoretical Results

In this section, we present our main theorems. In Theorem 5, we first establish a strong duality reformulation for the infinite-horizon worst-case expected regret in operator form. Exploiting the dual formulation, we reduce solving Problem 3 into solving a suboptimal Problem 6. In Theorem 9, we present the suboptimal controller and argue that it is stabilizing. Due to space constraints, we defer the proofs of our theorems to the extended version (Kargin et al., 2023).

3.1. Reduction to a suboptimal Problem via Strong Duality

In the finite-horizon DR-RO problem, Theorem 2 in Taha et al. (2023) establishes an equivalent formulation for the worst-case expected regret (12) as a single-parameter optimization problem via strong duality. In Theorem 5, we establish an analogous dual reformulation for the infinite-horizon worst-case expected regret (13) as a single-parameter search problem. For ease of notation and clarity of results, we make the following assumption.

Assumption 4 *For any finite-horizon interval $\mathbb{I}_T$, the nominal distribution, $\mathbf{w}_{\circ,T} \sim \mathbb{P}_\circ$, is absolutely continuous wrt the Lebesgue measure with $\mathbb{E}_{\mathbb{P}_\circ}[\mathbf{w}_{\circ,T}\mathbf{w}_{\circ,T}^*] = I$.*

Theorem 5 (Strong Duality for (13)) *Let $\mathcal{C}_\mathcal{K} := \mathcal{T}_\mathcal{K}^*\mathcal{T}_\mathcal{K} - \mathcal{T}_{\mathcal{K}_\circ}^*\mathcal{T}_{\mathcal{K}_\circ}$ and $\mathcal{K} \in \mathscr{K}$ be a a causal and time-invariant policy. Under assumption 4, the infinite-horizon worst-case expected regret (13) incurred by $\mathcal{K}$ attains a finite value and is equivalent to the following dual problem:*

$$\inf_{\gamma \geq 0} \gamma(r^2 - \operatorname{Tr}\mathcal{I}) + \gamma \operatorname{Tr}(\mathcal{I} - \gamma^{-1}\mathcal{C}_\mathcal{K})^{-1} \quad \text{s.t.} \quad \gamma\mathcal{I} \succ \mathcal{C}_\mathcal{K}. \tag{15}$$

Furthermore, the worst-case disturbance, $\mathsf{w}_\star$, can be identified from the nominal disturbance, $\mathsf{w}_\circ$, as $\mathsf{w}_\star = (\mathcal{I} - \gamma_\star^{-1}\mathcal{C}_\mathcal{K})^{-1}\mathsf{w}_\circ$ where $\gamma_\star$ is the optimal solution to (15), which satisfies the following:

$$\operatorname{Tr}((\mathcal{I} - \gamma_\star^{-1}\mathcal{C}_\mathcal{K})^{-1} - \mathcal{I})^2 = r^2. \tag{16}$$

Using the dual problem (15), we can rewrite the DR-RO problem (14) as

$$\inf_{\mathcal{K} \in \mathscr{K}} \inf_{\gamma \geq 0} \gamma(r^2 - \operatorname{Tr}\mathcal{I}) + \gamma \operatorname{Tr}(\mathcal{I} - \gamma^{-1}\mathcal{C}_\mathcal{K})^{-1} \quad \text{s.t.} \quad \gamma\mathcal{I} \succ \mathcal{C}_\mathcal{K}. \tag{17}$$

By exchanging the infima and fixing γ, we can first find a suboptimal solution $\mathcal{K}_\gamma$ to (17). Using the suboptimal solutions $\mathcal{K}_\gamma$, we can search for the optimal $\gamma_\star$ by solving equation (16). Therefore, we restrict our attention to the suboptimal DR-RO problem stated below.

Problem 6 (Suboptimal DR-RO Control) *For a fixed $\gamma > \gamma_{RO} := \inf_{\mathcal{K} \in \mathscr{K}} \|\mathcal{C}_\mathcal{K}\|_{\text{op}}$, find a causal control policy, $\mathcal{K}_\gamma$, that minimizes the suboptimal objective function (15) i.e.,*

$$\inf_{\mathcal{K} \in \mathscr{K}} \operatorname{Tr}(\mathcal{I} - \gamma^{-1}\mathcal{C}_\mathcal{K})^{-1} \quad \text{s.t.} \quad \gamma\mathcal{I} \succ \mathcal{C}_\mathcal{K}. \tag{18}$$

Remark 7 *Note that as $r \to \infty$, the optimal $\gamma_\star$ approaches the lower bound $\|\mathcal{C}_\mathcal{K}\|_{\text{op}}$, i.e., the worst-case expected regret (13) reaches to the worst-case regret as in Sabag et al. (2021) and the optimal DR-RO controller recovers the optimal RO controller. The optimal regret, γ_{RO}, acts as a global lower bound on γ. Conversely, as $r \to 0$, $\gamma_\star \to \infty$, leading the worst-case expected regret (13) to nominal expected regret and the optimal DR-RO controller recovers the optimal H_2 controller. Adjusting r enables the DR-RO controller to interpolate between the RO and H_2 controllers.*

3.2. Solution for the Suboptimal Problem 6

In its present form, Problem 6 is challenging since the controller appears both in an operator inverse, as well as in the constraint $\gamma\mathcal{I}\succ\mathcal{C}_{\mathcal{K}}$. An alternative formulation via Fenchel duality follows.

Lemma 8 (Duality for the Suboptimal Problem 6) *Let $\gamma > \gamma_{RO}$ be fixed and let assumption 4 hold. The γ-optimal DR-RO control Problem 6 is equivalent to the following dual problem*

$$\sup_{\mathcal{M}\succ 0}\inf_{\mathcal{K}\in\mathscr{K}} -\operatorname{Tr}(\mathcal{M}) + 2\operatorname{Tr}(\sqrt{\mathcal{M}}) + \gamma^{-1}\operatorname{Tr}(\mathcal{C}_{\mathcal{K}}\mathcal{M}). \tag{19}$$

The concave-convex problem (19) is more manageable since the inner minimization wrt $\mathcal{K}\in\mathscr{K}$ can be solved via the Wiener-Hopf technique (Kailath et al., 2000). Introducing the spectral factorization $\Delta^*\Delta = \mathcal{I} + \mathcal{F}^*\mathcal{F}$ with causal and causally invertible Δ, we present our second main result in Theorem 9, the solution to the suboptimal DR-RO Problem 6.

Theorem 9 (Suboptimal DR-RO Controller) *The γ-suboptimal DR-RO controller $\mathcal{K}_\gamma$ of DR-RO Problem 6 coincides with the saddle point $(\mathcal{K}_\gamma, \mathcal{M}_\gamma)$ of the dual problem (19). Furthermore, let $\mathcal{L}_\gamma$ denote the causal and causally invertible spectral factor of $\mathcal{M}_\gamma$ such that $\mathcal{M}_\gamma = \mathcal{L}_\gamma\mathcal{L}_\gamma^*$. Then $(\mathcal{K}_\gamma, \mathcal{M}_\gamma)$ uniquely satisfies the following set of equations:*

$$\mathcal{K}_\gamma = \Delta^{-1}\left\{\Delta\mathcal{K}_\circ\mathcal{L}_\gamma\right\}_+\mathcal{L}_\gamma^{-1}, \quad \text{and} \quad \mathcal{L}_\gamma^*\mathcal{L}_\gamma = \frac{1}{4}\left(\mathcal{I}+\sqrt{\mathcal{I}+4\gamma^{-1}\{\Delta\mathcal{K}_\circ\mathcal{L}_\gamma\}_-^*\{\Delta\mathcal{K}_\circ\mathcal{L}_\gamma\}_-}\right)^2. \tag{20}$$

The proof of Theorem 9, given in the extended version (Kargin et al., 2023), is built upon the KKT conditions for (19) and the Wiener-Hopf technique (Kailath et al., 2000). Note that for $\gamma > \gamma_{\text{RO}}$, the worst-case expected regret (13) is finite, which allows us to present Corollary 10.

Corollary 10 *For any fixed $\gamma > \gamma_{RO}$, the suboptimal controller $\mathcal{K}_\gamma$ stabilizes the system dynamics.*

4. An Algorithm for Irrational Controller Synthesis

In section 4.1, we first show that, in the frequency domain, the KKT conditions (20) are uniquely determined by a finite-dimensional parameter, $\overline{B}_\gamma$. This allows us to argue that the suboptimal controller is *irrational*, and thus does not admit a finite-dimensional state-space realization. In section 4.2, for any fixed γ, we propose a fixed-point iteration to find $\overline{B}_\gamma$ and thereby to compute the suboptimal controller, $K_\gamma(\mathrm{e}^{j\omega})$. The optimal $\gamma_\star$, and thus the optimal DR-RO controller $K_{\gamma_\star}(\mathrm{e}^{j\omega})$, can be found by using the bisection method on equation (16).

4.1. Finite-Dimensional Parametrization of the Suboptimal Controller

Defining $S_{\gamma,-}(\mathrm{e}^{j\omega}) \coloneqq \{\Delta K_\circ L_\gamma\}_-(\mathrm{e}^{j\omega})$, and $N_\gamma(\mathrm{e}^{j\omega}) \coloneqq L_\gamma(\mathrm{e}^{j\omega})^* L_\gamma(\mathrm{e}^{j\omega})$, and using the identity $\{\mathcal{X}\}_+ = \mathcal{X} - \{\mathcal{X}\}_-$, we restate the KKT equations (20) in the frequency domain as follows:

$$K_\gamma(\mathrm{e}^{j\omega}) = K_\circ(\mathrm{e}^{j\omega}) - \Delta^{-1}(\mathrm{e}^{j\omega})S_{\gamma,-}(\mathrm{e}^{j\omega})L_\gamma^{-1}(\mathrm{e}^{j\omega}), \tag{21}$$

$$N_\gamma(\mathrm{e}^{j\omega}) = \frac{1}{4}\left(I+\sqrt{I+4\gamma^{-1}S_{\gamma,-}^*(\mathrm{e}^{j\omega})S_{\gamma,-}(\mathrm{e}^{j\omega})}\right)^2. \tag{22}$$

Furthermore, we define the LQR controller $K_{\text{lqr}} \coloneqq (R + B_u^*PB_u)^{-1}B_u^*PA$ and the closed-loop matrix $A_K \coloneqq A - B_uK_{\text{lqr}}$ where $P\succ 0$ is the unique stabilizing solution to the LQR Riccati equation $P = Q + A^*PA - A^*PB_u(R + B_u^*PB_u)^{-1}B_u^*PA$. In Lemma 11, we show that the strictly anticausal transfer function $S_{\gamma,-}(\mathrm{e}^{j\omega})$ admits a finite-dimensional state-space representation.

Lemma 11 *Let* $\overline{A} \coloneqq A_K^*$, $\overline{D} \coloneqq A_K^* P B_w$, *and* $\overline{C} \coloneqq -(R + B_u^* P B_u)^{-*/2} B_u^*$. *We have that*

$$S_{\gamma,-}(\mathrm{e}^{j\omega}) = \overline{C}(\mathrm{e}^{-j\omega} I - \overline{A})^{-1} \overline{B}_\gamma, \quad \textit{where} \quad \overline{B}_\gamma \coloneqq \frac{1}{2\pi}\int_0^{2\pi} (I - \mathrm{e}^{j\omega}\overline{A})^{-1}\overline{D} L_\gamma(\mathrm{e}^{j\omega}) d\omega. \tag{23}$$

Notice that the rhs of (22) for $N_\gamma(\mathrm{e}^{j\omega})$ involves the square-root of rational term $S_{\gamma,-}(\mathrm{e}^{j\omega})^* S_{\gamma,-}(\mathrm{e}^{j\omega})$. In general, square root does not preserve rationality. We thus get Corollary 12.

Corollary 12 *For any fixed* $\gamma \in (\gamma_{RO}, \infty)$, $N_\gamma(\mathrm{e}^{j\omega})$ *and the suboptimal DR-RO controller,* $K_\gamma(\mathrm{e}^{j\omega})$, *are irrational. Thus,* $K_\gamma(\mathrm{e}^{j\omega})$ *does not admit a finite-dimensional state-space form.*

Even though $K_\gamma(\mathrm{e}^{j\omega})$ does not admit a finite-dimensional state-space form, Lemma 11 suggests a finite-dimensional *parametrization* of $N_\gamma(\mathrm{e}^{j\omega})$ through $\overline{B}_\gamma$. Theorem 13 establishes that $\overline{B}_\gamma$ uniquely determines $N_\gamma(\mathrm{e}^{j\omega})$, and thus the suboptimal controller $K_\gamma(\mathrm{e}^{j\omega})$.

Theorem 13 (Fixed-Point Solution) *Fix* $\gamma > \gamma_{RO}$ *and consider the following set of mappings:*

$$F_{1,\gamma} : \overline{B} \mapsto \frac{1}{4}\left(I + \sqrt{I + 4\gamma^{-1}\overline{B}^\intercal (\mathrm{e}^{-j\omega} I - \overline{A})^{-*}\overline{C}^\intercal \overline{C}(\mathrm{e}^{-j\omega} I - \overline{A})^{-1}\overline{B}}\right)^2 \tag{24}$$

$$F_2 : N(\mathrm{e}^{j\omega}) \mapsto L(\mathrm{e}^{j\omega}), \quad F_3 : L(\mathrm{e}^{j\omega}) \mapsto \overline{B} \coloneqq \frac{1}{2\pi}\int_0^{2\pi} (I - \mathrm{e}^{j\omega}\overline{A})^{-1}\overline{D} L(\mathrm{e}^{j\omega}) d\omega. \tag{25}$$

where F_2 *returns a unique spectral factor of* $N(\mathrm{e}^{j\omega}) > 0$. *The composition* $F_3 \circ F_2 \circ F_{1,\gamma} : \overline{B} \mapsto \overline{B}$ *admits a unique fixed-point* $\overline{B}_\gamma$, *and* $N_\gamma(\mathrm{e}^{j\omega}) \coloneqq F_{1,\gamma}(\overline{B}_\gamma)$ *satisfies the KKT conditions (20).*

4.2. Algorithm Description

Motivated by Theorem 13, we introduce Algorithm 1 to compute the suboptimal controller $K_\gamma(\mathrm{e}^{j\omega})$ at uniformly sampled points on the unit circle. We start the algorithm with an initial estimate of the parameter $\overline{B}_\gamma^{(0)}$. At the n^{th} iteration, we construct the functions $S_{\gamma,-}^{(n)}(\mathrm{e}^{j\omega})$ and $N_\gamma^{(n)}(\mathrm{e}^{j\omega})$ from $\overline{B}_\gamma^{(n)}$ using the mappings in (24), (25). We subsequently compute the the spectral factor $L_\gamma^{(n)}(\mathrm{e}^{j\omega})$ at uniformly sampled points on the unit circle,from which we compute the next iterate $\overline{B}_\gamma^{(n+1)}$ via numerical integration of F_4. Upon convergence upto a tolerance, we ascertain the *suboptimal* controller $K_\gamma(\mathrm{e}^{j\omega})$ for a fixed $\gamma > \gamma_{\text{RO}}$ at every sampled frequency point using (21).

Spectral Factorization: Since there is no general closed-form formula for spectral factorization of irrational spectra, we can compute $L_\gamma(\mathrm{e}^{j\omega})$ only at finitely many frequencies. Focusing on single-input systems, we employ a discrete Fourier transform (DFT) based factorization method by Rino (1970), highlighted in Algorithm 2 in Kargin et al. (2023), to approximate $L_\gamma(\mathrm{e}^{j\omega})$ at uniformly sampled points on the unit circle. This method, tailored for *scalar* irrational functions, proves efficient as the associated error term, featuring as a multiplicative phase factor, rapidly diminishes with increasing number of samples.

5. Experimental Results

In this section, we showcase the applicability of the DR-RO controller, and its performance, compared to H_2, H_∞, and regret-optimal controllers. We focus our investigation on a set of 4 diverse systems from Leibfritz and Lipinski (2003), and, in particular, use the chemical reactor system, [REA4], as our main benchmark. The system has 8 states and is SISO. We perform all experiments using

Algorithm 1: *Distributionally Robust Regret-Optimal Control via Fixed-Point Iteration*

Data: $\gamma > \gamma_{\text{RO}}$, initial $\overline{B}_{\gamma}^{(0)}$, and 2^k equally spaced values of $\omega \in [0, 2\pi)$

for $n \geq 0$ **do**

  Set $S_{\gamma,-}^{(n)}(\mathrm{e}^{j\omega}) \leftarrow \overline{C}(\mathrm{e}^{j\omega}I - \overline{A})^{-1}\overline{B}_{\gamma}^{(n)}$

  Set $N_{\gamma}^{(n)}(\mathrm{e}^{j\omega}) \leftarrow \dfrac{1}{4}\left(I + \sqrt{I + 4\gamma^{-1}S_{\gamma,-}^{(n)}(\mathrm{e}^{j\omega})^{*}S_{\gamma,-}^{(n)}(\mathrm{e}^{j\omega})}\right)^2$

  Compute $L_{\gamma}^{(n)}(\mathrm{e}^{j\omega}) \leftarrow \text{SpectralFactorization}(N_{\gamma}^{(n)}(\mathrm{e}^{j\omega}))$

  Compute $\overline{B}_{\gamma}^{(n+1)} \leftarrow \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}(I - \mathrm{e}^{j\omega}\overline{A})^{-1}\overline{D}L_{\gamma}^{(n)}(\mathrm{e}^{j\omega})d\omega$

  if $\|\overline{B}_{\gamma}^{(n+1)} - \overline{B}_{\gamma}^{(n)}\| < tol$ **then**

    Compute $K_{\gamma}(\mathrm{e}^{j\omega}) \leftarrow K_{\circ}(\mathrm{e}^{j\omega}) - \Delta^{-1}(\mathrm{e}^{j\omega})S_{\gamma,-}^{(n)}(\mathrm{e}^{j\omega})L_{\gamma}^{(n)}(\mathrm{e}^{j\omega})^{-1}$

    break

  end

end

MATLAB, on a Macbook Air with Apple M1 processor and 8 GB of RAM. We specify the nominal distribution as Gaussian, with zero mean and identity covariance. We investigate various values for the radius r, and for each solve the optimization problem using the algorithm outlined in section 4.

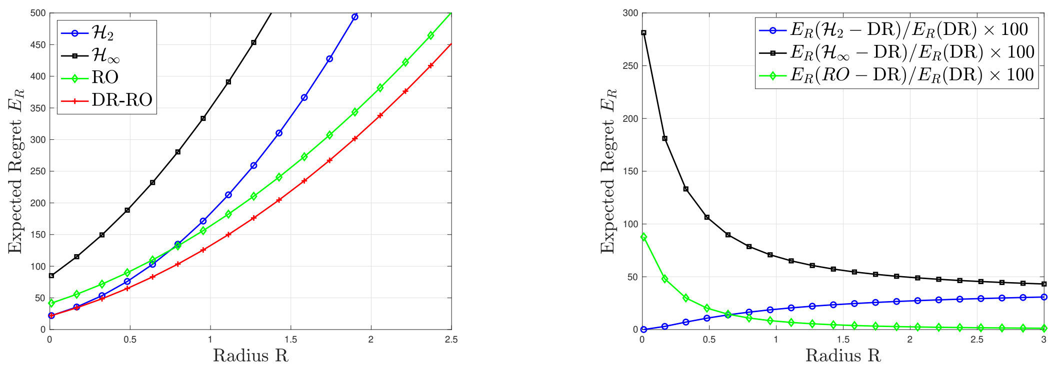
For the system [REA4], a comparative analysis of worst-case expected regret cost as defined in (2) is conducted against the H_2, H_∞ Hassibi et al. (1999), and RO Sabag et al. (2021) controllers, considering the unique worst-case distribution associated with each controller. The results are depicted in Figures 1(*a*) and 1(*b*). We redo the analysis considering 3 other systems (described in Leibfritz and Lipinski (2003)), and we show the results in Table 1 in the extended version Kargin et al. (2023). Another performance metric considered is the operator norm of T_K minimized by the H_∞ controller, which is expressed, in the frequency domain as: $\|\mathcal{T}_{\mathcal{K}}\|_{\text{op}}^2 = \max_{0\leq\omega\leq 2\pi} \sigma_{\max}(T_K(\mathrm{e}^{j\omega})^{*}T_K(\mathrm{e}^{j\omega}))$. This metric is visualized across all frequencies in Figure 2.

Figures 1(*a*), and 1(*b*) emphasize the robust performance of the DR controller in minimizing worst-case expected regret under worst-case disturbance conditions for any given parameter r. Notably, the DR controller exhibits a versatile nature, closely mirroring the H_2 controller for smaller r while converging towards the behavior of the RO controller for larger r. This dual capability underscores its adaptability to different robustness requirements, and aligns with the theoretical insights outlined in Remark 7. Moreover, in Figure 2, the performance of the DR controller exhibits an interpolation between the H_2 and RO controllers *across all frequencies*.

Finally, we note that Algorithm 1, coupled with the bisection technique, exhibits notable efficiency; the execution time is 5.8 seconds for [REA4] system, for $r = 0.79$. This highlights the significance of our approach compared to other DR control methods that rely on an SDP that scales with the time-horizon and number of states (e.g., Taha et al. (2023) and Hajar et al. (2023a) could only address systems with smaller dimensions and a time-horizon of only 10 steps).

6. Conclusion and Future Works

We studied DR-RO control for discrete-time linear dynamical systems over an infinite horizon. Focusing on regret as a measure of performance introduces a nuanced perspective, while the incorporation of uncertainties within a Wasserstein-2 ambiguity set provides a robust framework for handling unpredictable disturbances. The infinite horizon setting guarantees stability and robustness, and aligns



(a) (b)

Figure 1: (a) The worst-case expected regret cost of each controller for different values of r, for system [REA4]. (b) The percentage difference in the worst case regret relative to the DR-RO controller. (a) and (b) show that DR-RO minimizes the cost at all $r's$, and for small (large) r, the cost of DR-RO controller matches that of $\mathcal{H}_2$ (RO). The cost of the DR controller is less than that of H_2 and RO by 14.5%, and of H_∞ by 89.7% for $r = 0.639$.

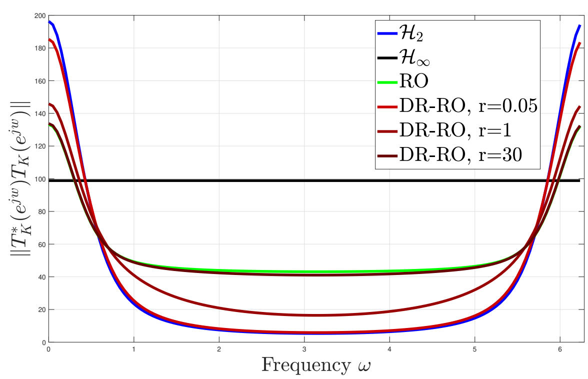


Figure 2: The operator norm, $\|T_K^*(e^{j\omega})T_K(e^{j\omega})\|$, of each controller at different frequency values, for system [REA4]. The cost of the DR-RO controller interpolates between H_2 and RO according to the value of r, across all frequencies. For a small (large) r, DR matches H_2 (RO) *across all frequencies*.

the framework with real-world demands since finite-horizon methods are hampered by ill-scaled SDPs. A key departure from prior research is our deliberate consideration of dependencies among disturbances over time. Thus, this approach better captures the essence of distributional robustness. Even though the optimal controller is irrational, we introduce a computationally efficient numerical method based on fixed-point iterations to find the controller in the frequency domain. Validation through numerical experiments demonstrates the effectiveness of our framework. Looking forward, avenues for future research include finding good low-dimensional rational approximations for the controller, providing convergence guarantees for the fixed point method, extending the algorithm to MIMO systems by exploring irrational matrix spectral factorization (see Nurdin (2005); Ephremidze (2010)), and extending the current framework to the partially observable case.

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