

# Learning Robust Policies for Uncertain Parametric Markov Decision Processes

**Luke Rickard**

**Alessandro Abate**

**Kostas Margellos**

*Department of Engineering Science, University of Oxford, Oxford, UK*

LUKE.RICKARD@ENG.OX.AC.UK

ALESSANDRO.ABATE@CS.OX.AC.UK

KOSTAS.MARGELLOS@ENG.OX.AC.UK

**Editors:** A. Abate, M. Cannon, K. Margellos, A. Papachristodoulou

## Abstract

Synthesising verifiably correct controllers for dynamical systems is crucial for safety-critical problems. To achieve this, it is important to account for uncertainty in a robust manner, while at the same time it is often of interest to avoid being overly conservative with the view of achieving a better cost. We propose a method for verifiably safe policy synthesis for a class of finite state models, under the presence of structural uncertainty. In particular, we consider uncertain parametric Markov decision processes (upMDPs), a special class of Markov decision processes, with parameterised transition functions, where such parameters are drawn from a (potentially) unknown distribution. Our framework leverages recent advancements in the so-called scenario approach theory, where we represent the uncertainty by means of scenarios, and provide guarantees on synthesised policies satisfying probabilistic computation tree logic (PCTL) formulae. We consider several common benchmarks/problems and compare our work to recent developments for verifying upMDPs.

**Keywords:** Markov decision processes; Robust optimization; Verification; Scenario approach.

## 1. Introduction

Verifying the safety of complex dynamical systems is an important challenge (Knight, 2002), with applications including unmanned aerial vehicles (UAVs) (Yu and Dimarogonas, 2021) and robotics (Brunke et al., 2022). Ensuring a safety property may require verifying satisfaction of complex and rich formal specifications in the process of uncertainty arising from, for example, inaccurate modelling or process noise. A vital aspect of verification lies in finding abstractions that encompass this uncertainty, whilst accurately modelling system dynamics to aid optimal control of such systems.

In this paper we do not consider how such an abstraction may be generated and focus solely on learning a good controller/policy. There are a number of techniques for abstracting dynamical systems, such as the widely celebrated counter-example guided abstraction/refinement approach (Clarke et al., 2003). Alternatively, techniques make use of the so-called scenario approach in order to develop their models (Badings et al., 2023; Rickard et al., 2023b).

One useful abstract model is that of parametric Markov decision processes (pMDPs) (Daws, 2004a; Hahn et al., 2011b; Junges et al., 2019), and particularly their probabilistic counterpart uncertain pMDPs (upMDPs) (Badings et al., 2022; Scheftelowitsch et al., 2017). Parametric MDPs extend MDPs by parameterising their transition function; any choice of parameters induces a standard MDP. Hence, a pMDP represents a family of MDPs, differing only in their transition function. By drawing these parameters from a (possibly unknown) distribution, we define an uncertain parametric MDP. The parameterisation of the model allows for the introduction of some structure, thus

avoiding overly conservative models. Here, we make no assumptions on the structure of the parameterisation, and could equivalently have a direct distribution over MDPs within an uncertain set, hence arriving at a problem closely related to that of robust MDPs (Iyengar, 2005; Nilim and Ghaoui, 2005; Wiesemann et al., 2013)

A useful tool for expressing specifications on (up)MDPs lies in temporal logic (Platzer, 2012), a rich language for specifying behaviours of a system (Belta et al., 2017). One particular language of interest is that of Probabilistic Computation Tree Logic (PCTL (Hansson and Jonsson, 1994)), an extension of Computation Tree Logic which allows for probabilistic quantification of properties. This language can be used to describe probabilistic specifications on system behaviour, with these probabilities allowing for uncertainty. Often, it is of interest to learn a policy which ensures satisfaction of the specification (Hahn et al., 2011a), even under different realisations of the uncertainty.

One common approach for verifying pMDPs is to solve the so-called *parameter synthesis* problem, finding parameter sets that satisfy the specification. Typically, only a single set of parameters is of interest (Cubuktepe et al., 2022; Daws, 2004b; Meedeniya et al., 2014). Instead, we wish to synthesise policies that are probabilistically robust to uncertainty over the entire parameter space.

In this paper, we investigate the following problem. Given a upMDP, with a possibly unknown distribution over parameters, we aim at synthesising a policy, accompanied by a probabilistic certificate, such that, for an MDP defined with a randomly drawn parameter set, the probability that the policy satisfies a given specification on that MDP is guaranteed by the computed certificate. To solve this problem, we capitalize on advancements from the so-called *scenario approach* literature (Calafiore and Campi, 2006; Campi et al., 2009; Campi and Garatti, 2018; Garatti and Campi, 2022). We sample a finite number of parameter sets, and compute a policy that is robust to any sample within that set. Then, by leveraging results from Garatti and Campi (2022) we provide probably approximately correct (PAC) guarantees that the policy will be robust to a newly sampled parameter set. Importantly, these techniques allow us to provide guarantees based only on finite samples, without knowledge of the underlying distribution, and with no assumptions on the shape of this distribution or the geometry of its support set.

Since we are optimising a policy for multiple MDPs, our work is similar to developments on multiple environment MDPs (MEMDPs (Raskin and Sankur, 2014; van der Vegt et al., 2023)). These techniques consider finding policies which are optimal for a finite set of MDPs. In contrast to our approach, these methods assume to have complete knowledge of possible environments, whereas we wish to be robust to new, unseen, environments. Previous work in this area typically required overly conservative assumptions on the parameterisation, restricting to models where the uncertainty is independent across different states (Puggelli et al., 2013), or across different state-action pairs (Kozine and Utkin, 2002). In the robust MDP literature, such assumptions are referred to as  $s$ -rectangular or  $(s, a)$ -rectangular ambiguity sets (Wiesemann et al., 2013). Some recent approaches (Badings et al., 2022) have relaxed this assumption, but required full knowledge of the parameter sets at runtime in order to apply a policy.

Our work makes no assumption on the parameterisation of the upMDP, and learns a single policy which may be applied at runtime with no explicit parameter knowledge, whilst still offering PAC-type guarantees. Our main contributions can be summarised as follows

1. We formally discuss different policy classes for an MDP, and investigate their associated probabilistic guarantees. This policy classification is interesting per se as it clarifies subtle differences among them.

2. We propose a new gradient-based algorithm to learn an optimal *robust* policy for a given specification in an upMDP; we demonstrate the benefits of this scheme over more traditional solution paradigms via extensive numerical investigation and set the basis for a theoretical convergence analysis in future work.
3. We use recent advancements in scenario approach theory to accompany our learned policy with non-trivial guarantees on the probability that newly sampled MDPs meet certain specifications.

## 2. Background

### 2.1. Markov Decision Processes

A Markov Decision Process (MDP) is a tuple  $\mathcal{M} = (\mathcal{S}, \mathcal{A}, T, \rho, \gamma)$ . Where  $\mathcal{S} = \{s_0, \dots, s_N\}$  is a finite set of states, with initial distribution  $\rho : \mathcal{S} \rightarrow (0, 1)$ ,  $\mathcal{A} = \{a_0, \dots, a_M\}$  a finite set of actions,  $T : \mathcal{S} \times \mathcal{A} \rightarrow \text{Dist}(\mathcal{S})$  is a probabilistic transition function, with  $\text{Dist}(\mathcal{S})$  the set of all probability distributions over  $\mathcal{S}$  (Baier and Katoen, 2008), and  $\gamma \in (0, 1)$  is some discount factor.

We call a tuple  $(s, a, s')$  with probability  $T(s, a)(s') > 0$  a *transition*. By absorbing state, we refer to a state  $s \in \mathcal{S}$  in which all transitions return to that state with probability 1 so that  $T(s, a)(s) = 1$ , for all  $a \in \mathcal{A}$ , these typically being some goal or critical states. An infinite trajectory of an MDP is a sampled sequence of states and actions  $\zeta = s_I a_0 s_1 a_1 \dots$ , where actions are chosen according to some *policy*, which we define in the sequel.

#### 2.1.1. POLICY CLASSES

We consider three distinct classes of policy to determine actions in this MDP. Namely, deterministic (also called pure), mixed (also called randomized), and behavioural policies. These policies are denoted by  $\pi^D \in \Pi^D, \pi^M \in \Pi^M, \pi^B \in \Pi^B$ , referring to a deterministic, mixed and behavioural policy respectively. Specifically,

$$\pi^D : \mathcal{S} \rightarrow \mathcal{A}, \quad \pi^M = \text{Dist}(\Pi^D), \quad \pi^B : \mathcal{S} \rightarrow \text{Dist}(\mathcal{A}). \quad (1)$$

In other words, a deterministic policy shows which action to pick at each state; a mixed policy provides directly a distribution over deterministic policies, so that we sample one policy at the start of a trajectory and follow it throughout; a behavioural policy gives us a distribution over actions at each state, so that we sample one action at each state of a trajectory. For MDPs, it can be shown that deterministic policies suffice for optimality, but this doesn't hold for more complex models.

We denote by  $\pi^M(\pi^D)$  the probability of choosing  $\pi^D$  under the distribution defined for the mixed policy, and by  $\pi^B(s)(a)$  the probability of choosing action  $a$  in a state  $s$ , under the policy defined by  $\pi^B$ . Mixed and behavioural policies offer two semantically different options to resolve the non-determinism in a probabilistic manner. For MDPs, there *is* a link between behavioural and mixed policies through Kuhn's theorem (Arrow et al., 1953). Further, for an MDP, it is always possible to find memoryless behavioural policies that are realization-equivalent to mixed policies (see (Rickard et al., 2023a, Appendix A) for a proof). In Uncertain Parametric MDPs (introduced in the sequel) the link between behavioural and mixed policies may not be satisfied.

#### 2.1.2. UNCERTAIN PARAMETRIC MDP

An Uncertain Parametric MDP (upMDP) is a tuple  $\mathcal{M}_v^{\mathbb{P}} = (\mathcal{S}, \mathcal{A}, \mathcal{V}, \mathcal{T}, \mathbb{P}, \rho, \gamma)$ . Similar to an MDP, but the probabilistic transition function is now  $\mathcal{T} : \mathcal{S} \times \mathcal{A} \times \mathcal{V} \rightarrow \text{Dist}(\mathcal{S})$ , so that we denote

by  $\mathcal{T}_v(s, a)(s')$  the probability of transitioning to state  $s'$ , given action  $a$  in  $s$ , with parameters  $v \in \mathcal{V}$  (i.e. the transition function is *parameterised* by  $v \in \mathcal{V}$ ), and with  $\mathbb{P}$  defining a probability distribution over the parameter space. Thus, when a set of parameters  $v$  is sampled, we extract a concrete MDP  $\mathcal{M}[v]$ . We do not impose any structure for this parameterisation, and only require that  $\mathcal{M}[v]$  be a well-defined MDP. To uncover a trajectory in this upMDP we first sample a set of parameters  $v$  from  $\mathbb{P}$ , and then follow a trajectory in the resulting MDP. We denote by  $\mathbb{P}^N$  the product measure associated with  $N$  sampled parameter sets.

## 2.2. Probabilistic Computation Tree Logic

Probabilistic computation tree logic (PCTL) depends on the following syntax:

$$\Phi ::= \text{true} \mid p \mid \neg\Phi \mid \Phi \wedge \Phi \mid P_{\sim\lambda}(\psi) \quad \psi ::= \Phi \cup \Phi. \quad (2)$$

Here,  $\sim \in \{<, \leq, \geq, >\}$  is a comparison operator and  $\lambda \in [0, 1]$  a probability threshold; PCTL formulae  $\Phi$  are state formulae, which can in particular depend on path formulae  $\psi$ . Informally, the syntax consists of state labels  $p \in AP$  in a set of atomic propositions  $AP$ , propositional operators negation  $\neg$  and conjunction  $\wedge$ , and temporal operator until  $\cup$ . The probabilistic operator  $P_{\sim\lambda}(\psi)$  requires that paths generated from the initial distribution satisfy a path formula  $\psi$  with total probability exceeding (or below, depending on  $\sim$ ) some given threshold  $\lambda$ . We consider only infinite horizon until, but note that extensions to include the finite time bounded-until operator can be achieved as discussed in [Osband et al. \(2019\)](#).

The satisfaction relation  $\mathcal{M}[v] \models_{\pi} \Phi$  defines whether a PCTL formula  $\Phi$  holds true, when following policy  $\pi$  in the concrete MDP. Formal definitions for semantics and model checking are provided in [Hansson and Jonsson \(1994\)](#); [Baier and Katoen \(2008\)](#).

### 2.2.1. PCTL SATISFACTION

This satisfaction relation defines slightly semantics depending on the class of policy. We now explore these semantics for the policy classes identified in Section 2.1.1, namely, deterministic, mixed and behavioural, respectively.

$$\begin{aligned} \mathcal{M} \models_{\pi^D} P_{\geq\lambda}(\psi) &\iff \sum_{\{\zeta: \zeta \models \psi\}} \rho(s_0)T(s_1 | s_0, \pi^D(s_0))T(s_2 | s_1, \pi^D(s_1)) \cdots \geq \lambda \\ &= \sum_{\{\zeta: \zeta \models \psi\}} P_{\pi^D}(\zeta) \geq \lambda, \end{aligned} \quad (3)$$

$$\mathcal{M} \models_{\pi^M} P_{\geq\lambda}(\psi) \iff \sum_{\pi^D \in \Pi^D} \pi^M(\pi^D) \cdot \left( \sum_{\{\zeta: \zeta \models \psi\}} P_{\pi^D}(\zeta) \right) \geq \lambda, \quad (4)$$

$$\begin{aligned} \mathcal{M} \models_{\pi^B} P_{\geq\lambda}(\psi) &\iff \sum_{\{\zeta: \zeta \models \psi\}} \rho(s_0)T(s_1 | s_0, a_0)\pi^B(a_0 | s_0)T(s_2 | s_1, a_1)\pi^B(a_1 | s_1) \cdots \geq \lambda \\ &= \sum_{\{\zeta: \zeta \models \psi\}} P_{\pi^B}(\zeta) \geq \lambda. \end{aligned} \quad (5)$$

We use the  $P_\pi(\zeta)$  to refer to the probability of uncovering a trajectory when playing a given policy. Note that both eqs. (4) and (5) contain terms relating to the probability distributions introduced in the policy classes. Instead, eq. (3) only considers uncertainty in the model itself.

### 2.3. Robust Policies

We are interested in synthesising *robust policies* for upMDPs. That is, policies that maximise the probability of satisfying a PCTL formula  $\psi$ , within some given risk tolerance. Thus, we are interested in solving the chance-constrained optimisation program

$$\max_{\pi \in \Pi, \lambda \in [0,1]} \lambda \quad \text{subject to} \quad \mathbb{P} \left\{ v \in \mathcal{V} | \mathcal{M}[v] \models_{\pi} P_{\geq \lambda}(\psi) \right\} \geq 1 - \epsilon. \quad (6)$$

for some a priori chosen risk level  $\epsilon \in (0, 1)$ . There is an inherent tradeoff here between risk and satisfaction probability: namely, for a small risk we might obtain a small satisfaction probability, and vice versa. For brevity, we discuss only maximising  $\lambda$ , but note that the minimisation for  $\sim \in \{<, \leq\}$  can be obtained trivially. Note that here we optimise over policies  $\pi$  in a generic set  $\Pi$ ; the exact policy class from the ones of Section 2.1.1 will be specified in the sequel.

## 3. Robust Policy Synthesis

**Problem 1** *Given an upMDP  $\mathcal{M}_v^{\mathbb{P}}$ , PCTL formula  $\psi$  and risk level  $\epsilon$ , find a robust policy  $\pi$  and maximum satisfaction probability  $\lambda$ , such that, with probability  $1 - \epsilon$  playing policy  $\pi$  on a newly sampled MDP will satisfy the PCTL formula with satisfaction probability at least  $\lambda$ .*

Due to the chance constraint on the parameter distribution  $\mathbb{P}$ , this problem is a semi-infinite optimization program (having a finite number of optimisation variables, but infinite constraints), and is generally intractable. Further, we may not have access to an analytical form for  $\mathbb{P}$ . Thus, we turn to a sample based analogue to this problem. In Section 4, we investigate how solutions to this problem can provide guarantees to the original chance-constrained problem.

Consider an upMDP  $\mathcal{M}_v^{\mathbb{P}}$  and  $N$  i.i.d. sampled parameter sets (or *scenarios*)  $\mathcal{U}_N = \{u_1, \dots, u_N\}$  sampled from  $\mathbb{P}$ . In this section, we investigate how to learn a robust policy which maximises the probability of satisfying a PCTL formula  $\psi$ , under the worst case realisation of the parameters, i.e. we investigate the following *scenario program* associated to eq. (6)

$$\max_{\pi \in \Pi, \lambda \geq 0} \lambda \quad \text{subject to} \quad \text{sol}_{\mathcal{M}}^{\pi}(u; \psi) \geq \lambda, \quad \forall u \in \mathcal{U}_N. \quad (7)$$

Where we use the notation  $\text{sol}_{\mathcal{M}}^{\pi}(u; \psi)$  to refer to the maximum probability of satisfying a formula  $\psi$ , in the concrete MDP  $\mathcal{M}[v]$ , under policy  $\pi$ ,

$$\text{sol}_{\mathcal{M}}^{\pi}(v; \psi) = \arg \max_{\lambda \in [0,1]} \mathcal{M}[v] \models_{\pi} P_{\geq \lambda}(\psi). \quad (8)$$

This scenario program problem coincides closely with the concept of Multiple Environment MDPs (MEMDPs) (Raskin and Sankur, 2014), but differs in some key ways. When finding an optimal policy for MEMDPs, one wishes to find the policy that is optimal for the given environments. Instead, our goal to find a policy that will be robust to an unseen environment. Furthermore, the

concrete chance constrained problem we study is drastically different to MEMDPs since it involves continuous parameters rather than a finite set of environments.

It is shown in [Raskin and Sankur \(2014\)](#) that infinite memory policies are required for optimality in MEMDPs. In our setting, infinite memory policies will indeed lead to optimal satisfaction probabilities in the sample set, but suffer when generalising to new samples. This can be seen as a problem of overfitting, and may also lead to a deterioration in our guarantees.

In the sequel, we provide different solution methodologies for eq. (7), where each methodology is based on a different class of policies.

### 3.1. Solution by Interval MDPs (under deterministic policies)

A straightforward approach to solve this problem is to ignore the structure induced by the parameterisation, and to build instead an interval MDP, where each transition is only known up to an interval. Each interval can be constructed by looking at each transition in turn and finding the minimum and maximum probabilities from the sampled MDPs. Such techniques are examined further (generally when no parameterisation is available) in [Kozine and Utkin \(2002\)](#) and are also closely aligned with the concept of  $(s, a)$ -rectangularity ([Wiesemann et al., 2013](#)). The resulting policy is in general very conservative, since it considers the worst case probability for every single transition. In reality, it is likely to be the case that the worst case transitions are unlikely to co-occur (for example, in a UAV motion planning problem like the one considered in [Table 1](#) the wind may push us left or right into an obstacle, but is unlikely to do both).

### 3.2. MaxMin Game (under mixed policies)

The problem in eq. (7) can be rewritten as a MaxMin problem  $\max_{\pi \in \Pi} \min_{u \in \mathcal{U}_N} \text{sol}_{\mathcal{M}}^{\pi}(u; \psi)$ , which can be seen as a two player zero-sum game. In which we have a policy player, and the samples act as an adversary. The policy player's actions are the set of all deterministic policies  $\Pi^D$  for the upMDP, providing a finite, but potentially very large  $\mathcal{O}(|\mathcal{A}|^{|S|})$ , action set. The sample adversary's actions are the set of samples. Given sample  $u$  and policy  $\pi^D$ , the reward to the policy player is  $r_p(\pi^D, u) = \text{sol}_{\mathcal{M}}^{\pi^D}(u; \psi)$ . We may solve this as a Stackelberg game ([Stackelberg, 1952](#); [Yousefimanesh et al., 2023](#)), with the adversary as the leader, to obtain an optimal deterministic policy. Details on this method, and associated results, are available in ([Rickard et al., 2023a](#), Appendix E).

Alternatively, a mixed strategy set  $(s_p, s_{\sigma})$ , contains finite probability distributions over possible actions of each player, respectively, and returns the reward  $r_p(s_p, s_{\sigma}) = \sum_{\pi^D \in \Pi^D} \sum_{u \in \mathcal{U}_N} s_p(\pi^D) \cdot s_{\sigma}(u) \cdot \text{sol}_{\mathcal{M}}^{\pi^D}(u; \psi)$ , or in matrix form  $s_p R s_{\sigma}$ , where matrix  $R \in \mathbb{R}^{|\Pi^D| \times N}$  has elements  $R_{\pi^D, u_j} = \text{sol}_{\mathcal{M}}^{\pi^D}(u_j; \psi)$ . Since the game consists of a finite number of players, each with a finite set of pure strategies, then by allowing players to play mixed strategies, it can be shown that a Nash equilibrium of the game will exist, and further, that all Nash equilibria have the same value. Note that the mixed strategy defines a mixed policy  $s_p = \pi^M$ .

**Theorem 1 (Nash Equilibrium Nash (1989); Frihauf et al. (2012))** *For a two player zero-sum game, there exists at least one mixed strategy profile  $s^* = (s_p^*, s_{\sigma}^*)$ , such that*

$$r_p(s_p^*, s_{\sigma}^*) \geq r_p(s_p, s_{\sigma}^*), \forall s_p \in S_p, \quad r_p(s_p^*, s_{\sigma}^*) \leq r_p(s_p^*, s_{\sigma}), \forall s_{\sigma} \in S_{\sigma}. \quad (9)$$

*If both inequalities are strict, then there is a single unique Nash equilibrium, called a strict Nash equilibrium. Otherwise, there is a set of Nash equilibria, all having equal value.*

Finding Nash equilibrium strategies in this game is relatively straightforward, and there are a number of existing methods for solving the game (for example, the Porter-Nudelman-Shoham (PNS) algorithm (Porter et al., 2008, Algorithm 1) or fictitious play methods (Heinrich et al., 2015)). Thus, we can simply build the reward matrix  $R$ , by considering each deterministic policy and sample in turn, and pass this to one of these algorithms. Unfortunately, this method is computationally very expensive, with the size of the reward matrix being  $\mathbb{R}^{N \times |S|^{|A|}}$ , and algorithms to solve such games being exponential in the size of this matrix.

### 3.3. Subgradient Ascent (under behavioural policies)

Under the choice of behavioural policies, we propose an alternative method that avoids computing a full payoff matrix for every deterministic policy. Taking inspiration from Bhandari and Russo (2019), we rewrite the solution function as  $\text{sol}_{\mathcal{M}}^{\pi^B}(u; \psi) = \sum_{s \in S} \eta_{\pi^B}^u(s) \sum_{a \in A} Q_{\pi^B}^u(s, a) \pi^B(s)(a)$ , where  $Q_{\pi^B}^u(s, a)$  is a Q-function, defined to model the PCTL requirement (Rickard et al., 2023a, Appendix B) and  $\eta_{\pi^B}^u(s)$  is the discounted state-occupancy measure  $\eta_{\pi^B}^u(s) = (1-\gamma) \sum_{k=0}^{\infty} \gamma^k P_k^{\pi^B}(s)$ , with  $P_k^{\pi^B}(s)$  the probability of uncovering state  $s$  at time  $k$ . Intuitively,  $\eta_{\pi^B}^u(s)$  defines the (discounted) fraction of time the system spends in a given state. We fix  $\eta_{\pi^B}^u$  and  $Q_{\pi^B}^u$  (as in policy iteration (Foster, 1962)), and find the gradient (given analytically in Algorithm 1).

This provides us with a (sub)gradient for a single solution function, however, we are primarily interested in the pointwise minimum amongst a set of solution functions. Hence, we turn to subdifferentials (or *subgradients*) (Kiwiel, 2001; Aussel et al., 1995). One simple way of finding a valid subdifferential is to find the gradient of one of the current minimum functions.

**Algorithm 1:** Projected Subgradient Ascent

**Data:** upMDP  $\mathcal{M}$ , samples  $\mathcal{U}_N$ , formula  $\psi$ , step-size sequence  $\{\alpha_0, \alpha_1, \dots\}$

**Result:** Optimal Policy  $\pi$

$k \leftarrow 0$     $\pi_0^B \leftarrow$  uniform random

**while** *not converged* **do**

$u^* \leftarrow$ random choice( $\arg \min_{u \in \mathcal{U}_N} \text{sol}_{\mathcal{M}}^{\pi_k^B}(u; \psi)$ )     // Select worst case sample $\nabla_{k+1}(s, a) = \eta_{\pi^B}^{u^*}(s) Q_{\pi^B}^{u^*}(s, a)$ // Find gradient for worst case $\pi_{k+1}^B(s)(a) \leftarrow \text{proj}_{\sum \pi^B(\cdot s)=1}[\pi_k^B + \alpha_k \nabla_{k+1}(s, a)]$ // Gradient step $f_*^{k+1} = \max\{f_*^k, f(\pi_{k+1}^B)\}$ // Store record objective $k \leftarrow k + 1$	
--	--

**end**

We initialise with a uniform random policy, select a minimising (worst-case) sample from the set of minimisers (which may not be a singleton), then find the gradient for this sample, take a step in this direction, and project onto the constraint set. We use a diminishing, non-summable step size  $\alpha_k$ , typically employed in subgradient methods. We give numerical evidence that this algorithm exhibits a convergent behaviour in Section 5, and further analysis in (Rickard et al., 2023a, Appendix D).

## 4. Guarantees

Once we have synthesised a policy using one of the methods above, we can accompany each of the policies synthesized according to the aforementioned methodologies with PAC-type guarantees on

the satisfaction of the PCTL property under consideration. To achieve this, we use the following recent result in the scenario approach theory.

Given the solution to a scenario program, we are interested in quantifying the risk associated with the solution (i.e. the probability that our solution violates a new sampled constraint). The scenario approach provides us with the techniques to achieve this by considering the number of support constraints. In an optimisation problem, if the removal of a constraint leads to a changed solution, then this constraint is said to be a support constraint. Further, if a solution returned when considering only the support constraints is the same as the solution obtained when employing all samples, then the problem is a non-degenerate problem.

**Theorem 2 (PAC Guarantees Garatti and Campi (2022))** *Consider the optimisation problem in eq. (7), with  $N$  sampled parameter sets. Given a confidence parameter  $\beta \in (0, 1)$ , for any  $k = 0, 1, \dots, N$ , consider the polynomial equation in the variable  $t$*

$$\xi_k(t) = \begin{cases} 1 - \frac{\beta}{6N} \sum_{i=N+1}^{4N} \binom{i}{k} t^{i-k}, & \text{if } k = N, \\ \binom{N}{k} t^{N-k} - \frac{\beta}{2N} \sum_{i=k}^{N-1} \binom{i}{k} t^{i-k} - \frac{\beta}{6N} \sum_{i=N+1}^{4N} \binom{i}{k} t^{i-k}, & \text{otherwise.} \end{cases} \quad (10)$$

Solving  $\xi_k(t) = 0$  for  $t \in [0, +\infty)$ , for  $k = 0, 1, \dots, N-1$ , we find exactly two solutions, which we denote with  $\underline{t}(k), \bar{t}(k)$  with  $\underline{t}(k) \leq \bar{t}(k)$ . For  $k = N$ , we find a single solution, which we denote by  $\bar{t}(N)$ , and define  $\underline{t}(N) = 0$ . Let  $\underline{\epsilon}(k) := \max\{0, 1 - \bar{t}(k)\}$  and  $\bar{\epsilon}(k) := 1 - \underline{t}(k)$ .

Assume the problem is non-degenerate<sup>1</sup>, and has a unique solution. Let  $\pi_N^*$  be the optimal policy,  $\lambda_N^*$  the optimal satisfaction probability, and  $\tilde{s}_N^*$  is the number of support samples. Then, for any  $\mathbb{P}$  it holds that

$$\mathbb{P}^N \{ \underline{\epsilon}(\tilde{s}_N^*) \leq \mathbb{P} \left\{ v \in \mathcal{V} : \mathcal{M}[v] \not\stackrel{\pi_M}{\models} P_{\geq \lambda^*}(\psi) \right\} \leq \bar{\epsilon}(\tilde{s}_N^*) \} \geq 1 - \beta, \quad (11)$$

where  $\mathbb{P} \{ v \in \mathcal{V} : \mathcal{M}[v] \not\stackrel{\pi_M}{\models} P_{\geq \lambda^*}(\psi) \}$  is the risk (the probability that our solution violates the specification for another sample  $v$ ).

Note that in  $\pi_N^*$  we do not specify the problem class; this is considered in the sequel when we deploy this theorem for each solution methodology from the previous section. Moreover, to determine the number of support samples  $\tilde{s}_N^*$ , we exploit our problem's structure and, based on the solution methodology adopted, we discuss how an upper-bound on their number can be obtained.

To this end, we provide guarantees for the policy generated by each of the methods of Section 3. Consider first the case of the mixed policy determined using a Nash equilibrium solver as per the developments of Section 3.2. In this case, finding the number of support samples consists of finding samples which are included in the mixed strategy of the sampled adversary.

**Corollary 3 (PAC Guarantees for Mixed Policies)** *Consider the MaxMin game of Section 3.2, and let  $\pi_N^*$  be the optimal mixed policy  $\pi^M$  returned by a Nash equilibrium solver. The number of support constraints may be found as  $\tilde{s}_N^* = |\{u \in \mathcal{U}_N : s_\sigma(u) > 0\}|$ , while the satisfaction relation  $\mathcal{M}[v] \not\stackrel{\pi_M}{\models} P_{\geq \lambda^*}(\psi)$  is as defined in eq. (4).*

1. Non-degeneracy implies that solving the problem using only the samples that are of support results in the same solution (policy in our case) had all the samples been employed. We say that a sample (which gives rise to a constraint in eq. (7)) is of support, if removing only that sample results in a different solution. Repeating this procedure, removing samples one-by-one allows identifying the support samples of a given problem.



Consider now the case of a behavioural policy obtained using the subgradient methodology (Section 3.3). We determine the number of support constraints by finding the active constraints (as the problem is assumed to be non-degenerate, the active constraints are also the support constraints).

**Corollary 4 (PAC Guarantees for Behavioural Policies)** *Consider the subgradient algorithm of Section 3.2, and let  $\pi_N^*$  be the optimal behavioural policy  $\pi^B$  returned by Algorithm 1. The number of support samples is given by the active constraints, which are in turn the ones for which the obtained satisfaction probability  $\lambda^*$  is tight, i.e.,  $\tilde{s}_N^* = |\{u \in \mathcal{U}_N : \text{sol}_{\mathcal{M}}^{\pi^B}(u; \psi) = \lambda^*\}|$ , while the satisfaction relation  $\mathcal{M}[v] \not\equiv_{\pi^B} P_{\geq \lambda^*}(\psi)$  is as defined in eq. (5).*

Finally, consider the construction of an interval MDP under the class of deterministic policies. In this case, our problem is degenerate, since it may be necessary to remove multiple constraints for another policy to become optimal, thus prohibiting the use of Theorem 2. Therefore, we leverage techniques from Campi et al. (2018) that do not require imposing a non-degeneracy assumption.

**Corollary 5 (PAC Guarantees for iMDP Policies)** *Fix  $\beta \in (0, 1)$ , and as in (Campi et al., 2018, Theorem 1), define  $\mu(N) := 1$ , and for  $\tilde{s}_N^* < N$  let*

$$\mu(\tilde{s}_N^*) := 1 - \sqrt[N - \tilde{s}_N^*]{\frac{\beta}{N(\tilde{s}_N^*)}}. \quad (12)$$

*Support samples are those which define the intervals:  $\overline{\tilde{s}_N^*} = |\{u \in \mathcal{U}_N : \exists(s, a, s'), T_u(s, a)(s') \leq T_v(s, a)(s') \vee T_u(s, a)(s') \geq T_v(s, a)(s'), \forall v \in \mathcal{U}_N\}|$ , i.e., those with at least one transition probability at an extremum of its interval. Then (with satisfaction relation defined in eq. (3)) we have*

$$\mathbb{P}^N \left\{ \mathbb{P} \left\{ v \in \mathcal{V} : \mathcal{M}[v] \not\equiv_{\pi^D} P_{\geq \lambda^*}(\psi) \right\} \leq \mu(\overline{\tilde{s}_N^*}) \right\} \geq 1 - \beta. \quad (13)$$

## 5. Numerical Experiments

We implemented our techniques in Python and made use of the probabilistic model checker Storm (Hensel et al., 2022) to verify PCTL formulae on MDPS, and PRISM (Kwiatkowska et al., 2011) for iMDPs. Experiments were run on a server with 80 2.5 GHz CPUs and 125 GB of RAM. The codebase is available at [https://github.com/lukearcus/robust\\_upMDP](https://github.com/lukearcus/robust_upMDP)

We evaluate our techniques on a number of benchmark models available in the literature (Quatmann et al., 2016), as well as a few simpler examples, for a complete description see (Rickard et al., 2023a, Appendix C). We compare our subgradient algorithm (Section 3.3) to existing techniques from Badings et al. (2022) (column 1) and an implementation of the synthesis based on iMDP, as in Kozine and Utkin (2002) (column 2). Unless otherwise stated, we use a numerical tolerance of  $10^{-4}$ , a confidence parameter of  $\beta = 10^{-5}$ , take  $N = 200$  parameter samples, and allow 1 hour of computation time. We provide numerical values for the optimal satisfaction probability  $\lambda^*$ , the theoretical risk upper bound  $\bar{\epsilon}$ , the runtimes, the empirical values of the risk  $\tilde{\epsilon}$ , and the empirical values of the risk without access to the true parameter set (if parameter access is needed to synthesise a policy, we draw a sample  $v_1 \sim \mathbb{P}$  for synthesis, but test on sample,  $v_2 \sim \mathbb{P}$ ). We highlight in grey cases where we discuss next how the methodologies under comparison are compared/outperformed by our proposed subgradient algorithm either in terms of the quality of their probabilistic guarantees and/or the computational requirements, and use red text for empirical values which violate a bound.

Model	Instance	$ \mathcal{S}  \cdot  \mathcal{A} $	Badings et al. (2022)				iMDP Solver				Subgradient (Algorithm 1)						
			Sat. Prob.	Risk U.B.	Time (s)	Emp. Risk (no pars)	Sat. Prob.	Risk U.B.	Time (s)	Emp. Risk (no pars)	Sat. Prob.	Risk U.B.	Time (s)	Emp. Risk (no pars)			
			$\lambda^*$	$\bar{\epsilon}$	$\tilde{\epsilon}$	$\tilde{\epsilon}$	$\lambda^*$	$\bar{\mu}$	$\tilde{\mu}$	$\tilde{\mu}$	$\lambda^*$	$\bar{\epsilon}$	$\tilde{\epsilon}$	$\tilde{\epsilon}$			
<b>consensus</b> (min)	(2,2)	306	.967	.056	1	.004	.008	.967	1	3	.007	.008	.967	.187	4	.009	.011
	(2,32)	5 586	.996	.056	136	.000	.014	.996	1	76	.000	.000	.996	.516	202	.002	.000
	(4,2)	90 624	-	-	TO	-	-	.936	1	1887	.000	.000	-	-	TO	-	-
<b>brp</b> (max)	(256,5)	42 064	-	-	TO	-	-	.985	1	383	.001	.000	.985	1	923	.000	.001
<b>sav</b> (min)	(6,2,2)	1516	.834	.056	4	.015	.112	.923	1	9	.002	.004	.884	.180	218	.013	.007
	(6,2,2)	1516	.700	.056	5	.000	.017	.721	1	9	.017	.010	.720	.187	76	.011	.017
	(10,3,3)	7400	.446	.056	51	.009	.120	.524	1	50	.017	.015	.526	.147	109	.021	.018
<b>UAV</b> (max)	uniform	45 852	.301	.056	978	.003	.749	.065	1	238	.015	.013	.195	.245	85719 <sup>2</sup>	.077	.089
	x-neg	45 852	.198	.056	804	.002	.802	.012	1	210	.000	.000	.102	.440	58612	.068	.071
	y-pos	45 852	.564	.056	811	.003	.755	.050	1	196	.001	.001	.444	.163	92440	.095	.101
<b>Toy Model</b>	(max)	8	.326	.056	.11	.008	.106	.323	.155	1.47	.008	.094	.325	.366	8.95	.003	.098

Table 1: Experimental Results; the results for the case of the MaxMin Solution are discussed in the text. TO indicates timed-out.

Our subgradient algorithm is less conservative than a naive iMDP solution, offering non-trivial theoretical risk bounds (for the iMDP approach these bounds are often equal to one), and superior satisfaction probabilities (the iMDP approach considers the worst case probability for every transition). By superior, we mean either higher or lower depending on the optimisation direction (maximisation or minimisation, respectively, denoted by (max) and (min) in Table 1). The approach in Badings et al. (2022) generally outperforms our methods in speed, risk bounds and satisfaction probability. The difference on the theoretical risk bounds lies in the fact that our approach involves solving a problem with a non-trivial number of support constraints, while in Badings et al. (2022) they have only one support constraint. As such, it offers tighter guarantees, however, these refer to an existential statement. To see this, note that Badings et al. (2022) compute a different policy per sample, and provide guarantees only on the existence of a policy for a new sample. On the contrary, we construct a policy that is robust with respect to all samples, and at the same time is accompanied by guarantees on its feasibility properties for unseen samples. Moreover, Badings et al. (2022) require access to true parameters at runtime, and a non-trivial amount of online computation to solve the MDP. Without this, their risk bounds may be violated (as seen in red).

Our mixed policy solution (Section 3.2), timed out on all but the toy model, for which the results are: Sat. Prob: .325, Risk U.B.: .147, runtime: 0.75s Emp. risk: .003, Emp. risk (no pars): .101.

## 6. Concluding Remarks and Future Directions

We presented a novel method for learning a single robust policy for upMDPs, providing PAC guarantees on satisfaction of PCTL formulae. We have considered several policy classes, and provided guarantees for each. Our experiments demonstrate the efficacy of our methods on a number of benchmarks, and provide comparisons to previous methods.

One avenue for future work is discarding constraints (Campi and Garatti, 2011). Relatedly, improving the runtimes of our algorithms (perhaps at the cost of guarantees) is of interest. Finally, we leave a full technical analysis of the convergence of our subgradient method to later work.

2. For the UAV benchmark, we allowed up to 2 days of computation time.

## Acknowledgments

This work was supported by the EPSRC Centre for Doctoral Training in Autonomous Intelligent Machines and Systems EP/S024050/1.

## References

- K. J. Arrow, E. W. Barankin, D. Blackwell, R. Bott, N. Dalkey, M. Dresher, D. Gale, D. B. Gillies, I. Glicksberg, O. Gross, S. Karlin, H. W. Kuhn, J. P. Mayberry, J. W. Milnor, T. S. Motzkin, J. Von Neumann, H. Raiffa, L. S. Shapley, M. Shiffman, F. M. Stewart, G. L. Thompson, and R. M. Thrall. *Contributions to the Theory of Games (AM-28), Volume II*. Princeton University Press, 1953. ISBN 978-0-691-07935-6.
- Didier Aussel, Jean-Noel Corvellec, and Marc Lassonde. Mean Value Property and Subdifferential Criteria for Lower Semicontinuous Functions. *Transactions of the American Mathematical Society*, 347(10):4147–4161, 1995. ISSN 0002-9947.
- Thom S. Badings, Murat Cubuktepe, Nils Jansen, Sebastian Junges, Joost-Pieter Katoen, and Ufuk Topcu. Scenario-based verification of uncertain parametric MDPs. *Int. J. Softw. Tools Technol. Transf.*, 24(5):803–819, 2022.
- Thom S. Badings, Licio Romao, Alessandro Abate, David Parker, Hasan A. Poonawala, Mariëlle Stoelinga, and Nils Jansen. Robust Control for Dynamical Systems with Non-Gaussian Noise via Formal Abstractions. *J. Artif. Intell. Res.*, 76:341–391, 2023.
- Christel Baier and Joost-Pieter Katoen. *Principles of Model Checking*. MIT Press, 2008. ISBN 978-0-262-02649-9.
- Calin Belta, Boyan Yordanov, and Ebru Aydin Gol. *Formal Methods for Discrete-Time Dynamical Systems*, volume 89 of *Studies in Systems, Decision and Control*. Springer International Publishing, Cham, 2017. ISBN 978-3-319-50762-0 978-3-319-50763-7. doi: 10.1007/978-3-319-50763-7.
- Jalaj Bhandari and Daniel Russo. Global optimality guarantees for policy gradient methods. *CoRR*, abs/1906.01786, 2019.
- Lukas Brunke, Melissa Greeff, Adam W. Hall, Zhaocong Yuan, Siqi Zhou, Jacopo Panerati, and Angela P. Schoellig. Safe learning in robotics: From learning-based control to safe reinforcement learning. *Annu. Rev. Control. Robotics Auton. Syst.*, 5:411–444, 2022.
- Giuseppe Carlo Calafiore and Marco C. Campi. The scenario approach to robust control design. *IEEE Trans. Autom. Control.*, 51(5):742–753, 2006.
- Marco C. Campi and Simone Garatti. A Sampling-and-Discarding Approach to Chance-Constrained Optimization: Feasibility and Optimality. *J. Optim. Theory Appl.*, 148(2):257–280, 2011.
- Marco C. Campi and Simone Garatti. Wait-and-judge scenario optimization. *Math. Program.*, 167(1):155–189, 2018.

- Marco C. Campi, Simone Garatti, and Maria Prandini. The scenario approach for systems and control design. *Annu. Rev. Control.*, 33(2):149–157, 2009. doi: 10.1016/j.arcontrol.2009.07.001.
- Marco Claudio Campi, Simone Garatti, and Federico Alessandro Ramponi. A General Scenario Theory for Nonconvex Optimization and Decision Making. *IEEE Trans. Autom. Control.*, 63(12):4067–4078, 2018.
- Edmund M. Clarke, Ansgar Fehnker, Zhi Han, Bruce H. Krogh, Olaf Stursberg, and Michael Theobald. Verification of Hybrid Systems Based on Counterexample-Guided Abstraction Refinement. In Hubert Garavel and John Hatcliff, editors, *Tools and Algorithms for the Construction and Analysis of Systems, 9th International Conference, TACAS 2003, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2003, Warsaw, Poland, April 7-11, 2003, Proceedings*, volume 2619 of *Lecture Notes in Computer Science*, pages 192–207. Springer, 2003. doi: 10.1007/3-540-36577-X\\_14.
- Murat Cubuktepe, Nils Jansen, Sebastian Junges, Joost-Pieter Katoen, and Ufuk Topcu. Convex Optimization for Parameter Synthesis in MDPs. *IEEE Trans. Autom. Control.*, 67(12):6333–6348, 2022.
- Conrado Daws. Symbolic and Parametric Model Checking of Discrete-Time Markov Chains. In *ICTAC*, volume 3407 of *Lecture Notes in Computer Science*, pages 280–294. Springer, 2004a.
- Conrado Daws. Symbolic and Parametric Model Checking of Discrete-Time Markov Chains. In *ICTAC*, volume 3407 of *Lecture Notes in Computer Science*, pages 280–294. Springer, 2004b.
- F. G. Foster. Dynamic Programming and Markov Processes. By R. A. Howard. Pp. 136. 46s. 1960. (John Wiley and Sons, N.Y.). *The Mathematical Gazette*, 46(358):340–341, December 1962. ISSN 0025-5572, 2056-6328.
- Paul Frihauf, Miroslav Krstic, and Tamer Basar. Nash equilibrium seeking in noncooperative games. *IEEE Trans. Autom. Control.*, 57(5):1192–1207, 2012.
- Simone Garatti and Marco C. Campi. Risk and complexity in scenario optimization. *Math. Program.*, 191(1):243–279, 2022. doi: 10.1007/s10107-019-01446-4.
- Ernst Moritz Hahn, Tingting Han, and Lijun Zhang. Synthesis for PCTL in Parametric Markov Decision Processes. In Mihaela Gheorghiu Bobaru, Klaus Havelund, Gerard J. Holzmann, and Rajeev Joshi, editors, *NASA Formal Methods - Third International Symposium, NFM 2011, Pasadena, CA, USA, April 18-20, 2011. Proceedings*, volume 6617 of *Lecture Notes in Computer Science*, pages 146–161. Springer, 2011a. doi: 10.1007/978-3-642-20398-5\\_12.
- Ernst Moritz Hahn, Holger Hermanns, and Lijun Zhang. Probabilistic reachability for parametric Markov models. *Int. J. Softw. Tools Technol. Transf.*, 13(1):3–19, 2011b.
- Hans Hansson and Bengt Jonsson. A logic for reasoning about time and reliability. *Formal Aspects of Computing*, 6(5):512–535, September 1994. ISSN 1433-299X. doi: 10.1007/BF01211866.
- Johannes Heinrich, Marc Lanctot, and David Silver. Fictitious Self-Play in Extensive-Form Games. In *ICML*, volume 37 of *JMLR Workshop and Conference Proceedings*, pages 805–813. JMLR.org, 2015.

- Christian Hensel, Sebastian Junges, Joost-Pieter Katoen, Tim Quatmann, and Matthias Volk. The probabilistic model checker Storm. *Int. J. Softw. Tools Technol. Transf.*, 24(4):589–610, 2022.
- Garud N. Iyengar. Robust Dynamic Programming. *Math. Oper. Res.*, 30(2):257–280, 2005.
- Sebastian Junges, Erika Ábrahám, Christian Hensel, Nils Jansen, Joost-Pieter Katoen, Tim Quatmann, and Matthias Volk. Parameter Synthesis for Markov Models. *CoRR*, abs/1903.07993, 2019.
- Krzysztof C. Kiwiel. Convergence and efficiency of subgradient methods for quasiconvex minimization. *Math. Program.*, 90(1):1–25, 2001.
- John C. Knight. Safety critical systems: Challenges and directions. In Will Tracz, Michal Young, and Jeff Magee, editors, *Proceedings of the 24th International Conference on Software Engineering, ICSE 2002, 19-25 May 2002, Orlando, Florida, USA*, pages 547–550. ACM, 2002. doi: 10.1145/581339.581406.
- Igor Kozine and Lev V. Utkin. Interval-Valued Finite Markov Chains. *Reliab. Comput.*, 8(2):97–113, 2002.
- Marta Z. Kwiatkowska, Gethin Norman, and David Parker. PRISM 4.0: Verification of probabilistic real-time systems. In *CAV*, volume 6806 of *Lecture Notes in Computer Science*, pages 585–591. Springer, 2011.
- Indika Meedeniya, Irene Moser, Aldeida Aleti, and Lars Grunske. Evaluating probabilistic models with uncertain model parameters. *Softw. Syst. Model.*, 13(4):1395–1415, 2014.
- John Nash. Non-cooperative games. *Cournot Oligopoly*, pages 82–94, January 1989. doi: 10.1017/CBO9780511528231.007.
- Arnab Nilim and Laurent El Ghaoui. Robust Control of Markov Decision Processes with Uncertain Transition Matrices. *Oper. Res.*, 53(5):780–798, 2005.
- Ian Osband, Benjamin Van Roy, Daniel J. Russo, and Zheng Wen. Deep exploration via randomized value functions. *J. Mach. Learn. Res.*, 20:124:1–124:62, 2019.
- André Platzer. Logics of Dynamical Systems. In *Proceedings of the 27th Annual IEEE Symposium on Logic in Computer Science, LICS 2012, Dubrovnik, Croatia, June 25-28, 2012*, pages 13–24. IEEE Computer Society, 2012. doi: 10.1109/LICS.2012.13.
- Ryan Porter, Eugene Nudelman, and Yoav Shoham. Simple search methods for finding a Nash equilibrium. *Games Econ. Behav.*, 63(2):642–662, 2008.
- Alberto Puggelli, Wenchao Li, Alberto L. Sangiovanni-Vincentelli, and Sanjit A. Seshia. Polynomial-Time Verification of PCTL Properties of MDPs with Convex Uncertainties. In *CAV*, volume 8044 of *Lecture Notes in Computer Science*, pages 527–542. Springer, 2013.
- Tim Quatmann, Christian Dehnert, Nils Jansen, Sebastian Junges, and Joost-Pieter Katoen. Parameter synthesis for markov models: Faster than ever. In *ATVA*, volume 9938 of *Lecture Notes in Computer Science*, pages 50–67, 2016.

- Jean-François Raskin and Ocan Sankur. Multiple-Environment Markov Decision Processes. In *FSTTCS*, volume 29 of *LIPICs*, pages 531–543. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2014.
- Luke Rickard, Alessandro Abate, and Kostas Margellos. Learning robust policies for uncertain parametric markov decision processes. Technical report, 2023a. URL <https://www.lukerickard.co.uk/pubs/RAM23.pdf>.
- Luke Rickard, Thom Badings, Licio Romao, and Alessandro Abate. Formal Controller Synthesis for Markov Jump Linear Systems with Uncertain Dynamics, May 2023b.
- Dimitri Scheftelowitsch, Peter Buchholz, Vahid Hashemi, and Holger Hermanns. Multi-Objective Approaches to Markov Decision Processes with Uncertain Transition Parameters. In *VALUE-TOOLS*, pages 44–51. ACM, 2017.
- H. Von. Stackelberg. The Theory of Market Economy. Translated from the German and with an Introduction by A.T. Peacock. London, Edinburgh, Glasgow, W. Hodge & Co, Ltd., 1952, xxiii p. 328 p., 25/- . *Recherches Économiques de Louvain/ Louvain Economic Review*, 18(5):543–543, 1952. ISSN 1373-9719. doi: 10.1017/S0770451800047382.
- Marck van der Vegt, Nils Jansen, and Sebastian Junges. Robust Almost-Sure Reachability in Multi-Environment MDPs. In *TACAS (1)*, volume 13993 of *Lecture Notes in Computer Science*, pages 508–526. Springer, 2023.
- Wolfram Wiesemann, Daniel Kuhn, and Berç Rustem. Robust Markov Decision Processes. *Math. Oper. Res.*, 38(1):153–183, 2013.
- Niloofer Yousefimanesh, Iwan Bos, and Dries Vermeulen. Strategic rationing in Stackelberg games. *Games Econ. Behav.*, 140:529–555, 2023.
- Pian Yu and Dimos V. Dimarogonas. Distributed motion coordination for multi-robot systems under LTL specifications, March 2021.