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Non-Clashing Teaching Maps for Balls in Graphs

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Abstract

Recently, Kirkpatrick et al. [ALT 2019] and Fallat et al. [JMLR 2023] introduced non-clashing teaching and showed it to be the most efficient machine teaching model satisfying the benchmark for collusion-avoidance set by Goldman and Mathias. A teaching map T for a concept class $\mathcal{C}$ assigns a (teaching) set $T(C)$ of examples to each concept $C \in \mathcal{C}$. A teaching map is non-clashing if no pair of concepts are consistent with the union of their teaching sets. The size of a non-clashing teaching map (NCTM) T is the maximum size of a teaching set $T(C)$, $C \in \mathcal{C}$. The non-clashing teaching dimension $\mathrm{NCTD}(\mathcal{C})$ of $\mathcal{C}$ is the minimum size of an NCTM for $\mathcal{C}$. NCTM$^+$ and NCTD$^+(\mathcal{C})$ are defined analogously, except the teacher may only use positive examples.

We study NCTMs and NCTM$^+$s for the concept class $\mathcal{B}(G)$ consisting of all balls of a graph G. We show that the associated decision problem B-NCTD$^+$ for NCTD$^+$ is NP-complete in split, co-bipartite, and bipartite graphs. Surprisingly, we even prove that, unless the ETH fails, B-NCTD$^+$ does not admit an algorithm running in time $2^{2^{o(\mathtt{vc})}} \cdot n^{\mathcal{O}(1)}$, nor a kernelization algorithm outputting a kernel with $2^{o(\mathtt{vc})}$ vertices, where vc is the vertex cover number of G. These are extremely rare results: it is only the second (fourth, resp.) problem in NP to admit such a double-exponential lower bound parameterized by vc (treewidth, resp.), and only one of very few problems to admit such an ETH-based conditional lower bound on the number of vertices in a kernel. We complement these lower bounds with matching upper bounds. For trees, interval graphs, cycles, and trees of cycles, we derive NCTM$^+$s or NCTMs for $\mathcal{B}(G)$ of size proportional to its VC-dimension. For Gromov-hyperbolic graphs, we design an approximate NCTM$^+$ for $\mathcal{B}(G)$ of size 2, in which only pairs of balls with Hausdorff distance larger than some constant must satisfy the non-clashing condition.

Keywords: Non-clashing teaching, VC-dimension, balls in graphs, parameterized complexity, vertex cover, kernelization, double-exponential lower bounds, ETH lower bounds, hyperbolic graphs

1. Introduction

Machine teaching is a core paradigm in computational learning theory that has attracted significant attention due to its applications in diverse areas such as trustworthy AI Mei and Zhu (2015); Zhang et al. (2018), inverse reinforcement learning Brown and Niekum (2019); Ho et al. (2016), robotics Akgun et al. (2012); Thomaz and Cakmak (2009), and education Chen et al. (2018); Zhu (2015) (see Zhu et al. (2018) for an overview). In machine teaching models, given a concept class $\mathcal{C}$, a teacher presents to a learner a carefully chosen set $T(C)$ of correctly labeled examples from a concept $C \in \mathcal{C}$ in such a way that the learner can reconstruct C from $T(C)$. This defines the *teaching map (TM)* T and the *teaching sets* $T(C), C \in \mathcal{C}$. The goal is to find a TM that minimizes the size of a largest teaching set. The examples selected in $T(C)$ by the teacher are the most useful to the learner to reconstruct C, in contrast to models of learning (like the classical PAC-learning) where the learner must reconstruct a concept of $\mathcal{C}$ from randomly chosen examples.

There are a multitude of formal models of machine teaching Balbach (2008); Gao et al. (2016, 2017); Goldman and Kearns (1995); Goldman and Mathias (1996); Mansouri et al. (2019); Shinohara and Miyano (1991); Zilles et al. (2011), which differ by the conditions imposed on the teacher and learner. Several of these are *batch teaching models*, where the examples proposed by the teacher to the learner are sets. This is in contrast to *sequential teaching models*, where the examples are not presented all at once, but rather in an order chosen by the teacher. A key notion in formal models of machine teaching is that the teacher and learner should not *collude*. The benchmark for preventing this is the *Goldman-Mathias (GM) collusion-avoidance criterion* Goldman and Mathias (1996), which essentially demands a teaching map T to admit a learner that returns the concept C whenever it is shown any set of labeled examples that include $T(C)$ and are consistent with C. Recently, a batch teaching model called *non-clashing teaching* (*NC-teaching*) was proposed Fallat et al. (2023); Kirkpatrick et al. (2019). Given a concept class $\mathcal{C}$, a TM T on $\mathcal{C}$ is *non-clashing* if, for any two distinct concepts $C, C' \in \mathcal{C}$, either $T(C)$ is not consistent with C' or $T(C')$ is not consistent with C, or both. They proved NC-teaching to be the most efficient model (in terms of the worst-case number of examples required) satisfying the GM collusion-avoidance criterion. It is common to restrict the teacher to only presenting positive examples (see Angluin (1980a,b) for early successes of this approach). These models are vastly studied due to their pertinence in, *e.g.*, grammatical inference Denis (2001); Stolcke and Omohundro (1994), computational biology Wang et al. (2006); Yousef et al. (2008), and recommendation systems Schwab et al. (2000). For these reasons, Fallat et al. (2023); Kirkpatrick et al. (2019) also introduced and studied *positive NC-teaching*, in which the teacher may only use positive examples.

As with PAC-learning, where the VC-dimension $\mathrm{VCD}(\mathcal{C})$ of $\mathcal{C}$ drives the number of randomly chosen examples that are sufficient to learn the concepts of $\mathcal{C}$, various models of machine teaching lead to different notions of teaching dimension which bound the teaching set sizes. The definitive teaching dimension (DTD) Goldman and Kearns (1995); Shinohara and Miyano (1991) is a prototypical one. A *definitive teaching set (DTS)* of a concept $C \in \mathcal{C}$ is a $\mathcal{C}$-sample (a sample consistent with a concept of $\mathcal{C}$) for which C is the only consistent concept in $\mathcal{C}$. $\mathrm{DTD}(\mathcal{C})$ is the maximum size of a DTS over all $C \in \mathcal{C}$. Note also the important recursive teaching dimension (RTD) Zilles et al. (2008, 2011). NC-teaching and positive NC-teaching also have dimension parameters. The size of a TM T on $\mathcal{C}$ is the maximum size of $T(C)$ over all $C \in \mathcal{C}$. The *non-clashing teaching dimension* $\mathrm{NCTD}(\mathcal{C})$ (*positive non-clashing teaching dimension* $\mathrm{NCTD}^+(\mathcal{C})$, resp.) is the minimum size of a non-clashing TM (NCTM) for $\mathcal{C}$ (positive NCTM (NCTM^+) for $\mathcal{C}$, resp.) Fallat et al. (2023); Kirkpatrick et al. (2019). An important research direction for various notions of teaching dimension is their relationship with the VC-dimension. For NC-teaching, Fallat et al. (2023); Kirkpatrick et al. (2019) say that "The most fundamental open question resulting from our paper is probably whether NCTD is upper-bounded by VCD in general", and Simon (2023) also mentions this open question.

NCTMs are signed versions of representation maps (RMs) for concept classes. Kuzmin and Warmuth (2007) introduced RMs to design unlabeled sample compression schemes (SCSs) for maximum concept classes. They proved that the existence of RMs of size $d = \mathrm{VCD}(\mathcal{C})$ for a maximum class $\mathcal{C}$ is equivalent to the existence of unlabeled SCSs of size d for $\mathcal{C}$. Chalopin et al. (2022) generalized this equivalence to ample classes, and they also constructed RMs of size d for maximum classes. The main difference between NCTMs and RMs is that RMs assign to each concept $C \in \mathcal{C}$ a set shattered by $\mathcal{C}$. Littlestone and Warmuth (1986) introduced SCSs, which have been vastly studied due to their importance in computational machine learning.

In this paper, we consider NCTMs and NCTM^+s for the concept class $\mathcal{B}(G)$ consisting of *all balls of a graph* G. For several graph classes, we derive NCTMs for $\mathcal{B}(G)$ of size proportional to $\mathrm{VCD}(\mathcal{B}(G))$. Further, we study the computational complexity of the following decision problems:

NCTD FOR BALLS IN GRAPHS (B-NCTD)	
Input:	A graph G on n vertices and a positive integer k.
Question:	Is $\mathrm{NCTD}(\mathcal{B}(G)) \leq k$?

NCTD^+ FOR BALLS IN GRAPHS (B-NCTD^+)	
Input:	A graph G on n vertices and a positive integer k.
Question:	Is $\mathrm{NCTD}^+(\mathcal{B}(G)) \leq k$?

Motivation for balls in graphs. The *combinatorial* and *geometric* aspects of balls in graphs motivate B-NCTD and B-NCTD^+. The combinatorial one ensures that the study of balls in graphs is as general as that of arbitrary concept classes. Notably, to any set-family $\mathcal{C} \subseteq 2^V$, one can associate a set of balls of a graph G as follows. $V(G) = V \cup \{x_C : C \in \mathcal{C}\}$, the vertices of $\{x_C : C \in \mathcal{C}\}$ form a clique, and x_C and $v \in V$ are adjacent if and only if $v \in C$. For any $C \in \mathcal{C}$, $B_1(x_C) = C \cup \{x_{C'} : C' \in \mathcal{C}\}$. On the other hand, one may hope that for graphs G with a rich metric structure, the geometric structure of $\mathcal{B}(G)$ may allow to efficiently construct NCTMs, which our results confirm. Further, in light of the open question of Fallat et al. (2023); Kirkpatrick et al. (2019); Simon (2023), $\mathcal{B}(G)$ may provide graph classes where $\mathrm{NCTD}(\mathcal{B}(G)) > \mathrm{VCD}(\mathcal{B}(G))$, *e.g.*, we prove that, for trees of cycles, $\mathrm{NCTD}(\mathcal{B}(G)) \leq 4$, while $\mathrm{VCD}(\mathcal{B}(G)) \leq 3$. Trees of cycles are planar, and, for planar graphs, $\mathrm{VCD}(\mathcal{B}(G)) \leq 4$ Bousquet and Thomassé (2015); Chepoi et al. (2007), but it is unclear how to bound $\mathrm{NCTD}(\mathcal{B}(G))$ by a small constant (it is at most 615 as, for any set-family $\mathcal{C}$ with $\mathrm{VCD}(\mathcal{C}) = d$, $\mathrm{NCTD}(\mathcal{C}) \leq \mathrm{RTD}(\mathcal{C}) \leq 39.3752d^2 - 3.633d$ Fallat et al. (2023); Hu et al. (2017)). Lastly, SCSs for balls in graphs were studied in Chalopin et al. (2023).

Our Results. Our focus is twofold: we show that 1) B-NCTD^+ is computationally hard and exhibits rare properties from parameterized complexity, and 2) for several graph classes, we derive NCTMs of size proportional to $\mathrm{VCD}(\mathcal{B}(G))$. We begin with the first direction, proving that:

1. B-NCTD^+ is NP-complete in split and co-bipartite graphs with a universal vertex, and bipartite graphs of diameter 3.

Note that Kirkpatrick et al. (2019) proved that it is NP-hard to decide, for a concept class $\mathcal{C}$, whether $\mathrm{NCTD}(\mathcal{C}) = k$ or $\mathrm{NCTD}^+(\mathcal{C}) = k$, even if $k = 1$, but their results do not apply to $\mathcal{B}(G)$ as they rely on the fact that deciding whether $\mathrm{NCTD}^+(\mathcal{C}) = 1$ is NP-hard.[1]

B-NCTD^+ being NP-hard in these graph classes motivates studying its parameterized complexity, as was done for other problems in learning theory Brand et al. (2023); Downey et al. (1993); Ganian and Korchemna (2021); Li and Liang (2018). This leads to our first *main result*, which exhibits the extreme computational complexity of the problem. Recently, Foucaud et al. (2024a) developed

1. $\mathrm{NCTD}^+(\mathcal{B}(G)) = 1$ if and only if G is edgeless. Let $uv \in E(G)$. $T(B_1(v))$ must contain a vertex in $N(v)$ as $T(B_0(v)) \subset \{v\}$, say $u \in T(B_1(v))$. Then, $T(B_1(v))$ must contain another vertex in $N[v]$ as $T(B_0(u)) \subset \{u\}$.

a technique to prove double-exponential dependence on the treewidth ($\texttt{tw}$) and/or the vertex cover number ($\texttt{vc}$) of the graph in the running time of FPT algorithms for problems in NP. For these classic structural parameters, they proved that, unless the ETH fails,[2] even on bounded-diameter graphs METRIC DIMENSION and GEODETIC SET do not admit $2^{2^{o(\texttt{tw})}} \cdot n^{\mathcal{O}(1)}$-time algorithms, STRONG METRIC DIMENSION does not admit a $2^{2^{o(\texttt{vc})}} \cdot n^{\mathcal{O}(1)}$-time algorithm, and these bounds are tight. Notably, these were the *only* problems in NP known to admit such double-exponential lower bounds, *until now*.[3] Applying this technique, we obtain our first *main result*:

2. Unless the ETH fails, B-NCTD$^+$ does not admit a $2^{2^{o(\texttt{vc})}} \cdot n^{\mathcal{O}(1)}$-time algorithm, even in diameter-3 graphs.

Our lower bound is robust as all the traditional structural parameters like treewidth, pathwidth, and treedepth are at most $\texttt{vc} + 2$. Hence, B-NCTD$^+$ is only the *second* (*fourth*, resp.) problem in NP to admit a double-exponential dependence in $\texttt{vc}$ ($\texttt{tw}$, resp.). Thus, B-NCTD$^+$ is also incredibly interesting from a purely theoretical perspective, and it is strongly linked with other metric graph problems. The same reduction yields two more results, the first of which is also extremely rare:

3. Unless the ETH fails, B-NCTD$^+$ does not admit a kernelization algorithm outputting a kernel with $2^{o(\texttt{vc})}$ vertices, even in diameter-3 graphs.

4. Unless the ETH fails, B-NCTD$^+$ does not admit a $2^{o(n)}$-time algorithm, even in diameter-3 graphs.

Indeed, such ETH-based conditional lower bounds on the number of vertices in a kernel are very rare as they were only priorly known for five other problems Chandran et al. (2016); Cygan et al. (2016); Foucaud et al. (2024a,b). We show that our lower bounds concerning $\texttt{vc}$ are tight by giving matching upper bounds:

5. B-NCTD$^+$ admits a $2^{2^{\mathcal{O}(\texttt{vc})}} \cdot n^{\mathcal{O}(1)}$-time algorithm.

6. B-NCTD$^+$ admits a kernelization algorithm outputting a kernel with $2^{\mathcal{O}(\texttt{vc})}$ vertices.

For B-NCTD$^+$, we also give a $2^{\mathcal{O}(n^2 \cdot \mathrm{diam})}$-time algorithm, yielding a $2^{\mathcal{O}(n^2)}$-time algorithm in bounded-diameter graphs. We then focus on our second goal: designing NCTMs for $\mathcal{B}(G)$ that are linear in $\mathrm{VCD}(\mathcal{B}(G))$, when G is restricted to certain graph classes. Proving that any ball in a tree or interval graph can be distinctly represented by two of its "farthest apart" vertices, we show that:

7. If G is a tree or an interval graph, then $\mathrm{NCTD}(\mathcal{B}(G)) \leq \mathrm{NCTD}^+(\mathcal{B}(G)) = \mathrm{VCD}(\mathcal{B}(G)) \leq 2$.

In contrast to trees and interval graphs, we prove that:

8. Cycles do not admit NCTM$^+$s of fixed size for $\mathcal{B}(G)$, but do admit NCTMs of size 2.

With this in mind, we search for NCTMs for $\mathcal{B}(G)$ for richer graph classes. This already proves difficult in trees of cycles, for which, by a technical proof, we get the following:

9. If G is a tree of cycles, then $\mathrm{NCTD}(\mathcal{B}(G)) \leq 4$, while $\mathrm{VCD}(\mathcal{B}(G)) \leq 3$.

In analogy to PAC-learning, in approximate NCTM$^+$s, only pairs of balls with Hausdorff distance larger than some constant must satisfy the non-clashing condition. Akin to our method in trees, we show that:

10. If G is a δ-hyperbolic graph, then $\mathcal{B}(G)$ admits a 2δ-approximate NCTM$^+$ of size 2.

2. Roughly, the Exponential Time Hypothesis (ETH) states that n-variable 3-SAT cannot be solved in time $2^{o(n)}$.
3. After a preprint of this paper appeared on arXiv, tight double-exponential dependence on the treewidth was also shown for the NP-complete problems TEST COVER and LOCATING-DOMINATING SET Chakraborty et al. (2024).

2. Preliminaries

This section consists of definitions and notation. For the ball $B_r(x)$, $T(x,r)$ denotes $T(B_r(x))$. *Omitted proofs of theorems and sketches of proofs (marked by $\star$) are in the appendix.*

Concept classes and samples. In machine learning, a *concept class* on a set V is any collection $\mathcal{C}$ of subsets of V. The *VC-dimension* $\mathrm{VCD}(\mathcal{C})$ of $\mathcal{C}$ is the size of a largest set $S \subseteq V$ shattered by $\mathcal{C}$, that is, such that $\{C \cap S : C \in \mathcal{C}\} = 2^S$. A *sample* is a set $X = \{(x_1, y_1), \dots, (x_m, y_m)\}$, where $x_i \in V$ and $y_i \in \{-1, +1\}$. A sample X is *realizable by a concept* $C \in \mathcal{C}$ if $y_i = +1$ when $x_i \in C$, and $y_i = -1$ when $x_i \notin C$. A sample X is a $\mathcal{C}$*-sample* if X is realizable by some concept C in $\mathcal{C}$. To encode concepts of a concept class $\mathcal{C}$ on V and $\mathcal{C}$-samples, we use the language of sign vectors from oriented matroids theory Björner et al. (1993). Let $\mathcal{L}$ be a non-empty *set of sign vectors*, *i.e.*, maps from V to $\{\pm 1, 0\} := \{-1, 0, +1\}$. For $X \in \mathcal{L}$, let $X^+ := \{v \in V : X_v = +1\}$ and $X^- := \{v \in V : X_v = -1\}$. The set $\underline{X} := X^+ \cup X^-$ is called the *support* of X. We denote by $\preceq$ the product ordering on $\{\pm 1, 0\}^V$ relative to the ordering of signs with $0 \preceq -1$ and $0 \preceq +1$. Any $\mathcal{C} \subseteq 2^V$ can be viewed as a set of sign vectors of $\{\pm 1\}^V$: each concept $C \in \mathcal{C}$ is encoded by the sign vector $X(C)$, where $X_v(C) = +1$ if $v \in C$ and $X_v(C) = -1$ if $v \notin C$. In what follows, we consider $\mathcal{C}$ simultaneously as a collection of sets and as a set of $\{\pm 1\}$-vectors. From the definition of a sample X, it follows that X is just a sign vector and that the samples realizable by a concept $C \in \mathcal{C}$ are all $X \in \{\pm 1, 0\}^V$ such that $X \preceq C$.

NCTMs and NCTD. For a concept class $\mathcal{C}$ on V, a TM T associates, to each $C \in \mathcal{C}$, a realizable sample $T(C)$ for C (the *teaching set* of C), *i.e.*, $T(C) \in \{\pm 1, 0\}^V$ and $T(C) \preceq C$. Rephrasing the original definitions of NCTMs and RMs, a TM $T : \mathcal{C} \to \{\pm 1, 0\}^V$ is *non-clashing* if whenever $T(C') \preceq C$ and $T(C) \preceq C'$ for $C, C' \in \mathcal{C}$, then $C = C'$. Equivalently, T is non-clashing if, for any two distinct concepts C, C' of $\mathcal{C}$, the *non-clashing condition* holds: for all $C, C' \in \mathcal{C}$ with $C \neq C'$, $C|_{(\underline{T}(C) \cup \underline{T}(C'))} \neq C'|_{(\underline{T}(C) \cup \underline{T}(C'))}$. If T also satisfies the *inclusion condition*: $\underline{T}(C) = T^+(C) \subseteq C^+$ for any $C \in \mathcal{C}$, then T is an NCTM^+. The *size* of a TM T for $\mathcal{C}$ is $\max\{|\underline{T}(C)| : C \in \mathcal{C}\}$. NCTD($\mathcal{C}$) (NCTD$^+$($\mathcal{C}$), resp.) is the minimum size of an NCTM for $\mathcal{C}$ (NCTM^+ for $\mathcal{C}$, resp.).

Graphs. In this paper, graphs are simple, connected, and undirected, and logarithms are to the base 2. For a positive integer k, $[k] := \{1, \dots, k\}$. Given a graph G, its vertex set is $V(G)$ and its edge set is $E(G)$. The *distance* $d_G(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest (u, v)-path in G. For any $r \in \mathbb{N}$ and $u \in V(G)$, the *ball of radius r centered at u* is $B_r(u) := \{v : d_G(u, v) \leq r\}$. For any $u \in V(G)$, $B_1(u) = N_G[u]$ and $N_G(u) := N_G[u] \setminus \{u\}$. Two balls are *distinct* if they are distinct as sets. $\mathcal{B}(G)$ is the set of all distinct balls of G. For $S \subseteq V(G)$, the *diameter* of S is $\mathrm{diam}(S) := \max_{u,v \in S} d_G(u, v)$, and a *diametral pair* of S is a pair $u, v \in S$ such that $d_G(u, v) = \mathrm{diam}(S)$. The *diameter* of G is $\mathrm{diam}(G) := \mathrm{diam}(V(G))$. For any $u, v \in V(G)$, the *interval* $I_G(u, v)$ between u and v in G is the set of vertices on a shortest path between u and v in G. The *vertex cover number* $\mathtt{vc}(G)$ of G is the minimum number of vertices that are incident to all edges of G. When the context is clear, G is omitted from some of these notations.

Parameterized complexity. An instance of a parameterized problem π consists of an input I of the non-parameterized problem and a parameter $k \in \mathbb{N}$. A *kernelization* algorithm for π transforms, in polynomial time, an instance (I, k) of π into an *equivalent* instance (I', k') of π with $|I'|, k' \leq f(k)$, for a computable function f. A reduction rule is *safe* if the input instance is a YES-instance if and only if the output instance is a YES-instance. See Cygan et al. (2015) for a book on the topic.

Examples. To illustrate the notion, we present NCTMs for some concept classes in graphs.

Example 1 *Let G be any graph of diameter 2 such that, for each edge $xy \in E(G)$, $B_1(x) \cup B_1(y) = V(G)$. Examples are the complete bipartite graph $K_{n,m}$ and the n-*octahedron*, which is the complete graph K_{2n} on $2n$ vertices minus a perfect matching. We define an NCTM T for $\mathcal{B}(G)$ of size 2 as follows:*

- *for all $x \in V(G)$, set $T^+(x,0) := \{x\}$ and $T^-(x,0) := \{y\}$ for some neighbor y of x;*
- *for all $x \in V(G)$ such that $V(G) \setminus B_1(x) \neq \varnothing$, set $T^+(x,1) := \{x\}$ and $T^-(x,1) := \{z\}$ for some vertex z at distance 2 from x. Also set $T^+(V(G)) := \varnothing$ and $T^-(V(G)) := \varnothing$.*

We show that T is non-clashing. For any $x \in V(G)$ and $r \in \{0,1\}$, if $B_r(x) \neq V(G)$, then $T^-(x,r) \neq \varnothing$, and thus, T is non-clashing for $B_r(x)$ and $V(G)$. Consider a ball $B_0(x)$ and let $\underline{T}(x,0) = \{x,y\}$. For any ball $B' \neq B_0(x)$ such that $x \in B'$ and $y \notin B'$, we have that $B' = B_1(y')$ for some neighbor y' of x distinct from y. Since $y' \in T^+(y',1) \setminus B_0(x)$, T is non-clashing for $B_0(x)$ and any other ball. Consider now two balls $B_1(x)$ and $B_1(y)$ such that $B_1(x) \neq V(G)$ and $B_1(y) \neq V(G)$. If $d(x,y) = 2$, then $x \in T^+(x,1) \setminus B_1(y)$ and T is non-clashing for $B_1(x)$ and $B_1(y)$. Suppose now that x and y are adjacent and let $\underline{T}(x,1) = \{x,z\}$. Then, z is adjacent to y, and thus, T is non-clashing for $B_1(x)$ and $B_1(y)$.

There is no NCTM^+ of constant size for the example above, even for the n-dimensional octahedron G. Indeed, for any $x \in V(G)$, there is a unique $\bar{x} \in V(G) \setminus B_1(x)$. Thus, in order to distinguish $B_1(x) = V(G) \setminus \{\bar{x}\}$ and $V(G)$, we must have $\bar{x} \in T^+(V(G))$, and thus, $|T^+(V(G))| = |V(G)|$.

Our second example is the concept class $\mathfrak{C}_5$ that does not come from the family of balls of a graph. It was given in Pálvölgyi and Tardos (2020) as an example of a concept class with VC-dimension 2 that does not admit an unlabeled sample compression scheme of size 2.

Example 2 *The ground set of $\mathfrak{C}_5$ is the vertex set of the 5-cycle C_5 (which we will suppose to be oriented counterclockwise). The concepts of $\mathfrak{C}_5$ are of two types: the sets $\{u,v\}$ of size 2 such that u and v are not adjacent in C_5, and the sets $\{x,y,z\}$ defining a path of length 2 in C_5. If $C = \{x,y,z\}$ is a path of length 2 in C_5 ordered counterclockwise, we call y the* middle vertex *of C, and xy the* first edge *of C. Consider the following NCTM^+ T of size 2: for any concept $C = \{u,v\}$ of size 2, set $T(C) := C = \{u,v\}$; for any concept $C = \{x,y,z\}$ of size 3, set $T(C) := \{x,y\}$.*

We show that T is non-clashing. Any two concepts C, C' of size 2 are clearly distinguished by their traces on $T(C) \cup T(C')$. Analogously, any two concepts C, C' of size 3 have different first edges, and thus, are distinguished by their traces on $T(C) \cup T(C')$. Finally, any concepts $C = \{u,v\}$ of size 2 and $C' = \{x,y,z\}$ of size 3 are distinguished by their traces on $T(C) \cup T(C')$ since $T(C) = C = \{u,v\} \neq \{x,y\} = T(C')$ as u and v are not adjacent in C_5, while x and y are.

3. B-NCTD$^+$ is NP-complete

In this section, we prove that B-NCTD$^+$ is NP-complete for split, co-bipartite, and bipartite graphs. We reduce from the well-known NP-hard SET COVER problem defined as follows: given a set of elements $X = \{1, \ldots, n\}$, a family $\mathcal{S} = \{S_1, \ldots, S_m\}$ of subsets of X that covers X (*i.e.*, whose union is X), and a positive integer k, does there exist $\mathcal{S}' \subset \mathcal{S}$ such that $\mathcal{S}'$ covers X and $|\mathcal{S}'| \leq k$?

Theorem 1 B-NCTD$^+$ *is* NP*-complete in split and co-bipartite graphs with a universal vertex, and bipartite graphs of diameter 3.*

Proof ($\star$) The problem is in NP as any NCTM$^+$ for $\mathcal{B}(G)$ has a set of at most n^2 distinct balls as a domain. We sketch the proof for split graphs. The proof for co-bipartite graphs is similar, while the one for bipartite graphs is more involved.

Let ϕ be an instance of SET COVER in which each element of X is in at most $m-2$ sets of $\mathcal{S}$. From ϕ, we construct the graph G as follows. Add the sets of vertices $V = \{v_1, \ldots, v_n\}$, $S = \{s_1, \ldots, s_m\}$, $U = \{u_1, \ldots, u_{m+1}\}$, and $W = \{w_1, \ldots, w_m\}$. For all $i \in [n]$ and $j \in [m]$, if $i \notin S_j$ in ϕ, then add the edge $v_i s_j$. For all $j, \ell \in [m]$ with $j \neq \ell$, add the edges $u_j w_\ell$ and $u_{m+1} w_\ell$. Make each vertex in U adjacent to each vertex in S. Make each vertex in V adjacent to each vertex in W. Lastly, make the vertices in $U \cup V$ form a clique. We prove that ϕ admits a set cover of size at most t if and only if there is an NCTM$^+$ of size at most $k = m + t$ for $\mathcal{B}(G)$.

Suppose that ϕ admits a set cover $\mathcal{S}' \subset \mathcal{S}$ of size at most t. Let $S' \subset S$ be such that, for all $j \in [m]$, $s_j \in S'$ if and only if $S_j \in \mathcal{S}'$ in ϕ. We define an NCTM$^+$ T of size at most k for $\mathcal{B}(G)$. Note that we only need to define T for balls of G of radius 0 or 1 as, for all $x \in V(G)$, $B_2(x) = B_1(u_{m+1}) = V(G)$. For all $x \in V(G)$, set $T(x, 0) := \{x\}$. For all $x \in V$, set $T(x, 1) := B_1(x) \cap S$. For all $x \in W \cup S$, set $T(x, 1) := \{x, u_{m+1}\}$. For all $x \in U$, set $T(x, 1) := S' \cup (B_1(x) \cap W)$. Clearly, T has size at most k and satisfies the inclusion condition. One can also check that T is non-clashing. Thus, T is an NCTM$^+$ of size at most k for $\mathcal{B}(G)$.

Now, suppose that ϕ does not admit a set cover of size at most t. For all $i \in [n]$ and $j \in [m]$, $B_1(u_{m+1}) = B_1(u_j) \cup \{w_j\}$, $B_1(v_i) \subset B_1(u_{m+1})$, and $(B_1(u_{m+1}) \setminus B_1(v_i)) \subset S$. Hence, for any NCTM$^+$ T for $\mathcal{B}(G)$, $W \subseteq T(u_{m+1}, 1)$ and $T(u_{m+1}, 1) \cap S$ corresponds to a set cover. Consequently, $|T(u_{m+1}, 1) \cap S| > t$, and thus, $|T(u_{m+1}, 1)| > k$ for any NCTM$^+$ T for $\mathcal{B}(G)$. ■

4. Tight bounds for parameterizations by the vertex cover number

In this section, we consider B-NCTD$^+$ parameterized by the vertex cover number $\texttt{vc}$ of G. For any $x \in V(G)$ and $r \in \mathbb{N}$, there are at most 2^n possibilities for $T(x, r)$, and there are at most $n \cdot \text{diam}$ unique balls in G (as it is connected). Thus, we get the following algorithm that will be needed later.

Proposition 2 B-NCTD$^+$ *and* B-NCTD *admit algorithms running in time* $2^{\mathcal{O}(n^2 \cdot \text{diam}(G))}$.

We use a recently introduced technique from Foucaud et al. (2024a) to prove the following:

Theorem 3 *Unless the ETH fails, even in graphs of diameter* 3*,* B-NCTD$^+$ *does not admit*

- *an algorithm running in time* $2^{2^{o(\texttt{vc})}} \cdot n^{\mathcal{O}(1)}$*, nor*
- *a kernelization algorithm outputting a kernel with* $2^{o(\texttt{vc})}$ *vertices, nor*
- *an algorithm running in time* $2^{o(n)}$.

Using this technique, we prove Theorem 3 via a reduction from 3-PARTITIONED-3-SAT, introduced in Lampis et al. (2023) and defined as follows. Given a 3-CNF formula ϕ and a partition of its variables into three disjoint sets X^α, X^β, X^γ such that $|X^\alpha| = |X^\beta| = |X^\gamma| = N$ and no clause contains more than one variable from any of X^α, X^β, and X^γ, is ϕ satisfiable? The crux of the technique is to replace edges between clause and variable vertices by a "small" separator, called a set representation gadget, that encodes these relationships. The proof of (Lampis et al., 2023, Theorem 3) along with the Sparsification Lemma Impagliazzo et al. (2001) implies the following:

Proposition 4 *Unless the ETH fails,* 3-PARTITIONED-3-SAT *does not admit an algorithm running in time $2^{o(M)}$, where M is the number of clauses.*

Set representation gadget. Let p be the smallest integer such that $3M \leq \binom{2p}{p}$, and observe that $p = \mathcal{O}(\log M)$. Let $\mathcal{F}_p$ be the collection of subsets of $[2p]$ that contain exactly p integers. Define $\mathsf{set\text{-}rep} : [3M] \to [\mathcal{F}_p]$ as a one-to-one function by arbitrarily assigning a set in $\mathcal{F}_p$ to each integer in $[3M]$. Consider the variables from X^α in ϕ. For each variable x_i^α ($i \in [N]$) in X^α in ϕ, there are two vertices t_{2i}^α and f_{2i-1}^α corresponding to the positive and negative literals of x_i^α, respectively. For each clause C_j ($j \in [M]$) in ϕ, there is a clause vertex c_j. Add a set of vertices $V^\alpha = \{v_1^\alpha, \ldots, v_{2p}^\alpha\}$. For each $i \in [N]$, add the edge $t_{2i}^\alpha v_{p'}^\alpha$ for each $p' \in \mathsf{set\text{-}rep}(2i)$. Similarly, for each $i \in [N]$, add the edge $f_{2i-1}^\alpha v_{p'}^\alpha$ for each $p' \in \mathsf{set\text{-}rep}(2i-1)$. Now, for all $i \in [N]$ and $j \in [M]$, if the variable x_i^α appears as a positive (negative, resp.) literal in the clause C_j in ϕ, then add the edge $c_j v_{p'}^\alpha$ for each $p' \in [2p] \setminus \mathsf{set\text{-}rep}(2i)$ ($p' \in [2p] \setminus \mathsf{set\text{-}rep}(2i-1)$, resp.). For all $j \in [M]$, if no variable from X^α appears in C_j in ϕ, then make c_j adjacent to all the vertices in V^α. See Fig. 1 (right). As a clause contains at most one variable from X^α in ϕ, t_{2i}^α (f_{2i-1}^α, resp.) and c_j do not share a common neighbor in V^α if and only if the clause C_j contains x_i^α as a positive (negative, resp.) literal in ϕ. We exploit this for the reduction, and since $p = \mathcal{O}(\log M)$, this ensures that $\mathtt{vc}(G) = \mathcal{O}(\log M)$.

Reduction. Let ϕ be an instance of 3-PARTITIONED-3-SAT on $3N$ variables and $M = \mathcal{O}(N)$ clauses such that $M > N$. For all $\delta \in \{\alpha, \beta, \gamma\}$, let the variables in X^δ be $x_1^\delta, \ldots, x_N^\delta$. From ϕ, we construct the graph G as follows. Add the vertex sets $C = \{c_1, \ldots, c_M\}$, $W = \{w_1, \ldots, w_{3M}\}$, and $U = \{u_1, \ldots, u_{3M}\}$, and the vertices u_{3M+1}, u'_{3M+1}, and z. For all $\delta \in \{\alpha, \beta, \gamma\}$ and $i \in [N]$, add the vertices t_{2i}^δ and f_{2i-1}^δ, and let $A^\delta = \{t_{2i}^\delta \mid i \in [N]\} \cup \{f_{2i-1}^\delta \mid i \in [N]\}$. For all $\delta \in \{\alpha, \beta, \gamma\}$, add two independent sets of $2p$ vertices $V^\delta = \{v_1^\delta, \ldots, v_{2p}^\delta\}$ and $V^{\delta,*} = \{v_1^{\delta,*}, \ldots, v_{2p}^{\delta,*}\}$. Also, add a clique of $2p$ vertices $V^W = \{v_1^W, \ldots, v_{2p}^W\}$. For all $i \in [N]$ and $\delta \in \{\alpha, \beta, \gamma\}$, add the edge $t_{2i}^\delta v_{p'}^\delta$ ($f_{2i-1}^\delta v_{p'}^\delta$, resp.) for all $p' \in \mathsf{set\text{-}rep}(2i)$ ($p' \in \mathsf{set\text{-}rep}(2i-1)$, resp.). For all $i \in [N]$, $j \in [M]$, and $\delta \in \{\alpha, \beta, \gamma\}$, if the variable x_i^δ appears as a positive (negative, resp.) literal in the clause C_j in ϕ, then add the edge $c_j v_{p'}^\delta$ for all $p' \in [2p] \setminus \mathsf{set\text{-}rep}(2i)$ ($p' \in [2p] \setminus \mathsf{set\text{-}rep}(2i-1)$, resp.). For all $j \in [M]$ and $\delta \in \{\alpha, \beta, \gamma\}$, if no variable from X^δ appears in C_j in ϕ, then make c_j adjacent to each vertex in V^δ. For all $\ell \in [3M]$, add the edge $w_\ell v_{p'}^W$ ($u_\ell v_{p'}^W$, resp.) for all $p' \in \mathsf{set\text{-}rep}(\ell)$ ($p' \in [2p] \setminus \mathsf{set\text{-}rep}(\ell)$, resp.).

Now, to ensure that, for all $i \in [N]$ and $\delta \in \{\alpha, \beta, \gamma\}$, exactly one of t_{2i}^δ and f_{2i-1}^δ is in $T(V(G))$ (each variable is assigned exactly one truth value), do the following: (i) Add a clause vertex c_i^δ, and let $C^\delta = \{c_i^\delta \mid i \in [N]\}$. (ii) Add the edges $t_{2i}^\delta v_{p'}^{\delta,*}$ and $f_{2i-1}^\delta v_{p'}^{\delta,*}$ for all $p' \in \mathsf{set\text{-}rep}(2i)$. Then, t_{2i}^δ and f_{2i-1}^δ have the same neighbors in $V^{\delta,*}$. (iii) Add the edge $c_i^\delta v_{p'}^{\delta,*}$ for all $p' \in [2p] \setminus \mathsf{set\text{-}rep}(2i)$. Note that c_i^δ and t_{2i}^δ (f_{2i-1}^δ, resp.) have no common neighbors in $V^{\delta,*}$. One can think of c_i^δ as a clause $(x_i^\delta \vee \overline{x}_i^\delta)$. (iv) Make c_i^δ adjacent to each vertex in $V^{\delta',*}$ for all $\delta' \in \{\alpha, \beta, \gamma\}$ such that $\delta' \neq \delta$.

Lastly, make z adjacent to each vertex in $U \cup A^\alpha \cup A^\beta \cup A^\gamma \cup \{u_{3M+1}, u'_{3M+1}\}$, u_{3M+1} adjacent to each vertex in $V(G) \setminus (U \cup A^\alpha \cup A^\beta \cup A^\gamma)$, and u'_{3M+1} adjacent to each neighbor of u_{3M+1}. See Fig. 1 (left). The reduction returns (G, k) as an instance of B-NCTD$^+$ where $k = 3N + 3M$.

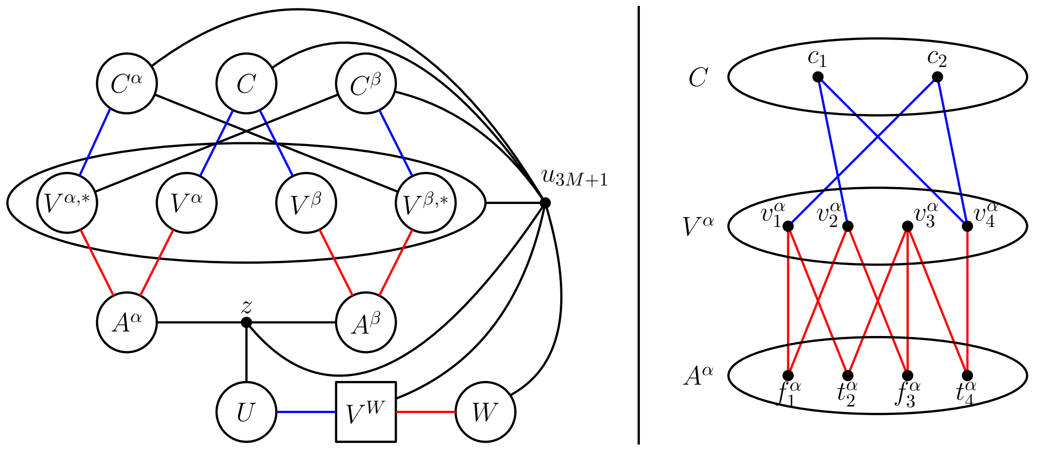

Figure 1: Graph G in the proof of Thm. 3 (left) and its sets A^α, V^α, C (right). For clarity, we omit $C^\gamma, V^\gamma, V^{\gamma,*}, A^\gamma, u'_{3M+1}$. In ϕ, x_1^α appears as a positive literal in C_1, and x_2^α as a negative literal in C_2. Red and blue edges are according to set-rep (in a complementary way).

Lemma 5 *If ϕ is satisfiable, then G admits an* NCTM^+ *for $\mathcal{B}(G)$ of size k.*

Proof ($\star$) Let $\pi : X^\alpha \cup X^\beta \cup X^\gamma \to \{\text{True}, \text{False}\}$ be a satisfying assignment for ϕ. We define the set $\pi' \subset A^\alpha \cup A^\beta \cup A^\gamma$ corresponding to π. For all $\delta \in \{\alpha, \beta, \gamma\}$ and x_i^δ in ϕ, if $\pi(x_i^\delta) = \text{True}$, then $t_{2i}^\delta \in \pi'$, and otherwise, $f_{2i-1}^\delta \in \pi'$. So, $|\pi'| = 3N$. We define an NCTM^+ T of size k for $\mathcal{B}(G)$ as follows. We need not define T for $B_2(z)$, $B_2(u'_{3M+1})$, and balls of radius at least 3 as, for all $x \in V(G)$, $B_3(x) = B_2(z) = B_2(u'_{3M+1}) = B_2(u_{3M+1}) = V(G)$.

For all $x \in V(G)$, set $T(x,0) := \{x\}$. For all $\delta \in \{\alpha, \beta, \gamma\}$ and $x \in A^\delta$, set $T(x,1) := B_1(x)$ and $T(x,2) := \{u_1, t_2^{\delta'}, t_2^{\delta''}\} \cup B_1(x)$, where $\{\delta, \delta', \delta''\} = \{\alpha, \beta, \gamma\}$. For all $\delta \in \{\alpha, \beta, \gamma\}$ and $x \in V^\delta \cup V^{\delta,*}$, set $T(x,1) := B_1(x) \setminus U$ and $T(x,2) := \{w_1, z\} \cup (B_1(x) \setminus U)$. For all $x \in V^W$, set $T(x,1) := \{u_{3M+1}, u'_{3M+1}\} \cup V^W \cup (B_1(x) \cap U)$ and $T(x,2) := \{u_{3M+1}, u'_{3M+1}, z\} \cup V^W \cup U$. For all $\delta \in \{\alpha, \beta, \gamma\}$ and $x \in C \cup C^\delta$, set $T(x,1) := B_1(x)$ and $T(x,2) := \{x, w_1\} \cup (B_2(x) \cap (A^\alpha \cup A^\beta \cup A^\gamma))$. For all $x \in U \cup \{u_{3M+1}, u'_{3M+1}\}$, set $T(x,1) := B_1(x) \setminus (C \cup C^\alpha \cup C^\beta \cup C^\gamma)$ and $T(x,2) := (B_2(x) \cap W) \cup \pi'$. For all $x \in W$, set $T(x,1) := B_1(x)$ and $T(x,2) := \{x, z, u_{3M+1}, u'_{3M+1}\} \cup (B_2(x) \cap U)$. Set $T(z,1) := \{z, u_1\}$. One can verify that T has size at most k and satisfies the inclusion condition for each ball in $\mathcal{B}(G)$ and the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$. Hence, T is an NCTM^+ of size at most k for $\mathcal{B}(G)$. ∎

Lemma 6 *If G admits an* NCTM^+ *for $\mathcal{B}(G)$ of size k, then ϕ is satisfiable.*

Proof ($\star$) Let T be an NCTM^+ for $\mathcal{B}(G)$ of size k. For all $i \in [N]$, $\ell \in [3M]$, and $\delta \in \{\alpha, \beta, \gamma\}$, as $B_2(u_{3M+1}) = V(G)$, $B_2(u_\ell) = V(G) \setminus \{w_\ell\}$, and $B_2(c_i^\delta) = V(G) \setminus \{t_{2i}^\delta, f_{2i-1}^\delta\}$, we have $|T(u_{3M+1}, 2) \cap W| = 3M$ and $|T(u_{3M+1}, 2) \cap \{t_{2i}^\delta, f_{2i-1}^\delta\}| \geq 1$. Since $k = 3N + 3M$, the latter inequality is an equality. From $T(u_{3M+1}, 2)$, we extract an assignment $\pi : X^\alpha \cup X^\beta \cup X^\gamma \to \{\text{True}, \text{False}\}$ for ϕ. For all $i \in [N]$ and $\delta \in \{\alpha, \beta, \gamma\}$, if $T(u_{3M+1}, 2) \cap \{t_{2i}^\delta, f_{2i-1}^\delta\} = \{t_{2i}^\delta\}$, then set $\pi(x_i^\delta) = \text{True}$, and otherwise, set $\pi(x_i^\delta) = \text{False}$. Thus, π assigns each variable in ϕ exactly one truth value. It is not hard to see that π is a satisfying assignment for ϕ. ∎

Proof of Theorem 3 By Lemmas 5 and 6, there is a reduction that takes an instance of 3-PARTITIONED-3-SAT and returns an equivalent instance (G, k) of B-NCTD$^+$ with $\mathrm{diam}(G) = 3$. As $|V(G)| = \mathcal{O}(M)$, unless the ETH fails, B-NCTD$^+$ does not admit a $2^{o(n)}$-time algorithm by Prop. 4. $S = \{u_{3M+1}, u'_{3M+1}, z\} \cup V^W \cup V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}$ is a vertex cover of G of size $\mathcal{O}(\log M)$. Hence, a $2^{2^{o(vc)}} \cdot n^{\mathcal{O}(1)}$-time algorithm for B-NCTD$^+$ would imply a $2^{o(M)}$-time algorithm for 3-PARTITIONED-3-SAT, contradicting the ETH by Prop. 4. Toward a contradiction, suppose that B-NCTD$^+$ admits a kernelization algorithm outputting a kernel with $2^{o(\mathtt{vc})}$ vertices. Consider the following algorithm for B-NCTD$^+$. Given an instance of 3-PARTITIONED-3-SAT with M clauses, it applies the reduction to obtain an equivalent instance (G, k) of B-NCTD$^+$ with $\mathtt{vc}(G) = \mathcal{O}(\log M)$. Then, it applies the assumed kernelization algorithm on (G, k), outputting a kernel with $2^{o(\mathtt{vc})}$ vertices. Finally, it applies the algorithm from Prop. 2 on the kernel, which takes $2^{\mathcal{O}((2^{o(\mathtt{vc})})^2 \cdot \mathrm{diam})} = 2^{o(M)}$ time, contradicting the ETH by Prop. 4. ∎

We now show that our vc lower bounds are tight:

Theorem 7 B-NCTD$^+$ *admits*

- *an algorithm running in time* $2^{2^{\mathcal{O}(\mathtt{vc})}} \cdot n^{\mathcal{O}(1)}$, *and*
- *a kernelization algorithm outputting a kernel with* $2^{\mathcal{O}(\mathtt{vc})}$ *vertices.*

Given a graph G, two vertices $u, v \in V(G)$ are *false twins* if $N(u) = N(v)$. The following reduction rule is used to design the kernelization algorithm in Theorem 7.

Reduction Rule 1 (RR1) *Given a graph G and a set $X \subseteq V(G)$ such that $I := V(G) \setminus X$ is an independent set (i.e., X is a vertex cover of G), if there exist $2^{|X|} + 2$ vertices in I that are pairwise false twins, then delete one of them.*

The idea for proving the forward direction of the next lemma is that, by the pigeonhole principle, for any set $S \subseteq I$ of $2^{|X|} + 2$ false twins, there exist $x, y \in S$ such that $x \in T(x, 1)$, $y \in T(y, 1)$, and $T(x, 1) \setminus \{x\} = T(y, 1) \setminus \{y\}$. Then, for any teaching set containing y, we can replace y by x or another vertex of S if x is already in the teaching set.

Lemma 8 *Reduction Rule 1 is safe for* B-NCTD$^+$.

Theorem 7 follows from exhaustively applying RR1 for the kernelization algorithm, and using the algorithm from Prop. 2 on the resulting kernel for the other algorithm.

5. NCTMs for classes of graphs

In this section, we construct NCTMs for balls of several simple classes of graphs: trees, interval graphs, cycles, and trees of cycles. We also design approximate NCTMs for balls in δ-hyperbolic graphs. In each of our NCTMs T, for any ball $B_r(x)$, $T^+(x, r)$ consists of two vertices of $B_r(x)$ that are "farthest apart". In each case except for (trees of) cycles, we set $T(x, r) = T^+(x, r)$.

5.1. Trees

For any ball $B_r(x)$ of a tree $\mathfrak{T}$, define $T(x,r)$ as any diametral pair $\{u,v\}$ of $B_r(x)$.

Proposition 9 *For a tree $\mathfrak{T}$, T is an NCTM^+ for $\mathcal{B}(\mathfrak{T})$, i.e., $\mathrm{NCTD}^+(\mathcal{B}(\mathfrak{T})) = \mathrm{VCD}(\mathcal{B}(\mathfrak{T})) \leq 2$.*

Proof For any ball $B_r(x)$, $T(x,r) \subseteq B_r(x)$. So, as $|T(x,r)| = 2$ for any ball $B_r(x)$ with $r \geq 1$, T is non-clashing for any pair of balls that includes a ball of radius 0. Now, suppose $B_{r_1}(x) \neq B_{r_2}(y)$ and assume that there exists $z \in B_{r_2}(y) \setminus B_{r_1}(x)$. Let $T(y,r_2) = \{u,v\}$. Then, $d(x,z)+d(u,v) \leq \max\{d(x,u)+d(v,z), d(x,v)+d(u,z)\}$, say $d(x,z)+d(u,v) \leq d(x,v)+d(u,z)$. Since $u,z \in B_{r_2}(y)$, $d(u,z) \leq d(u,v)$. Thus, $v \notin B_{r_1}(x)$ as $d(x,v) \geq d(x,z) \geq r_1$. So, T is non-clashing. ■

5.2. Interval graphs

We consider a representation of an interval graph $\mathfrak{I}$ by a set of segments J_v, $v \in V(\mathfrak{I})$, of $\mathbb{R}$ with pairwise distinct ends. For any $u \in V(\mathfrak{I})$, its segment is denoted by $J_u = [s_u, e_u]$, where s_u is the start of J_u, and e_u is the end of J_u, *i.e.*, $s_u \leq e_u$. We use the following property:

Lemma 10 *(Lemma 24, Chalopin et al. (2023)) If $u,v \in B_r(x)$, $s_u, s_z < s_v$, and $e_u < e_v, e_z$, then $z \in B_r(x)$.*

For a subgraph $\mathfrak{I}'$ of $\mathfrak{I}$, $\{u,v\}$ is a *farthest pair* of $\mathfrak{I}'$ if u is the vertex in $\mathfrak{I}'$ whose segment J_u ends farthest to the left, and v is the vertex in $\mathfrak{I}'$ whose segment J_v begins farthest to the right, *i.e.*, for any $w \in V(\mathfrak{I}') \setminus \{u,v\}$, we have $e_u < e_w$ and $s_w < s_v$. Define the map T on $\mathcal{B}(\mathfrak{I})$: for any ball $B_r(x)$ of $\mathfrak{I}$, set $T(x,r)$ to be the farthest pair $\{u,v\}$ of $B_r(x)$ if $r \geq 1$, and set $T(x,0) := \{x\}$.

Proposition 11 *For an interval graph $\mathfrak{I}$, T is an NCTM^+ for $\mathcal{B}(\mathfrak{I})$, i.e., $\mathrm{NCTD}^+(\mathcal{B}(\mathfrak{I})) = \mathrm{VCD}(\mathcal{B}(\mathfrak{I})) \leq 2$.*

Proof For any ball $B_r(x)$, $T(x,r) \subseteq B_r(x)$. So, as $|T(x,r)| = 2$ for any ball $B_r(x)$ with $r \geq 1$, T is non-clashing for any pair of balls that includes a ball of radius 0. Now, consider two balls $B_{r_1}(x)$ and $B_{r_2}(y)$ such that $T(x,r_1) = \{u,v\} \subseteq B_{r_2}(y)$. For any $z \in B_{r_1}(x)$, we have $e_z > e_u$ and $s_z < s_v$, and thus, $z \in B_{r_2}(y)$ by Lemma 10, establishing that $B_{r_1}(x) \subseteq B_{r_2}(y)$. Consequently, T sastisfies the non-clashing condition. Finally, $\mathrm{VCD}(\mathcal{B}(\mathfrak{I})) \leq 2$ Ducoffe et al. (2020). ■

5.3. Cycles

In contrast with trees and interval graphs, we prove that:

Proposition 12 *In cycles, the family of balls do not admit NCTM^+s of constant size.*

Proof Consider the cycle $\mathfrak{C}_n$ with $n = 2k+2 \geq 4$ and suppose that $\mathfrak{C}_n$ admits an NCTM^+ T of size at most k. Each ball $B_k(x)$ contains all the vertices of $\mathfrak{C}_n$ except the vertex $\overline{x}$ opposite to x in $\mathfrak{C}_n$. Hence, $T(x,k) \subset \mathfrak{C}_n \setminus \{\overline{x}\} = B_k(x)$. Since $|T(x,k)| \leq k$, $\overline{x} \in T(z,k)$ for at least $n-k-1$ vertices $z \neq x$. Thus, $\sum_{x \in \mathfrak{C}_n} |T(x,k)| \geq n(n-k-1)$. But, since $|T(x,k)| \leq k$, this sum is at most nk. Therefore, $nk \geq n(n-k-1)$, and thus, $n \leq 2k+1$, a contradiction. ■

However, any cycle $\mathfrak{C} = \mathfrak{C}_n$ with $n \geq 6$ admits an NTCM of size $2 < \mathrm{VCD}(\mathcal{B}(\mathfrak{C})) = 3$. Suppose that $\mathfrak{C}$ is oriented counterclockwise. Each ball $B_r(x)$ of $\mathfrak{C}$ either coincides with $\mathfrak{C}$ or is an arc of $\mathfrak{C}$. When $B_r(x)$ is an arc, we can speak about the *first* and *last vertices of* $B_r(x)$ in the counterclockwise order, and about the *first vertex outside* of $B_r(x)$ (this vertex is adjacent to the last vertex of $B_r(x)$). The NCTM T for $\mathcal{B}(\mathfrak{C})$ is as follows. If a ball $B_r(x)$ covers $\mathfrak{C}$, then $T(x,r) = \varnothing$. Otherwise, $T^+(x,r)$ is the first vertex of $B_r(x)$ and $T^-(x,r)$ is the first vertex outside of $B_r(x)$.

Proposition 13 *For a cycle $\mathfrak{C} = \mathfrak{C}_n$, T is an NCTM for $\mathcal{B}(\mathfrak{C})$, and if $n \geq 6$, then NCTD$(\mathcal{B}(\mathfrak{C})) =$ $2 < \mathrm{VCD}(\mathcal{B}(\mathfrak{C})) = 3$.*

Proof First, we prove that T is an NCTM for $\mathcal{B}(\mathfrak{C})$. Let $B = B_r(x)$ and $B' = B_{r'}(y)$ be two different balls of $\mathfrak{C}$. The non-clashing condition is immediate if one of the balls coincides with $\mathfrak{C}$. Therefore, suppose that both balls are arcs. Let $T(x,r) = \{u, w\}$, where u is the first vertex of $B_r(x)$ and w is the first vertex outside of $B_r(x)$. Analogously, $T(y,r') = \{u', w'\}$, where u' is the first vertex of $B_{r'}(y)$ and w' is the first vertex outside of $B_{r'}(y)$. Also, let v and v' be the last vertices of $B_r(x)$ and $B_{r'}(y)$, respectively. Since $B_r(x)$ and $B_{r'}(y)$ are arcs of $\mathfrak{C}$, they are either (1) disjoint, or (2) one is a proper subset of another, or (3) they overlap on a proper arc, or (4) they overlap on two arcs and together cover $\mathfrak{C}$. If $B_r(x)$ and $B_{r'}(y)$ are disjoint, then $u \in T^+(x,r) \setminus B_{r'}(y)$. If $B_{r'}(y) \subseteq B_r(x)$, then $B_r(x)$ and $B_{r'}(y)$ differ with respect to at least at one of the vertices u or v. If $u \neq u'$, then $u \in T^+(x,r) \setminus B_{r'}(y)$. If $u = u'$ and $v' \neq v$, then $w' \in B_r(x)$, while $w' \in T^-(y,r')$. Now, suppose that $B_r(x)$ and $B_{r'}(y)$ overlap on a proper arc of each of $B_r(x)$ and $B_{r'}(y)$. With respect to the counterclockwise order, this can be either the arc between u' and v or the arc between u and v', say the first (the other case is symmetric). In this case, $u \in T^+(x,r) \setminus B_{r'}(y)$. Finally, suppose that $B_r(x)$ and $B_{r'}(y)$ overlap on two arcs and cover $\mathfrak{C}$. These two arcs are defined by u and v' and by u' and v (in the counterclockwise order). Then, $w \in B_{r'}(y)$, while $w \in T^-(x,r)$. Thus, in all cases, T satisfies the non-clashing condition. Hence, T is an NCTM for $\mathcal{B}(\mathfrak{C})$ of size 2.

To prove the lower bound, let $k = \lfloor n/2 \rfloor$ (*i.e.*, $n \in \{2k, 2k+1\}$) and assume that $\mathfrak{C}$ admits an NCTM T of size 1. There are $kn+1$ distinct balls in $\mathfrak{C}$ (kn proper balls and $B_k(x) = \mathfrak{C}$). Since at most one ball can have an empty teaching set, there are kn balls that have teaching sets of size 1. As there are $2n$ possible teaching sets of size 1 (each vertex has sign ± 1), by the pigeonhole principle, if $kn > 2n$, then there exist two different balls B, B' with $T^+(B) = T^+(B')$ and $T^-(B) = T^-(B')$. But then T does not satisfy the non-clashing condition for B and B', contrary to the assumption that T is an NCTM. Thus, for any $n \geq 6$, there is no NCTM of size 1 for $\mathcal{B}(\mathfrak{C})$. ■

5.4. Trees of cycles (cacti)

A *tree of cycles* (or *cactus*) is a graph $\mathfrak{K}$ in which each 2-connected component is a cycle or an edge. For a vertex v of $\mathfrak{K}$ that is not a cut vertex, let $C(v)$ be the unique cycle containing v. If v is a cut vertex, then $C(v) = \{v\}$. For any vertices u, v of $\mathfrak{K}$, let $C(u,v)$ be the union of all cycles and/or edges on the unique path of $B(\mathfrak{K})$ between $C(u)$ and $C(v)$. Note that $C(u,v)$ is a path of cycles. A set $S \subseteq V(G)$ is *gated* if, for any $u \in V(G)$, there exists $u' \in S$ (the gate of u, with $u' = u$ if $u \in S$) such that $u' \in I(u,v)$ for any $v \in S$. Given a triplet x, u, v of vertices of G, a vertex y is an *apex of x with respect to u and v* if $y \in I(x,u) \cap I(x,v)$ and $I(x,y)$ is maximal with respect to the inclusion. One can easily show that for any triplet x, u, v of a tree of cycles $\mathfrak{K}$, there exists a unique apex of x with respect to u and v and that any cycle and any path of cycles of $\mathfrak{K}$ are gated. Let $\mathfrak{K}$ be

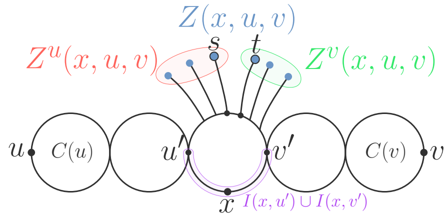

Figure 2: $Z(x,u,v)$, $Z^u(x,u,v)$, $Z^v(x,u,v)$, s, and t.

a tree of cycles and $\mathcal{B}$ its set of balls. For $B \in \mathcal{B}$, let $[B] = \{B_r(x) : B_r(x) = B\}$. We call a ball $B_r(x)$ in $[B]$ *minimal* if it has a minimal radius among all balls in $[B]$.

Lemma 14 *($\star$) If $\{u,v\}$ is a diametral pair of $B_r(x)$, x' is the apex of x with respect to u,v and $r' = r - d(x,x')$, then $B_{r'}(x') = B_r(x)$. In particular, if $B_r(x)$ is a minimal ball, then $x \in C(u,v)$.*

The NCTM. We define a map T for $\mathcal{B}$ as follows. Let $B \in \mathcal{B}$ and u,v be a diametral pair of B. We set $T^+(B) := \{u,v\}$ ($T^+(B) := B$ if $|B| = 1$). To define $T^-(B)$, let $B_r(x)$ be a minimal ball from $[B]$. By Lemma 14, x belongs to $C(u,v)$. If x is a cut vertex of $C(u,v)$, then we set $T^-(B) := \varnothing$. Otherwise, let C be the unique cycle of $C(u,v)$ containing x, and let u' and v' be the respective gates of u and v in C. For any vertex z of $\mathfrak{K}$, we denote by z' its gate in C. Consider the set $Z(x,u,v) = \{z \in V(\mathfrak{K}) : z' \notin I(x,u') \cup I(x,v') \text{ and } d(x,z) = r+1\}$. If $Z(x,u,v) = \varnothing$, then we set $T^-(B) = \varnothing$. If $Z(x,u,v) \neq \varnothing$, let $Z^u(x,u,v) = \{s \in Z(x,u,v) : u' \in I(x,s)\}$ and $Z^v(x,u,v) = \{t \in Z(x,u,v) : v' \in I(x,t)\}$. If $Z^u(x,u,v)$ ($Z^v(x,u,v)$, resp.) is not empty, pick $s \in Z^u(x,u,v)$ ($t \in Z^v(x,u,v)$, resp.) such that the distance $d(u',s')$ ($d(v',t')$, resp.) is maximized. We set $T^-(B) := \{s,t\}$ if s and t exist, and $T^-(B) := \{s\}$ or $T^-(B) := \{t\}$ if only one of s and t exists (see Fig. 2). Observe that if the cycle C is not completely included in $B_r(x)$, then s and t exist, they belong to C (possibly $s = t$), $s = s'$, $t = t'$, and $d(x,s) = d(x,t) = r+1$.

Theorem 15 *Let $B_1, B_2 \in \mathcal{B}$ be two balls of the same diameter in a tree of cycles $\mathfrak{K}$ such that, for all $q \in \underline{T}(B_1) \cup \underline{T}(B_2)$, $q \in B_1$ if and only if $q \in B_2$. Then, $B_2 = B_1$. Consequently, T is an* NCTM *of size 4 for $\mathcal{B}(\mathfrak{K})$, and thus, NCTD$(\mathcal{B}(\mathfrak{K})) \leq 4$, while* $\mathrm{VCD}(\mathcal{B}(\mathfrak{K})) \leq 3$.

Proof ($\star$) First, analogous to the case of trees, T is non-clashing for any pair of balls including a ball of radius 0. Let $B_{r_1}(x) \in [B_1]$ be a minimal ball for B_1 and u,v be the diametral pair of B_1 defining $T^+(B_1)$. Analogously, let $B_{r_2}(y) \in [B_2]$ be a minimal ball for B_2. By contradiction, assume that $B_1 \neq B_2$ and that T does not satisfy the non-clashing condition for B_1 and B_2. Without loss of generality, suppose that there exists a vertex $z \in B_2 \setminus B_1$. One can show that x cannot disconnect z and u (or v). So, x is not a cut vertex of $C(u,v)$. Hence, x belongs to a unique (gated) cycle C of $C(u,v)$. Since $\mathrm{diam}(B_1) = \mathrm{diam}(B_2)$ and $T^+(B_1) \subset B_2$, u,v is also a diametral pair of B_2.

If $z' \in I(x,u')$, one can show that $\mathrm{diam}(B_2) \geq d(v,z) > d(v,u) = \mathrm{diam}(B_1)$, a contradiction. Thus, $z' \notin I(x,u')$ and, similarly, $z' \notin I(x,v')$. Since x belongs to C, $I(x,u')$ and $I(x,v')$ intersect only in x, and their union is the (u',v')-path passing via x. By the previous assertion, z' belongs to the complementary (u',v')-path P of C and $I(x,z') \cap \{u',v'\} \neq \varnothing$. Hence, $Z(x,u,v)$ is non-empty. Indeed, let w be a vertex of $I(x,z)$ at distance r_1+1 from x. Then, either z' is the gate of w in C or w is a vertex of the path P, and so, $w \in Z(x,u,v)$. Hence, $Z(x,u,v) \neq \varnothing$, and thus, one of the vertices s,t exists. If s (t, resp.) exists, then its gate s' (t', resp.) in C belongs to P.

If $C \nsubseteq B_1$, then $s = s'$ and $t = t'$ disconnect x and any vertex of P. Since $x, z' \in B_2$ and $s, t \notin B_2$, necessarily $z' \in I(u', s') \cup I(v', t')$. If $C \subseteq B_1$, then by the definition of s and t, we also have $z' \in I(u', s') \cup I(v', t')$. Assume, w.l.o.g., that $z' \in I(u', s')$ and recall that $z' \neq u'$. Since $u' \in I(z', x) \subseteq I(s', x)$, $z' \in I(x, z)$, and $s' \in I(x, s)$, we get $d(x, z) = d(x, u') + d(u', z') + d(z', z)$ and $d(x, s) = d(x, u') + d(u', z') + d(z', s') + d(s', s)$. Since $d(x, z) \geq r_1 + 1 = d(x, s)$, we conclude that $d(z', z) \geq d(z', s') + d(s', s)$.

By Lemma 14 applied to $B_2 = B_{r_2}(y)$, we have $y \in C(u, v)$. Thus, $y = y' \in C$ or $y' \in \{u', v'\}$. So, $d(y, z) = d(y, y') + d(y', z') + d(z', z)$. As $s \notin B_2$, $r_2 < d(y, s) = d(y, y') + d(y', s') + d(s', s) \leq d(y, y') + d(y', z') + d(z', s') + d(s', s) \leq d(y, y') + d(y', z') + d(z', z) = d(y, z) \leq r_2$, a contradiction. This gives $B_1 = B_2$. The second assertion follows from the first one since two balls with distinct diameters are distinguished by diametral pairs. Finally, $\mathrm{VCD}(\mathcal{B}(\mathfrak{K})) \leq 3$ since trees of cycles $\mathfrak{K}$ cannot be contracted to K_4, and if a graph G does not contain K_{d+1} as a minor, then $\mathrm{VCD}(\mathcal{B}(G)) \leq d$ Bousquet and Thomassé (2015); Chepoi et al. (2007). ■

5.5. Hyperbolic graphs

Gromov's δ-hyperbolicity is important in metric geometry and geometric group theory, with applications in analyzing real networks. It stipulates how close, locally, the graph (or metric space) is to a tree metric. For $\delta \geq 0$, a metric space (X, d) is *δ-hyperbolic* Gromov (1987) if, for any four points u, v, x, y of X, $d(u, v) + d(x, y) \leq \max\{d(u, x) + d(v, y), d(u, y) + d(v, x)\} + 2\delta$. Two sets A and B of a metric space (X, d) are *ρ-identical* if the *Hausdorff distance* $d_H(A, B)$ between A and B is at most ρ. Otherwise, they are *ρ-distinct*. For balls, $B_{r_1}(x)$ and $B_{r_2}(y)$ are *ρ-identical* if and only if $B_{r_1}(x) \subseteq B_{r_2+\rho}(y)$ and $B_{r_2}(y) \subseteq B_{r_1+\rho}(x)$. A *$\rho$-approximate* NCTM^+ (NCTM^+_ρ) T associates, to each ball $B_r(x) \in \mathcal{B}(G)$, a set $T(x, r) \subseteq B_r(x)$ such that the non-clashing condition holds for each pair of ρ-distinct balls. For a δ-hyperbolic graph $\mathfrak{H}$ and any ball $B_r(x)$ of $\mathfrak{H}$, let $T(x, r)$ be any diametral pair of $B_r(x)$ if $r \geq 1$, and set $T(x, 0) := \{x\}$. Akin to our method in trees, we get that:

Theorem 16 *For a δ-hyperbolic graph $\mathfrak{H}$, T is an $\mathrm{NCTM}^+_{2\delta}$ of size 2 for $\mathcal{B}(\mathfrak{H})$.*

6. Further work

As is the case for general set-families, it would be interesting to know whether B-NCTD$^+$ and B-NCTD are also para-NP-hard parameterized by k. Further, it would be intriguing to know for which (other) structural parameterizations (*e.g.*, treewidth, feedback vertex number, feedback edge number, and treedepth) they are tractable, and whether $\mathrm{NCTD}(\mathcal{B}(G)) > \mathrm{VCD}(\mathcal{B}(G))$ for planar graphs. Lastly, as our NCTMs are simpler than the SCSs in Chalopin et al. (2023), it seems reasonable to study them for notable and rich metric graph classes like median and Helly graphs.

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Appendix A. Full proofs of omitted or sketched proofs

Theorem 1 B-NCTD$^+$ *is* NP*-complete in split and co-bipartite graphs with a universal vertex, and bipartite graphs of diameter 3.*

Proof The problem is in NP since any NCTM^+ T for $\mathcal{B}(G)$ has a set of at most n^2 distinct balls as a domain, and thus, it can be verified in polynomial time whether T satisfies both the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, and the inclusion condition for each ball in $\mathcal{B}(G)$. In all three cases, to prove that it is NP-hard, we give a reduction from SET COVER.

We begin with the case of split graphs. Let ϕ be an instance of SET COVER with $X = \{1, \dots, n\}$ and $\mathcal{S} = \{S_1, \dots, S_m\}$. We may also assume that ϕ is an instance in which each element of X is contained in at most $m-2$ sets of $\mathcal{S}$. Indeed, any element contained in all of the sets of $\mathcal{S}$ will be covered by any choice of the sets of $\mathcal{S}$, and so, could simply be removed from the instance. In the resulting instance, for each of the elements contained in exactly $m-1$ sets of $\mathcal{S}$, it suffices to duplicate the set not containing that element to obtain the property that any element is contained in at most $m-2$ sets of $\mathcal{S}$. From this instance ϕ, we construct the graph G as follows. For all $i \in [n]$

and $j \in [m]$, add a vertex v_i and a vertex s_j, and if $i \notin S_j$ in ϕ, then add the edge $v_i s_j$. Add the sets of vertices $U = \{u_1, \ldots, u_{m+1}\}$ and $W = \{w_1, \ldots, w_m\}$, and, for all $j, \ell \in [m]$ such that $j \neq \ell$, add the edge $u_j w_\ell$. For all $j \in [m]$, add the edge $u_{m+1} w_j$. Add edges so that every vertex in U is adjacent to every vertex in $S = \{s_1, \ldots, s_m\}$. Add edges so that every vertex in $V = \{v_1, \ldots, v_n\}$ is adjacent to every vertex in W. Lastly, add edges so that the vertices in $U \cup V$ form a clique. This completes the construction of G, which is clearly achieved in polynomial time. See Figure 3 for an illustration of G. Note that u_{m+1} is a universal vertex, and that the vertices in $W \cup S$ form an independent set, and thus, G is a split graph containing a universal vertex. We prove that ϕ admits a set cover of size at most t if and only if there is an NCTM^+ of size at most $k = m + t$ for $\mathcal{B}(G)$.

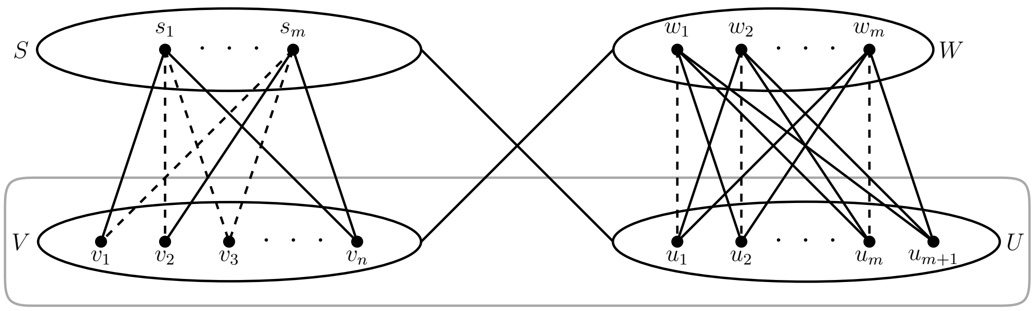

Figure 3: The split graph G constructed in the proof of Theorem 1. Vertices contained in a rectangle form a clique. An edge between two ellipses indicates that each vertex in one ellipse is adjacent to each vertex in the other. Dashed lines highlight some non-existing edges. In this example, $2, 3 \in S_1$ (but $1, n \notin S_1$) and $1, 3 \in S_m$ (but $2, n \notin S_m$) in ϕ.

First, suppose that ϕ admits a set cover of size at most t, and let $\mathcal{S}' \subset \mathcal{S}$ be such a set cover. Let $S' \subset S$ be such that, for all $j \in [m]$, $S_j \in \mathcal{S}'$ in ϕ if and only if $s_j \in S'$. We define an NCTM^+ T of size at most k for $\mathcal{B}(G)$ as follows. Also, note that we only need to define T for balls of G of radius at most 1 since, for all $x \in V(G)$, $B_2(x) = B_1(u_{m+1}) = V(G)$.

- For all $x \in V(G)$, set $T(x, 0) := \{x\}$.
- For all $x \in V$, set $T(x, 1) := B_1(x) \cap S$ and note that $|T(x, 1)| \geq 2$ since every element of X is contained in at most $m - 2$ sets of $\mathcal{S}$.
- For all $x \in W \cup S$, set $T(x, 1) := \{x, u_{m+1}\}$.
- Finally, for all $x \in U$, set $T(x, 1) := S' \cup (B_1(x) \cap W)$.

It is easy to verify that the map T has size at most k and satisfies the inclusion condition for each ball in $\mathcal{B}(G)$. We now show that T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$. For all $x \in V(G)$, $|B_0(x)| = |T(x, 0)| = 1$ and $|T(x, 1)| \geq 2$, and thus, T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$ where at least one of the balls has radius 0. For all $x \in W \cup S$, we have that $x \in T(x, 1)$, and $W \cup S$ is an independent set. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in $W \cup S$. For all $x, y \in V$, $B_1(x) \setminus S = B_1(y) \setminus S$, $T(x, 1) \cap S = B_1(x) \cap S$, and $T(y, 1) \cap S = B_1(y) \cap S$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in V. For all $x \in W \cup S$ and $y \in V$, $|B_1(x) \cap S| \leq 1$ and $|T(y,1) \cap S| \geq 2$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in $W \cup S \cup V$. For all $x, y \in U$, $B_1(x) \setminus W = B_1(y) \setminus W$, $T(x,1) \cap W = B_1(x) \cap W$, $T(y,1) \cap W = B_1(y) \cap W$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in U. For all $x \in U$ and $y \in W \cup S$, $|T(x,1) \cap W| \geq m-1$ and $B_1(y) \cap W \leq 1$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in $W \cup S \cup U$. For all $x \in V$, by the construction, we have that $B_1(x) \cap S' \neq S'$ since S' corresponds to the set cover $\mathcal{S}'$, and, for all $y \in U$, $S' \subset T(y,1)$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in $V \cup U$. Combining all this, we get that T is an NCTM^+ of size at most k for $\mathcal{B}(G)$.

Now, suppose that ϕ does not admit a set cover of size at most t. In this case, we prove that there is no NCTM^+ of size at most k for $\mathcal{B}(G)$. We first prove that $W \subseteq T(u_{m+1}, 1)$ for any NCTM^+ T for $\mathcal{B}(G)$. Indeed, for all $j \in [m]$, $B_1(u_{m+1}) = B_1(u_j) \cup \{w_j\}$, and so, to ensure that T satisfies the non-clashing condition for the pair $B_1(u_{m+1})$ and $B_1(u_j)$, we must have that $w_j \in T(u_{m+1}, 1)$. Now, we prove that $|T(u_{m+1}, 1) \cap S| > t$ for any NCTM^+ T for $\mathcal{B}(G)$ in this case, which completes the proof. Observe that, for all $i \in [n]$, $B_1(v_i) \subset B_1(u_{m+1})$ and $(B_1(u_{m+1}) \setminus B_1(v_i)) \subset S$. Hence, for each $i \in [n]$, to ensure that T satisfies the non-clashing condition for the pair $B_1(u_{m+1})$ and $B_1(v_i)$, it is necessary that $s_j \in T(u_{m+1}, 1)$ for some $j \in [m]$ such that $s_j \notin B_1(v_i)$. However, $s_j \notin B_1(v_i)$ if and only if $i \in S_j$ in ϕ. In other words, $T(u_{m+1}, 1) \cap S$ must correspond to a set cover in ϕ, and thus, $|T(u_{m+1}, 1)| > k$. This concludes the proof for split graphs.

We now proceed with the proof for co-bipartite graphs. Let ϕ be an instance of SET COVER with $X = \{1, \ldots, n\}$ and $\mathcal{S} = \{S_1, \ldots, S_m\}$. As in the proof for split graphs, we may assume that each element of X is contained in at most $m-2$ sets of $\mathcal{S}$. We may also assume that $m > n$ since we can simply duplicate sets in $\mathcal{S}$ to ensure this. From ϕ, we construct the graph G as in the proof for split graphs, except that we add the necessary edges so that the vertices in $W \cup S$ form a clique, and we add a vertex v^* and make it adjacent to each vertex in $V \cup W \cup U$. The graph G is clearly a co-bipartite graph with a universal vertex, that is constructed in polynomial time. See Figure 4 for an illustration of G. We prove that ϕ admits a set cover of size at most t if and only if there is an NCTM^+ of size at most $k = 2m + t + 1$ for $\mathcal{B}(G)$.

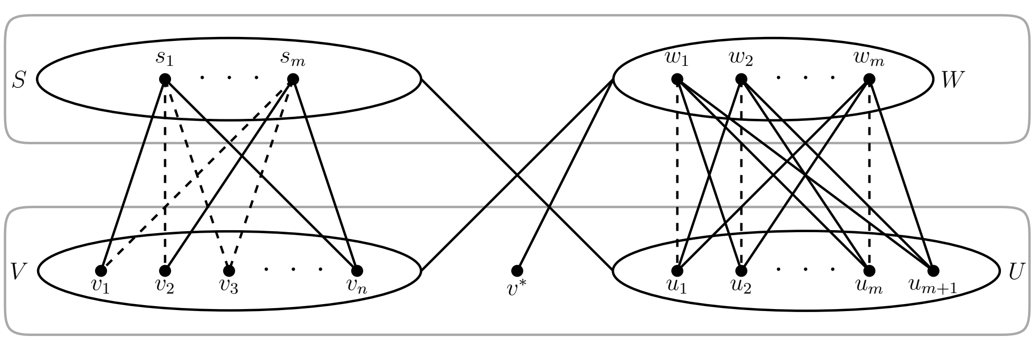

Figure 4: The co-bipartite graph G constructed in the proof of Theorem 1. An edge between a vertex and an ellipse indicates that vertex is adjacent to each vertex in the ellipse.

First, suppose that ϕ admits a set cover of size at most t, and let $\mathcal{S}' \subset \mathcal{S}$ be such a set cover. Let $S' \subset S$ be such that, for all $j \in [m]$, $S_j \in \mathcal{S}'$ in ϕ if and only if $s_j \in S'$. We define an NCTM^+ T of size at most k for $\mathcal{B}(G)$ as follows. As in the proof for split graphs, we only need to define T for balls of G of radius at most 1 since, for all $x \in V(G)$, $B_2(x) = B_1(u_{m+1}) = V(G)$.

- For all $x \in V(G)$, set $T(x,0) := \{x\}$.
- Set $T(v^*,1) := \{v^*, u_{m+1}\}$.
- For all $x \in V$, set $T(x,1) := \{v^*\} \cup U \cup (B_1(x) \cap S)$.
- For all $x \in S$, set $T(x,1) := \{x, u_{m+1}\} \cup (B_1(x) \cap V)$.
- For all $x \in W$, set $T(x,1) := \{v^*\} \cup S \cup (B_1(x) \cap U)$.
- Finally, for all $x \in U$, set $T(x,1) := \{v^*\} \cup S' \cup \{u_1, \ldots, u_m\} \cup (B_1(x) \cap W)$.

It is easy to verify that T satisfies the inclusion condition for each ball in $\mathcal{B}(G)$, and that T has size at most k since $m > n$. We now show that T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$. For all $x \in V(G)$, $|B_0(x)| = |T(x,0)| = 1$ and $|T(x,1)| \geq 2$, and thus, T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$ where at least one of the balls has radius 0. For all $x, y \in V$, $B_1(x) \setminus S = B_1(y) \setminus S$, $T(x,1) \cap S = B_1(x) \cap S$, and $T(y,1) \cap S = B_1(y) \cap S$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in V. For all $x, y \in U$, $B_1(x) \setminus W = B_1(y) \setminus W$, $T(x,1) \cap W = B_1(x) \cap W$, and $T(y,1) \cap W = B_1(y) \cap W$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in U. For all $x \in V$, by the construction, we have that $B_1(x) \cap S' \neq S'$ since S' corresponds to the set cover $\mathcal{S}'$, and, for all $y \in U$, $S' \subset T(y,1)$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in $V \cup U$. For all $x, y \in W$, $B_1(x) \setminus U = B_1(y) \setminus U$, $T(x,1) \cap U = B_1(x) \cap U$, and $T(y,1) \cap U = B_1(y) \cap U$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in W. For all $x \in V \cup U$ and $y \in W$, we have that $\{u_1, \ldots, u_m\} \subset T(x,1)$ and $B_1(y) \cap \{u_1, \ldots, u_m\} \neq \{u_1, \ldots, u_m\}$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in $V \cup U \cup W$. For all $x \in V \cup U \cup W$, $|T(x,1) \cap S| \geq 1$ (recall that each element of X is contained in at most $m-2$ sets of $\mathcal{S}$), and $|B_1(v^*) \cap S| = 0$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in $V \cup U \cup W \cup \{v^*\}$. For all $x, y \in S$, $B_1(x) \setminus V = B_1(y) \setminus V$, $T(x,1) \cap V = B_1(x) \cap V$, and $T(y,1) \cap V = B_1(y) \cap V$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in S. For all $x \in V \cup U \cup W \cup \{v^*\}$ and $y \in S$, we have that $v^* \in T(x,1)$ and $v^* \notin B_1(y)$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in $V(G)$. Thus, T is an NCTM^+ of size $\leq k$ for $\mathcal{B}(G)$.

Now, suppose that ϕ does not admit a set cover of size at most t. In this case, we prove that there is no NCTM^+ of size at most k for $\mathcal{B}(G)$. For all $x \in V \cup U$, it holds that $B_1(x) \setminus \{v^*\}$ is the same as in the graph G constructed in the proof for split graphs. Thus, since, for all $x \in V \cup U$, it holds that $v^* \in B_1(x)$, we get that $|T(u_{m+1},1) \cap (W \cup S)| > m + t$ for any NCTM^+ T for $\mathcal{B}(G)$ in this case. Now, we prove that $|T(u_{m+1},1) \cap \{u_1, \ldots, u_m\}| = m$ for any NCTM^+ T for $\mathcal{B}(G)$. Indeed, for all $j \in [m]$, $B_1(u_{m+1}) = B_1(w_j) \cup \{u_j\}$, and so, to ensure that T satisfies the non-clashing condition for the pair $B_1(u_{m+1})$ and $B_1(w_j)$, we must have that $u_j \in T(u_{m+1}, 1)$. Lastly, we prove that $|T(u_{m+1}, 1) \cap (V \cup \{v^*\})| \geq 1$ for any NCTM^+ T for $\mathcal{B}(G)$. Indeed, $B_1(u_{m+1}) = V(G)$ and, for all $x \in S$, $B_1(u_{m+1}) \setminus (V \cup \{v^*\}) = B_1(x) \setminus (V \cup \{v^*\})$, and so, to ensure that T satisfies the non-clashing condition for the pair $B_1(u_{m+1})$ and $B_1(x)$, we must have that $|T(u_{m+1}, 1) \cap (V \cup \{v^*\})| \geq 1$. Thus, $|T(u_{m+1}, 1)| > k$. This concludes the proof for co-bipartite graphs.

We now proceed with the proof for bipartite graphs. Let ϕ be an instance of SET COVER with $X = \{1, \ldots, n\}$ and $\mathcal{S} = \{S_1, \ldots, S_m\}$. As in the proof for co-bipartite graphs, we may assume that $m > n$, and that ϕ is an instance in which each element of X is contained in at most $m-2$ sets of $\mathcal{S}$. From this instance ϕ, we construct the graph G as follows. For all $i \in [n]$ and $j \in [m]$, add a vertex v_i and a vertex s_j, and if $i \notin S_j$ in ϕ, then add the edge $v_i s_j$. Add the sets of vertices $U = \{u_1, \ldots, u_{m+1}\}$ and $W = \{w_1, \ldots, w_m\}$, and, for all $j, \ell \in [m]$ such that $j \neq \ell$, add the edge $u_j w_\ell$. For all $j \in [m]$, add the edge $u_{m+1} w_j$. Add edges so that every vertex in U is adjacent to every vertex in $V = \{v_1, \ldots, v_n\}$. Lastly, add a vertex z, and add edges so that z is adjacent to every vertex in $U \cup S$, where $S = \{s_1, \ldots, s_m\}$. This completes the construction of G, which is clearly achieved in polynomial time. See Figure 5 for an illustration of G. Note that G has diameter 3 and is bipartite, as witnessed by the bipartition $(W \cup V \cup \{z\}) \cup (U \cup S)$ of its vertices. We prove that ϕ admits a set cover of size at most t if and only if there is an NCTM^+ of size at most $k = m + t$ for $\mathcal{B}(G)$.

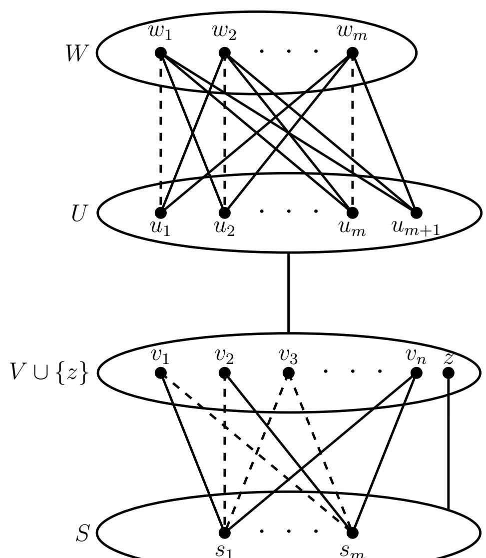

Figure 5: The bipartite graph G constructed in the proof of Theorem 1. An edge between a vertex and an ellipse indicates that vertex is adjacent to each vertex in the ellipse. See the caption of Figure 3 for more details.

First, suppose that ϕ admits a set cover of size at most t, and let $\mathcal{S}' \subset \mathcal{S}$ be such a set cover. Let $S' \subset S$ be such that, for all $j \in [m]$, $S_j \in \mathcal{S}'$ in ϕ if and only if $s_j \in S'$. We define an NCTM^+ T of size at most k for $\mathcal{B}(G)$ as follows. Also, note that we only need to define T for balls of G of radius at most 2 since, for all $x \in V(G)$, $B_3(x) = B_2(u_{m+1}) = V(G)$. Furthermore, we only need to define T for balls of radius at most 1 centered at z since $B_2(z) = B_2(u_{m+1})$ and we will define T for $B_2(u_{m+1})$.

- For all $x \in V(G)$, set $T(x, 0) := \{x\}$.
- For all $x \in V$, set $T(x, 1) := \{x\} \cup (B_1(x) \cap S)$ and $T(x, 2) := \{w_1\} \cup (B_1(x) \cap S)$, and note that $|T(x, 2)| = |T(x, 1)| \geq 3$ since, in ϕ, every element of X is contained in at most $m - 2$ sets of $\mathcal{S}$.
- Similarly, set $T(z, 1) := \{z\} \cup S$.
- For all $x \in S$, set $T(x, 1) := \{x, z\} \cup (B_1(x) \cap V)$, $T(x, 2) := \{u_{m+1}, z\} \cup (B_1(x) \cap V)$.
- For all $x \in W$, set $T(x, 1) := \{x\} \cup (B_1(x) \cap \{u_1, \ldots, u_m\})$, and $T(x, 2) := \{x, z\} \cup (B_1(x) \cap \{u_1, \ldots, u_m\})$.
- Finally, for all $x \in U$, set $T(x, 1) := \{x\} \cup (B_1(x) \cap W)$ and $T(x, 2) := S' \cup (B_1(x) \cap W)$.

It is easy to verify that T has size at most k and satisfies the inclusion condition for each ball in $\mathcal{B}(G)$.

We now show that T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$. For all $x \in V(G)$, $|B_0(x)| = |T(x, 0)| = 1$, $|T(x, 1)| \geq 2$, and $|T(x, 2)| \geq 2$, and thus, T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$ where at least one of the balls has radius 0. For all $x \in V(G)$, we have that $x \in T(x, 1)$. Furthermore, $W \cup S$, $W \cup V \cup \{z\}$, and $U \cup S$ are independent sets. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in

$$W \cup S; \tag{1}$$

$$W \cup V \cup \{z\}; \tag{2}$$

$$U \cup S. \tag{3}$$

For all $x \in W$ and $y \in U$, $|T(x, 1) \cap U| = m - 1$ and $|B_1(y) \cap U| = 1$. Similarly, for all $x' \in V \cup \{z\}$ and $y' \in U \cup S$, $|T(x', 1) \cap S| \geq 2$ and $|B_1(y') \cap S| \leq 1$. Hence, in combination with (1), (2), and (3), T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in

$$V(G). \tag{4}$$

For all $x \in W$ and $y \in U$, $|T(x, 2) \cap W| = 1$, $|T(y, 2) \cap W| \geq m - 1$, and $|B_1(z) \cap W| = 0$. Also, for all $x' \in V$, $|B_2(x') \cap S| < m$ and $|T(z, 1) \cap S| = m$. Lastly, $|B_1(z) \cap V| = 0$ and, for all $y' \in S$, either $|T(y', 2) \cap V| = 0$, in which case $B_1(z) = B_2(y')$, or $|T(y', 2) \cap V| \geq 1$. Recall that $B_2(u_{m+1}) = B_2(z)$, and hence, T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls is $B_1(z)$ and the other has radius 2 and is centered at a vertex in

$$V(G). \tag{5}$$

For all $x \in V(G)$, there exists a vertex $y \in T(x, 2)$ such that $d(x, y) = 2$. Hence, in combination with the construction of G, T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$W; \tag{6}$$

$$U; \tag{7}$$

$$V; \tag{8}$$

$$S. \tag{9}$$

For all $x \in S$ and $y \in W$, we have that $z \in T(x,2)$, $z \notin B_1(y)$, $|T(y,2) \cap U| = m-1$, and $|B_1(x) \cap U| = 0$. Hence, in combination with (6) and (9), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$W \cup S. \tag{10}$$

For all $x \in V$, $y \in W$, and $q \in \{1,2\}$, $|T(x,q) \cap S| \geq 2$ and $|B_q(y) \cap S| = 0$. Hence, in combination with (6) and (8), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$W \cup V. \tag{11}$$

For all $x \in W$ and $y \in U$, $|T(x,2) \cap U| = m-1$, $|B_1(y) \cap U| = 1$, $|T(y,2) \cap S| \geq 1$, and $|B_1(x) \cap S| = 0$. Hence, in combination with (6) and (7), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$W \cup U. \tag{12}$$

For all $x \in S$, $y \in U$, and $q \in \{1,2\}$, $|T(y,q) \cap W| \geq m-1$ and $|B_q(x) \cap W| = 0$. Hence, in combination with (7) and (9), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$U \cup S. \tag{13}$$

For all $x \in V$ and $y \in U$, $|T(y,2) \cap W| \geq m-1$, $|B_1(x) \cap W| = 0$, $|T(x,2) \cap S| \geq 2$, and $|B_1(y) \cap S| = 0$. Hence, in combination with (7) and (8), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$U \cup V. \tag{14}$$

For all $x \in V$ and $y \in S$, we have that $w_1 \in T(x,2)$, $w_1 \notin B_1(y)$, $z \in T(y,2)$, and $z \notin B_1(x)$. Hence, in combination with (8) and (9), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$V \cup S. \tag{15}$$

Combining (5) and (10)–(15), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$V(G). \tag{16}$$

It remains to prove that T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$. For all $j \in [m]$, $T(w_j,2) \cap \{u_1,\ldots,u_m\} = B_2(w_j) \cap \{u_1,\ldots,u_m\} = \{u_1 \ldots,u_m\} \setminus \{u_j\}$.

Hence, T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$, that are centered at vertices in

$$W. \tag{17}$$

For all $j \in [m]$, $T(u_j, 2) \cap W = B_2(u_j) \cap W = W \setminus \{w_j\}$. Also, $W \subset T(u_{m+1}, 2)$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$, that are centered at vertices in

$$U. \tag{18}$$

For any $x, y \in V$, $B_2(x) \setminus S = B_2(y) \setminus S$, $T(x, 2) \cap S = B_2(x) \cap S$, and $T(y, 2) \cap S = B_2(y) \cap S$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$, that are centered at vertices in

$$V. \tag{19}$$

For any $x, y \in S$, $B_2(x) \setminus V = B_2(y) \setminus V$, $T(x, 2) \cap V = B_2(x) \cap V$, and $T(y, 2) \cap V = B_2(y) \cap V$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$, that are centered at vertices in

$$S. \tag{20}$$

For all $x \in W$ and $y \in S$, we have that $x \in T(x, 2)$ and $|B_2(y) \cap W| = 0$. Hence, in combination with (17) and (20), T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$, that are centered at vertices in

$$W \cup S. \tag{21}$$

For all $x \in V$ and $y \in W$, $|T(x, 2) \cap S| \geq 2$ and $|B_2(y) \cap S| = 0$. Hence, in combination with (17) and (19), T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$, that are centered at vertices in

$$W \cup V. \tag{22}$$

For all $x \in W$ and $y \in U$, $|T(y, 2) \cap S| \geq 1$ and $|B_2(x) \cap S| = 0$. Hence, in combination with (17) and (18), T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$, that are centered at vertices in

$$W \cup U. \tag{23}$$

For all $x \in S$ and $y \in U$, $|T(y, 2) \cap W| \geq m - 1$ and $|B_2(x) \cap W| = 0$. Hence, in combination with (18) and (20), T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$, that are centered at vertices in

$$U \cup S. \tag{24}$$

For all $x \in V$ and $y \in S$, we have that $w_1 \in T(x, 2)$ and $w_1 \notin B_2(y)$. Hence, in combination with (19) and (20), T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$, that are centered at vertices in

$$V \cup S. \tag{25}$$

For all $x \in V$, by the construction, $B_2(x) \cap S' \neq S'$ since S' corresponds to the set cover $\mathcal{S}'$, and, for all $y \in U$, $S' \subset T(y, 2)$. Hence, in combination with (18) and (19), T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$, that are centered at vertices in

$$U \cup V. \tag{26}$$

Combining (21)–(26), we get that T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$, that are centered at vertices in

$$V(G). \tag{27}$$

Combining (4), (16), and (27), we get that T is an NCTM^+ of size at most k for $\mathcal{B}(G)$.

Now, suppose that ϕ does not admit a set cover of size at most t. In this case, we prove that there is no NCTM^+ of size at most k for $\mathcal{B}(G)$. We first prove that $|T(u_{m+1}, 2) \cap W| = m$ for any NCTM^+ T for $\mathcal{B}(G)$. Indeed, for all $1 \leq j \leq m$, $B_2(u_{m+1}) = B_2(u_j) \cup \{w_j\}$, and so, to ensure that T satisfies the non-clashing condition for the pair $B_2(u_{m+1})$ and $B_2(u_j)$, we must have that $w_j \in T(u_{m+1}, 2)$. Now, we prove that $|T(u_{m+1}, 2) \cap S| > t$ for any NCTM^+ T for $\mathcal{B}(G)$ in this case, which completes the proof. Observe that, for all $i \in [n]$, $B_2(v_i) \subset B_2(u_{m+1})$ and $(B_2(u_{m+1}) \setminus B_2(v_i)) \subset S$. Hence, for each $i \in [n]$, to ensure that T satisfies the non-clashing condition for the pair $B_2(u_{m+1})$ and $B_2(v_i)$, it is necessary that $s_j \in T(u_{m+1}, 1)$ for some $j \in [m]$ such that $s_j \notin B_1(v_i)$. However, $s_j \notin B_1(v_i)$ if and only if $i \in S_j$ in ϕ. In other words, $T(u_{m+1}, 2) \cap S$ must correspond to a set cover in ϕ, and thus, $|T(u_{m+1}, 2)| > k$. This concludes the proof for bipartite graphs. ■

Proposition 2 B-NCTD^+ *and* B-NCTD *admit algorithms running in time* $2^{\mathcal{O}(n^2 \cdot \mathrm{diam}(G))}$.

Proof For any $x \in V(G)$ and $r \in \mathbb{N}$, there are at most 2^n possible choices for $T(x, r)$, and there are at most $n \cdot \min\{(\mathrm{diam}+1), n\}$ unique balls in G. Thus, for each possible (positive) NCTM, it can be checked in polynomial time whether it satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, and the inclusion condition (for B-NCTD^+) for each ball in $\mathcal{B}(G)$. Hence, there is a brute-force algorithm running in time $2^{\mathcal{O}(n^2 \cdot \min\{\mathrm{diam}+1, n\})} = 2^{\mathcal{O}(n^2 \cdot \mathrm{diam}(G))}$ (since G is connected). ■

Lemma 5 *If* ϕ *is satisfiable, then* G *admits an* NCTM^+ *for* $\mathcal{B}(G)$ *of size* k.

Proof Suppose that $\pi : X^\alpha \cup X^\beta \cup X^\gamma \to \{\text{True}, \text{False}\}$ is a satisfying assignment for ϕ. Let us define the set π' of vertices in $A^\alpha \cup A^\beta \cup A^\gamma$ corresponding to π. Initially, set $\pi' := \varnothing$. Now, for each $\delta \in \{\alpha, \beta, \gamma\}$ and x_i^δ in ϕ, if $\pi(x_i^\delta) = \text{True}$ ($\pi(x_i^\delta) = \text{False}$, respectively), then add t_{2i}^δ (f_{2i-1}^δ, respectively) to π'. Thus, $|\pi'| = 3N$ and π' corresponds to a satisfying assignment for ϕ in the sense that, from π', we can extract the satisfying assignment π for ϕ. Using π', we define an NCTM^+ T of size k for $\mathcal{B}(G)$ as follows. Also, note that we only need to define T for balls of G of radius at most 2 since, for all $x \in V(G)$, $B_3(x) = B_2(u_{3M+1}) = V(G)$. Furthermore, we do not need to define T for $B_2(u'_{3M+1})$ nor $B_2(z)$ since $B_2(z) = B_2(u'_{3M+1}) = B_2(u_{3M+1})$, and we will define T for $B_2(u_{3M+1})$.

- For all $x \in V(G)$, set $T(x, 0) := \{x\}$.
- For each $\delta \in \{\alpha, \beta, \gamma\}$ and $x \in A^\delta$, set $T(x, 1) := B_1(x)$ and $T(x, 2) := \{u_1, t_2^{\delta'}, t_2^{\delta''}\} \cup B_1(x)$, where $\delta', \delta'' \in \{\alpha, \beta, \gamma\}$ such that $\delta \notin \{\delta', \delta''\}$ and $\delta' \neq \delta''$. Note that $T(x, 1) \subset T(x, 2)$ and $|T(x, 2)| = \mathcal{O}(\log M)$.
- For each $\delta \in \{\alpha, \beta, \gamma\}$ and $x \in V^\delta \cup V^{\delta,*}$, set $T(x, 1) := B_1(x) \setminus U$ and $T(x, 2) := \{w_1, z\} \cup (B_1(x) \setminus U)$. Note that $T(x, 1) \subset T(x, 2)$ and $|T(x, 2)| < 2N + 3M + 3$.

- For each $x \in V^W$, set $T(x,1) := \{u_{3M+1}, u'_{3M+1}\} \cup V^W \cup (B_1(x) \cap U)$ and $T(x,2) := \{u_{3M+1}, u'_{3M+1}, z\} \cup V^W \cup U$. Note that $T(x,1) \subset T(x,2)$ and $|T(x,2)| = 3M + \mathcal{O}(\log M)$.

- For each $\delta \in \{\alpha, \beta, \gamma\}$ and $x \in C \cup C^\delta$, set $T(x,1) := B_1(x)$ and $T(x,2) := \{x, w_1\} \cup (B_2(x) \cap (A^\alpha \cup A^\beta \cup A^\gamma))$. Note that $|T(x,1)| = \mathcal{O}(\log M)$ and $|T(x,2)| \leq 6N + 2 < 3N + 3M = k$.

- For each $x \in U \cup \{u_{3M+1}, u'_{3M+1}\}$, set $T(x,1) := B_1(x) \setminus (C \cup C^\alpha \cup C^\beta \cup C^\gamma)$ and $T(x,2) := (B_2(x) \cap W) \cup \pi'$. Note that $|T(x,1)| \leq 3M + \mathcal{O}(\log M)$ and $|T(x,2)| \leq 3N + 3M = k$.

- For each $x \in W$, set $T(x,1) := B_1(x)$ and $T(x,2) := \{x, z, u_{3M+1}, u'_{3M+1}\} \cup (B_2(x) \cap U)$. Note that $|T(x,1)| = \mathcal{O}(\log M)$ and $|T(x,2)| \leq 3M + 4$.

- Finally, set $T(z,1) := \{z, u_1\}$.

Hence, T has size at most k, and it is easy to verify that T satisfies the inclusion condition for each ball in $\mathcal{B}(G)$.

We now show that T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$. For all $x \in V(G)$, $|B_0(x)| = |T(x,0)| = 1$, $|T(x,1)| \geq 2$, and $|T(x,2)| \geq 2$, and thus, T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$ where at least one of the balls has radius 0. For all $x \in V(G)$, $x \in T(x,1)$. Furthermore, $W \cup U \cup C \cup C^\alpha \cup C^\beta \cup C^\gamma \cup A^\alpha \cup A^\beta \cup A^\gamma$, $\{z\} \cup W \cup V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}$, $\{u_{3M+1}, u'_{3M+1}\} \cup U \cup A^\alpha \cup A^\beta \cup A^\gamma$, $C \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}$, $C^\alpha \cup C^\beta \cup C^\gamma \cup V^\alpha \cup V^\beta \cup V^\gamma$, and $\{z\} \cup C \cup C^\alpha \cup C^\beta \cup C^\gamma$ are independent sets. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in

$$W \cup U \cup C \cup C^\alpha \cup C^\beta \cup C^\gamma \cup A^\alpha \cup A^\beta \cup A^\gamma; \tag{1}$$

$$\{z\} \cup W \cup V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}; \tag{2}$$

$$\{u_{3M+1}, u'_{3M+1}\} \cup U \cup A^\alpha \cup A^\beta \cup A^\gamma; \tag{3}$$

$$C \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}; \tag{4}$$

$$C^\alpha \cup C^\beta \cup C^\gamma \cup V^\alpha \cup V^\beta \cup V^\gamma; \tag{5}$$

$$\{z\} \cup C \cup C^\alpha \cup C^\beta \cup C^\gamma. \tag{6}$$

For all $x \in W$ and $y \in V^W$, $|B_1(x) \cap V^W| = p$, $|T(y,1) \cap V^W| = 2p$, and $|T(u_{3M+1}, 1) \cap W| = |T(u'_{3M+1}, 1) \cap W| = 3M$. Hence, in combination with (1) and (2), T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$, where one of the balls is centered at a vertex in

$$W. \tag{7}$$

For all $x \in V^W$ and $x' \in U \cup \{u_{3M+1}, u'_{3M+1}\}$, we have that $z \in T(x',1)$ and $z \notin B_1(x)$. Further, for all $x \in V^W$ and $x'' \in V(G) \setminus (V^W \cup W \cup U \cup \{u_{3M+1}, u'_{3M+1}\})$, we have that $x'' \in T(x'',1)$ and $x'' \notin B_1(x)$. Lastly, for all $x, y \in V^W$, $B_1(x) \setminus (W \cup U) = B_1(y) \setminus (W \cup U)$, $(B_1(x) \cap U) \subset T(x,1)$, $(B_1(y) \cap U) \subset T(y,1)$, and $B_1(x) \cap U = B_1(y) \cap U$ if and only if $B_1(x) \cap W = B_1(y) \cap W$ by the construction. Hence, in combination with (7), T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$, where one of the balls is centered at a vertex in

$$V^W. \tag{8}$$

For all $x \in U \cup \{u_{3M+1}, u'_{3M+1}\}$, $|T(x,1) \cap V^W| \geq p$. Further, for all $y \in V(G) \setminus (V^W \cup W \cup U \cup \{u_{3M+1}, u'_{3M+1}\})$, $|B_1(y) \cap V^W| = 0$. Hence, in combination with (3), (7), and (8), this implies that T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$, where one of the balls is centered at a vertex in

$$U \cup \{u_{3M+1}, u'_{3M+1}\}. \tag{9}$$

For all $x \in C$, $y \in C^\alpha \cup C^\beta \cup C^\gamma$, and $\delta \in \{\alpha, \beta, \gamma\}$, $|T(x,1) \cap V^\delta| \geq p$ and $|T(y,1) \cap V^{\delta,*}| \geq p$. Hence, in combination with (1), (4), (5), (6), (8), and (9), T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$, where one of the balls is centered at a vertex in

$$C \cup C^\alpha \cup C^\beta \cup C^\gamma. \tag{10}$$

For all $x \in A^\alpha \cup A^\beta \cup A^\gamma$ and $\delta \in \{\alpha, \beta, \gamma\}$, $|T(x,1) \cap V^\delta| \geq p$ and $z \in T(x,1)$. Hence, in combination with (1), (2), and (7)–(10), T satisfies the non-clashing condition for all pairs of balls of radius 1 in $\mathcal{B}(G)$ centered at vertices in

$$V(G). \tag{11}$$

For all $x \in V(G) \setminus (U \cup A^\alpha \cup A^\beta \cup A^\gamma \cup \{u_{3M+1}, u'_{3M+1}\})$, we have that $x \in T(x,2)$ and $x \notin B_1(z)$. For all $\delta \in \{\alpha, \beta, \gamma\}$ and $x' \in A^\delta$, $|T(x',2) \cap V^\delta| = p$ and $|B_1(z) \cap V^\delta| = 0$. Lastly, for all $x'' \in U \cup \{u_{3M+1}\}$, $|T(x'',2) \cap W| \geq 3M-1$ and $|B_1(z) \cap W| = 0$. Recall that $B_2(u_{3M+1}) = B_2(u'_{3M+1}) = B_2(z)$, and hence, T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls is $B_1(z)$ and the other has radius 2 and is centered at a vertex in

$$V(G). \tag{12}$$

For all $x \in V(G)$, there exists a vertex $y \in T(x,2)$ such that $d(x,y) = 2$. Hence, in combination with the construction of G, T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$W; \tag{13}$$

$$V^W; \tag{14}$$

$$U \cup \{u_{3M+1}, u'_{3M+1}\}; \tag{15}$$

$$C \cup C^\alpha \cup C^\beta \cup C^\gamma; \tag{16}$$

$$V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}; \tag{17}$$

$$A^\alpha \cup A^\beta \cup A^\gamma. \tag{18}$$

For all $x \in W$ and $y \in V^W$, we have that $z \in T(x,2)$, $z \notin B_1(y)$, $z \in T(y,2)$, and $z \notin B_1(x)$. Hence, in combination with (13) and (14), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$W \cup V^W. \tag{19}$$

For all $x \in W$ and $y \in U \cup \{u_{3M+1}, u'_{3M+1}\}$, $|T(x,2) \cap \{u_{3M+1}, u'_{3M+1}\}| = 2$, $|B_1(y) \cap \{u_{3M+1}, u'_{3M+1}\}| \leq 1$, $|T(y,2) \cap A^\alpha| = N$, and $|B_1(x) \cap A^\alpha| = 0$. Hence, in combination with (13) and (15), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$W \cup U \cup \{u_{3M+1}, u'_{3M+1}\}. \tag{20}$$

For all $x \in W$ and $y \in V(G) \setminus (W \cup V^W \cup U \cup \{u_{3M+1}, u'_{3M+1}, z\})$, we have that $x \in T(x,2)$, $x \notin B_1(y)$, $y \in T(y,2)$, and $y \notin B_1(x)$. Hence, in combination with (13) and (16)–(18), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$W \cup C \cup C^\alpha \cup C^\beta \cup C^\gamma; \tag{21}$$

$$W \cup V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}; \tag{22}$$

$$W \cup A^\alpha \cup A^\beta \cup A^\gamma. \tag{23}$$

For all $x \in V^W$ and $y \in U \cup \{u_{3M+1}, u'_{3M+1}\}$, $|T(x,2) \cap \{u_{3M+1}, u'_{3M+1}\}| = 2$, $|B_1(y) \cap \{u_{3M+1}, u'_{3M+1}\}| \leq 1$, $|T(y,2) \cap A^\alpha| = N$, and $|B_1(x) \cap A^\alpha| = 0$. Hence, in combination with (14) and (15), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$V^W \cup U \cup \{u_{3M+1}, u'_{3M+1}\}. \tag{24}$$

For all $x \in V^W$ and $y \in V(G) \setminus (W \cup V^W \cup U \cup \{u_{3M+1}, u'_{3M+1}, z\})$, we have that $x \in T(x,2)$, $x \notin B_1(y)$, $y \in T(y,2)$, and $y \notin B_1(x)$. Hence, in combination with (14) and (16)–(18), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$V^W \cup C \cup C^\alpha \cup C^\beta \cup C^\gamma; \tag{25}$$

$$V^W \cup V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}; \tag{26}$$

$$V^W \cup A^\alpha \cup A^\beta \cup A^\gamma. \tag{27}$$

For all $x \in U \cup \{u_{3M+1}, u'_{3M+1}\}$ and $y \in V(G) \setminus (W \cup V^W \cup U \cup \{u_{3M+1}, u'_{3M+1}, z\})$, $|T(x,2) \cap W| \geq 3M-1$, $|B_1(y) \cap W| = 0$, $|T(y,2) \cap (A^\alpha \cup A^\beta \cup A^\gamma)| \geq 1$, and $|B_1(x) \cap (A^\alpha \cup A^\beta \cup A^\gamma)| = 0$. Hence, in combination with (15)–(18), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$U \cup \{u_{3M+1}, u'_{3M+1}\} \cup C \cup C^\alpha \cup C^\beta \cup C^\gamma; \tag{28}$$

$$U \cup \{u_{3M+1}, u'_{3M+1}\} \cup V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}; \tag{29}$$

$$U \cup \{u_{3M+1}, u'_{3M+1}\} \cup A^\alpha \cup A^\beta \cup A^\gamma. \tag{30}$$

For all $x \in C \cup C^\alpha \cup C^\beta \cup C^\gamma$ and $y \in V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}$, we have that $w_1 \in T(x,2)$, $w_1 \notin B_1(y)$, $z \in T(y,2)$, and $z \notin B_1(x)$. Hence, in combination with (16) and (17), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$C \cup C^\alpha \cup C^\beta \cup C^\gamma \cup V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}. \tag{31}$$

For all $x \in C \cup C^\alpha \cup C^\beta \cup C^\gamma$ and $y \in A^\alpha \cup A^\beta \cup A^\gamma$, we have that $x \in T(x,2)$, $x \notin B_1(y)$, $y \in T(y,2)$, and $y \notin B_1(x)$. Hence, in combination with (16) and (18), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$C \cup C^\alpha \cup C^\beta \cup C^\gamma \cup A^\alpha \cup A^\beta \cup A^\gamma. \tag{32}$$

For all $x \in V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}$ and $y \in A^\alpha \cup A^\beta \cup A^\gamma$, we have that $w_1 \in T(x,2)$, $w_1 \notin B_1(y)$, $z \in T(y,2)$, and $z \notin B_1(x)$. Hence, in combination with (17) and (18), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*} \cup A^\alpha \cup A^\beta \cup A^\gamma. \tag{33}$$

Combining (12) and (19)–(33), T satisfies the non-clashing condition for all pairs of balls in $\mathcal{B}(G)$, where one of the balls has radius 1 and the other has radius 2, that are centered at vertices in

$$V(G). \tag{34}$$

It remains to prove that T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$. For all $\ell \in [3M]$, $T(w_\ell,2) \cap U = B_2(w_\ell) \cap U = U \setminus \{u_\ell\}$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$ that are centered at vertices in

$$W. \tag{35}$$

For any $x,y \in V^W$, $B_2(x) = B_2(y)$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$ that are centered at vertices in

$$V^W. \tag{36}$$

For all $\ell \in [3M]$, $T(u_\ell,2) \cap W = B_2(u_\ell) \cap W = W \setminus \{w_\ell\}$, and $W \subset T(u_{3M+1},2)$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$ that are centered at vertices in

$$U \cup \{u_{3M+1}\}. \tag{37}$$

For any $x,y \in C \cup C^\alpha \cup C^\beta \cup C^\gamma$, $B_2(x) \setminus (A^\alpha \cup A^\beta \cup A^\gamma) = B_2(y) \setminus (A^\alpha \cup A^\beta \cup A^\gamma)$, $B_2(x) \cap (A^\alpha \cup A^\beta \cup A^\gamma) \subset T(x,2)$, and $B_2(y) \cap (A^\alpha \cup A^\beta \cup A^\gamma) \subset T(y,2)$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$ that are centered at vertices in

$$C \cup C^\alpha \cup C^\beta \cup C^\gamma. \tag{38}$$

For any $x,y \in V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}$, $B_2(x) \setminus (A^\alpha \cup A^\beta \cup A^\gamma) = B_2(y) \setminus (A^\alpha \cup A^\beta \cup A^\gamma)$, $B_2(x) \cap (A^\alpha \cup A^\beta \cup A^\gamma) = B_1(x) \cap (A^\alpha \cup A^\beta \cup A^\gamma) \subset T(x,2)$, and $B_2(y) \cap (A^\alpha \cup A^\beta \cup A^\gamma) =$

$B_1(y) \cap (A^\alpha \cup A^\beta \cup A^\gamma) \subset T(y,2)$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$ that are centered at vertices in

$$V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}. \tag{39}$$

For any $x, y \in A^\alpha \cup A^\beta \cup A^\gamma$, we have that $B_1(x) \subset T(x,2)$, $B_1(y) \subset T(y,2)$, and $B_2(x) \neq B_2(y)$ if and only if $B_1(x) \neq B_1(y)$. Hence, T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$ that are centered at vertices in

$$A^\alpha \cup A^\beta \cup A^\gamma. \tag{40}$$

For all $x \in W$ and $y \in V^W$, $B_2(x) \setminus U = B_2(y) \setminus U$ and $U \subset T(y,2)$. Hence, in combination with (35) and (36), T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$ that are centered at vertices in

$$W \cup V^W. \tag{41}$$

For all $x \in W \cup V^W$ and $y \in V(G) \setminus (W \cup V^W \cup \{u'_{3M+1}\} \cup \{z\})$, $|T(y,2) \cap (A^\alpha \cup A^\beta \cup A^\gamma)| \geq 1$ and $|B_2(x) \cap (A^\alpha \cup A^\beta \cup A^\gamma)| = 0$. Hence, in combination with (37)–(41), T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$ that are centered at vertices in

$$W \cup V^W \cup U \cup \{u_{3M+1}\}; \tag{42}$$

$$W \cup V^W \cup C \cup C^\alpha \cup C^\beta \cup C^\gamma; \tag{43}$$

$$W \cup V^W \cup V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}; \tag{44}$$

$$W \cup V^W \cup A^\alpha \cup A^\beta \cup A^\gamma. \tag{45}$$

For all $x \in U \cup \{u_{3M+1}\}$ and $y \in C \cup C^\alpha \cup C^\beta \cup C^\gamma$, we have that $\pi' \subset T(x,2)$ and $B_2(y) \cap \pi' \neq \pi'$. Indeed, for all $y \in C \cup C^\alpha \cup C^\beta \cup C^\gamma$, the only vertices in $A^\alpha \cup A^\beta \cup A^\gamma$ that are not in $B_2(y)$ are those corresponding to the literals contained in the clause corresponding to y in ϕ (as mentioned before, for each $i \in [N]$ and $\delta \in \{\alpha, \beta, \gamma\}$, the vertex c_i^δ can be thought of as a clause containing only the positive and negative literals of x_i^δ). The property then follows since π' corresponds to the satisfying assignment π for ϕ. Hence, in combination with (37) and (38), T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$ that are centered at vertices in

$$U \cup \{u_{3M+1}\} \cup C \cup C^\alpha \cup C^\beta \cup C^\gamma. \tag{46}$$

For all $x \in U \cup \{u_{3M+1}\} \cup C \cup C^\alpha \cup C^\beta \cup C^\gamma$ and $y \in A^\alpha \cup A^\beta \cup A^\gamma$, $|T(x,2) \cap W| \geq 1$ and $|B_2(y) \cap W| = 0$. Hence, in combination with (40) and (46), T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$ that are centered at vertices in

$$U \cup \{u_{3M+1}\} \cup C \cup C^\alpha \cup C^\beta \cup C^\gamma \cup A^\alpha \cup A^\beta \cup A^\gamma. \tag{47}$$

For all $x \in U \cup \{u_{3M+1}\} \cup C \cup C^\alpha \cup C^\beta \cup C^\gamma \cup A^\alpha \cup A^\beta \cup A^\gamma$, $y \in V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}$, and $\delta \in \{\alpha, \beta, \gamma\}$, $|T(x,2) \cap A^\delta| \geq 1$ and, for some $\delta' \in \{\alpha, \beta, \gamma\}$, $|B_2(y) \cap A^{\delta'}| = 0$. Hence, in combination with (39) and (47), T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$ that are centered at vertices in

$$U \cup \{u_{3M+1}\} \cup C \cup C^\alpha \cup C^\beta \cup C^\gamma \cup A^\alpha \cup A^\beta \cup A^\gamma \cup V^\alpha \cup V^\beta \cup V^\gamma \cup V^{\alpha,*} \cup V^{\beta,*} \cup V^{\gamma,*}. \tag{48}$$

Combining (42)–(45) and (48), T satisfies the non-clashing condition for all pairs of balls of radius 2 in $\mathcal{B}(G)$ that are centered at vertices in

$$V(G). \tag{49}$$

Combining (11), (34), and (49), we get that T is an NCTM^+ of size at most k for $\mathcal{B}(G)$. ■

Lemma 6 *If G admits an* NCTM^+ *for $\mathcal{B}(G)$ of size k, then ϕ is satisfiable.*

Proof Suppose that T is an NCTM^+ for $\mathcal{B}(G)$ of size k. We first prove some properties of T. For each $\ell \in [3M]$, to ensure that T satisfies the non-clashing condition for the pair $B_2(u_{3M+1}) = V(G)$ and $B_2(u_\ell) = V(G) \setminus \{w_\ell\}$, we have that $|T(u_{3M+1}, 2) \cap W| = 3M$. For each $i \in [N]$ and $\delta \in \{\alpha, \beta, \gamma\}$, to ensure that T satisfies the non-clashing condition for the pair $B_2(u_{3M+1}) = V(G)$ and $B_2(c_i^\delta) = V(G) \setminus \{t_{2i}^\delta, f_{2i-1}^\delta\}$, we have that $|T(u_{3M+1}, 2) \cap \{t_{2i}^\delta, f_{2i-1}^\delta\}| \geq 1$. Since $k = 3N + 3M$, by the two previous arguments, it must be that $|T(u_{3M+1}, 2) \cap \{t_{2i}^\delta, f_{2i-1}^\delta\}| = 1$ for each $i \in [N]$ and $\delta \in \{\alpha, \beta, \gamma\}$.

From $T(u_{3M+1}, 2)$, we extract an assignment $\pi : X^\alpha \cup X^\beta \cup X^\gamma \to \{\text{True}, \text{False}\}$ for ϕ. For each $i \in [N]$ and $\delta \in \{\alpha, \beta, \gamma\}$, if $T(u_{3M+1}, 2) \cap \{t_{2i}^\delta, f_{2i-1}^\delta\} = \{t_{2i}^\delta\}$, then set $\pi(x_i^\delta) = \text{True}$, and otherwise, set $\pi(x_i^\delta) = \text{False}$. Thus, each variable in ϕ is assigned exactly one truth value by π. It remains to show that π is a satisfying assignment for ϕ.

Recall that, for each $\ell \in [M]$, we have that c_ℓ is the vertex in G corresponding to the clause C_ℓ in ϕ. By the construction, for each $i \in [N]$ and $\delta \in \{\alpha, \beta, \gamma\}$, if x_i^δ appears as a positive (negative, respectively) literal in C_ℓ, then $t_{2i}^\delta \notin B_2(c_\ell)$ ($f_{2i-1}^\delta \notin B_2(c_\ell)$, respectively). Moreover, these are the only vertices of G that are not in $B_2(c_\ell)$. Since, for all $\ell \in [M]$, T satisfies the non-clashing condition for the pair $B_2(u_{3M+1}) = V(G)$ and $B_2(c_\ell)$, we have that $T(u_{3M+1}, 2)$ contains at least one of the vertices missing from $B_2(c_\ell)$. Further, for each of these vertices in $T(u_{3M+1}, 2)$ that are missing from $B_2(c_\ell)$, π assigns the corresponding truth value of that vertex to the corresponding variable. Since this is true for the clause vertices in C corresponding to all the clauses in ϕ, we have that π is a satisfying assignment for ϕ. ■

Theorem 7 B-NCTD^+ *admits*

- *an algorithm running in time* $2^{2^{\mathcal{O}(\mathtt{vc})}} \cdot n^{\mathcal{O}(1)}$*, and*
- *a kernelization algorithm outputting a kernel with* $2^{\mathcal{O}(\mathtt{vc})}$ *vertices.*

Proof We begin by proving that B-NCTD^+ admits a kernelization algorithm outputting a kernel with $2^{\mathcal{O}(\mathtt{vc})}$ vertices. Given a graph G, let $X \subseteq V(G)$ be a minimum vertex cover of G, that is, $I =: V(G) \setminus X$ is an independent set. If a minimum vertex cover is not given, then we can compute a 2-approximate vertex cover in polynomial time. The kernelization algorithm exhaustively applies Reduction Rule 1 to G, which is safe by Lemma 8. Now, for any instance on which Reduction Rule 1 cannot be applied, it holds that, for any $Y \subseteq X$, there are at most $2^{\mathtt{vc}} + 1$ vertices in I whose open neighborhoods are exactly Y. Since there are $2^{\mathtt{vc}}$ distinct subsets of vertices of X, there are at most $2^{\mathtt{vc}} \cdot (2^{\mathtt{vc}} + 1) + \mathtt{vc} = 2^{\mathcal{O}(\mathtt{vc})}$ vertices in the reduced instance.

To obtain an algorithm running in time $2^{2^{\mathcal{O}(\mathtt{vc})}} \cdot n^{\mathcal{O}(1)}$, one can apply the above (polynomial-time) kernelization algorithm to the graph G, and then apply the algorithm from Proposition 2 to the resulting kernel. ■

Lemma 8 *Reduction Rule 1 is safe for* B-NCTD$^+$.

Proof Let $S \subseteq I$ be a set of $2^{|X|} + 2$ vertices that are pairwise false twins. Let T be an NCTM^+ of size at most k for $\mathcal{B}(G)$. Since T satisfies the non-clashing condition, for any $u, v \in S$, at least one of the inclusions $u \in T(u,1), v \in T(v,1)$ holds. Thus, there is at most one vertex $w \in S$ such that $w \notin T(w,1)$. Note that, for any $u \in S$, $T(u,1) \subseteq B_1(u) \subseteq X \cup \{u\}$. As there are at most $2^{|X|}$ distinct subsets of the vertices of X, and there is at most one vertex $w \in S$ such that $w \notin T(w,1)$, since $|S| = 2^{|X|} + 2$, there exist two vertices $x, y \in S$ such that $x \in T(x,1)$, $y \in T(y,1)$, and $T(x,1) \setminus \{x\} = T(y,1) \setminus \{y\}$. Pick any vertex $z \in V(G) \setminus \{y\}$ and any $r \in \mathbb{N}$ such that $y \in T(z,r)$. We assert that removing y from $T(z,r)$ and adding another carefully chosen vertex v to $T(z,r)$ maintains that T is an NCTM^+ of size at most k for $\mathcal{B}(G)$. If it was not the case that x was in $T(z,r)$, then $v = x$, and otherwise, v is any other vertex in $S \setminus \{y\}$ (if $S \setminus \{y\} \subseteq T(z,r)$, then y is simply removed from $T(z,r)$ and no vertex is added to it).

Namely, let T' be the map obtained from T by applying the above procedure for all $z \in V(G) \setminus \{y\}$ and any $r \in \mathbb{N}$ such that $y \in T(z,r)$. Note that $x \in T'(z,r)$ and $T(z,r) \setminus \{y\} \subseteq T'(z,r)$. Clearly, T' has size at most k, so it remains to show that T' is an NCTM^+ for $\mathcal{B}(G)$. The presence of y in $T(z,r)$ could only be used to satisfy the non-clashing condition between $B_r(z)$ (which contains S) and a ball B' that contains at most 1 vertex from $S \setminus \{y\}$ since any ball in G contains 0, 1 or $|S|$ vertices from S as the vertices of S are pairwise false twins. If $|B' \cap S| = 0$, then $T'(z,r)$ satisfies the non-clashing condition for the pair $B_r(z)$ and B' since $x \in T'(z,r) \cap S$. Otherwise, $|B' \cap S| = 1$, and so, B' is a ball of radius 0 or 1 centered at a vertex in S. In this case, $T'(z,r)$ clearly satisfies the non-clashing condition for the pair $B_r(z)$ and B', as long as B' is not the ball of radius 0 or 1 centered at x. Thus, let $B' \in \{B_0(x), B_1(x)\}$. For T to be an NCTM^+ for $\mathcal{B}(G)$, it must be that $T(z,r) \neq \{x\}$, and thus, $T'(z,r) \neq \{x\}$, since otherwise T would not satisfy the non-clashing condition for the pair $B_r(z)$ and $B_0(x)$. Hence, let $B' = B_1(x)$. If there exists $w_1 \in T(z,r)$ such that $w_1 \notin B' \cup \{y\}$, then T' satisfies the non-clashing condition for the pair $B_r(z)$ and B' since $T(z,r) \setminus \{y\} \subseteq T'(z,r)$. So, assume no such vertex w_1 exists. Similarly, if there exists $w_2 \in T(x,1)$ such that $w_2 \notin B_r(z)$, then T' satisfies the non-clashing condition for the pair $B_r(z)$ and B'. So, assume no such vertex w_2 exists. Then, it must be that $T(z,r) \setminus \{y\} \subseteq B_1(x)$ and $T(x,1) \subseteq B_r(z)$. Thus, $T(y,1) \subseteq B_r(z)$ since $T(x,1) \setminus \{x\} = T(y,1) \setminus \{y\}$ and $x, y \in B_r(z)$. Since T is an NCTM^+ for $\mathcal{B}(G)$, $T(z,r) \setminus B_1(y) \neq \varnothing$. Let the vertex s be in $T(z,r) \setminus B_1(y)$, and note that $s \in T'(z,r) \setminus B_1(y)$ since $T(z,r) \setminus \{y\} \subseteq T'(z,r)$. If $s \neq x$, then $T'(z,r)$ satisfies the non-clashing condition for the pair $B_r(z)$ and $B_1(x)$. If $s = x$, then there exists $t \in T'(z,r) \cap (S \setminus \{x,y\})$ since either $v = t$ or t was already in $T(z,r)$. In this case, $T'(z,r)$ satisfies the non-clashing condition for the pair $B_r(z)$ and $B_1(x)$. Consequently, T' is an NCTM^+ for $\mathcal{B}(G)$. Since y is not contained in $T'(z,r)$, then T' restricted to the vertices of $G \setminus \{y\}$ is an NCTM^+ of size at most k for $\mathcal{B}(G \setminus \{y\})$.

For the reverse direction, let T' be an NCTM^+ of size at most k for $\mathcal{B}(G \setminus \{y\})$. Without loss of generality, assume that $T'(z,r) \neq \varnothing$ for any $z \in V(G)$ and any integer $r \geq 0$. First, note that the addition of y does not make any two balls that were the same in $G \setminus \{y\}$ become distinct in G. Indeed, if both balls contained every vertex in $S \setminus \{y\}$, then they will both contain y; if both balls did not contain any vertex in $S \setminus \{y\}$, then neither of them will contain y; if both balls contained exactly one vertex in $S \setminus \{y\}$, then neither of them will contain y. Hence, it suffices to extend T' to an NCTM^+ T of size at most k for $\mathcal{B}(G)$, by defining $T(y,r)$ for all $r \in \mathbb{N}$ so that T satisfies the non-clashing condition for any pair of balls, where one ball is centered in y. Thus, let $T(z,r) := T'(z,r)$

for all $z \in V(G) \setminus \{y\}$ and $r \in \mathbb{N}$. As before, let $x \in S$ be such that $x \in T(x,1)$ (there must be such an x since there is at most one vertex $w \in S$ such that $w \notin T(w,1)$). Set $T(y,0) := \{y\}$, $T(y,1) := \{y\} \cup (T'(x,1) \setminus \{x\})$, and $T(y,r) = T'(x,r)$ for all integers $r \geq 2$. Note that $B_1(x) \setminus \{x\} = B_1(y) \setminus \{y\}$ and $B_r(x) = B_r(y)$ for all integers $r \geq 2$.

To show that T is an NCTM^+ for $\mathcal{B}(G)$, we have to show that it satisfies the non-clashing condition for a ball $B \in \{B_0(y), B_1(y)\}$ and any other ball $B' = B_r(z)$. First, let $z = y$. Since T' satisfies the non-clashing condition for $B_0(x)$ and $B_r(x)$ for any $r > 0$, $T'(x,r) \setminus \{x\} \neq \varnothing$, and thus, $T(y,r) \setminus \{y\} \neq \varnothing$ since $T'(x,r) \setminus \{x\} \subseteq T(y,r) \setminus \{y\}$ for all $r > 0$. Moreover, since T' satisfies the non-clashing condition for $B_1(x)$ and $B_r(x)$ for any $r > 1$, there exists $u \in T'(x,r) \setminus B_1(x)$. Note that $u \neq y$, and thus, $u \in T(y,r) \setminus B_1(y)$. Consequently, T satisfies the non-clashing condition for a ball B in $\{B_0(y), B_1(y)\}$ and any other ball $B_r(y)$. Now, let $z = x$. If $r \leq 1$, then $y \in T(y,r) \setminus B_r(x)$. If $r \geq 2$, then $B_r(x) = B_r(y)$ and, by the previous case, T satisfies the non-clashing condition for $B \in \{B_0(y), B_1(y)\}$ and $B_r(x)$. Finally, let $z \notin \{x,y\}$. Since $T(z,r) = T'(z,r)$ is non-empty and does not contain y, we have that $T(z,r) \setminus B_0(y) \neq \varnothing$. If $T(z,r) \setminus B_1(y) \neq \varnothing$, then T satisfies the non-clashing condition for $B_1(y)$ and $B_r(z)$. Suppose now that $T(z,r) \subseteq B_1(y)$. Since $y \notin T(z,r) = T'(z,r)$, $T'(z,r) \subseteq B_1(y) \setminus \{y\} \subseteq B_1(x)$. Since T' satisfies the non-clashing condition for $B_r(z)$ and $B_1(x)$, there exists $u \in T'(x,1) \setminus B_r(z)$. If $u \neq x$, then $u \in T(y,1) \setminus B_r(z)$, and if $u = x$, then $B_r(z)$ contains neither x nor y, and thus, $y \in T(y,1) \setminus B_r(z)$. In any case, T satisfies the non-clashing condition for $B_1(y)$ and $B_r(z)$. Hence, T is an NCTM^+ of size at most k for $\mathcal{B}(G)$. ■

We continue with Lemma 14, which we reformulate and prove in a more general form through the following lemma and its corollary:

Lemma 17 *(Lemma 11, Chalopin et al. (2023)) Let $B \in \mathcal{B}$, X be a realizable sample for B, and $\{u,v\}$ be a diametral pair of X^+ in $\mathfrak{K}$. If $B_r(x) \in [B]$, x' is the apex of x with respect to u and v, and $r' = r - d(x,x')$, then X is a realizable sample for $B_{r'}(x')$. Consequently, the path of cycles $C(u,v)$ contains a center of a ball realizing X.*

Corollary 18 *If $\{u,v\}$ is a diametral pair of a ball $B_r(x)$, x' is the apex of x with respect to u and v, and $r' = r - d(x,x')$, then $B_{r'}(x') = B_r(x)$. In particular, if $B \in \mathcal{B}$, $\{u,v\}$ is a diametral pair of B, and $B_r(x)$ is a minimal ball of $[B]$, then x belongs to $C(u,v)$.*

Proof Set $X := V(\mathfrak{K})$. By Lemma 17, $B_r(x) = X^+ \subset B_{r'}(x') \subseteq B_r(x)$, yielding $B_{r'}(x') = B_r(x)$. ■

Theorem 15 *Let $B_1, B_2 \in \mathcal{B}$ be two balls of the same diameter in a tree of cycles $\mathfrak{K}$ such that, for all $q \in \underline{T}(B_1) \cup \underline{T}(B_2)$, $q \in B_1$ if and only if $q \in B_2$. Then, $B_2 = B_1$. Consequently, T is an* NCTM *of size 4 for $\mathcal{B}(\mathfrak{K})$, and thus, NCTD$(\mathcal{B}(\mathfrak{K})) \leq 4$, while* $\mathrm{VCD}(\mathcal{B}(\mathfrak{K})) \leq 3$.

Proof First, analogous to the case of trees, T is non-clashing for any pair of balls including a ball of radius 0. Now, suppose that B_1 is defined by the minimal ball $B_{r_1}(x) \in [B_1]$ and let u, v be the diametral pair of B_1 defining $T^+(B_1)$. Analogously, suppose that B_2 is defined by the minimal ball $B_{r_2}(y) \in [B_2]$. By contradiction, assume that $B_1 \neq B_2$ and that T does not satisfy the non-clashing condition for B_1 and B_2. Without loss of generality, suppose that there exists a vertex $z \in B_2 \setminus B_1$.

Claim 19 *The vertex x is not a cut vertex of $C(u,v)$.*

Proof By Corollary 18, x belongs to $C(u,v)$. If x disconnects u and v, we can suppose, without loss of generality, that x also disconnects z and u. Consequently, $d(z,u) = d(z,x) + d(x,u) > r_1 + d(x,u) \geq d(v,x) + d(x,u) = d(v,u)$. Thus, $\mathrm{diam}(B_2) \geq d(u,z) > d(u,v) = \mathrm{diam}(B_1)$, contrary to the assumption that B_1 and B_2 have the same diameter. ■

Consequently, x belongs to a unique (gated) cycle C of $C(u,v)$. Let y' and z' be the gates of y and z in C. Recall also that u' and v' are the gates of u and v in C. Since $\mathrm{diam}(B_1) = \mathrm{diam}(B_2)$ and $T^+(B_1) \subset B_2$, u,v is also a diametral pair of B_2. By Corollary 18 applied to $B_2 = B_{r_2}(y)$ and the diametral pair u,v of B_2, we conclude that $y \in C(u,v)$. Therefore, either y belongs to the cycle C or the gate y' of y in C coincides with u' or v'.

Claim 20 $z' \notin I(x,u') \cup I(x,v')$.

Proof Suppose by way of contradiction that $z' \in I(x,u')$. Then, z' and v' separate v and z, and thus, $d(v,z) = d(v,v') + d(v',z') + d(z',z)$. First, suppose that $x \in I(v',z')$. Then, $x \in I(v,z)$. Since $u \in B_{r_1}(x)$ and $z \notin B_{r_1}(x)$, we obtain that $d(v,u) \leq d(v,x) + d(x,u) < d(v,x) + d(x,z) = d(v,z)$. Consequently, $\mathrm{diam}(B_2) > \mathrm{diam}(B_1)$, a contradiction. Now, suppose that $x \notin I(v',z')$. This implies that $u' \in I(z',v')$. Since u,v is a diametral pair of B_2, we obtain that $d(v,v') + d(v',u') + d(u',z') + d(z',z) = d(v,z) \leq d(v,u) = d(v,v') + d(v',u') + d(u',u)$, yielding $d(u',z') + d(z',z) \leq d(u',u)$. Since $z' \in I(x,u')$, $u \in B_{r_1}(x)$, and $z \notin B_{r_1}(x)$, we obtain that $d(x,z') + d(z',u') + d(u',u) = d(x,u) \leq r_1 < d(x,z) = d(x,z') + d(z',z)$, yielding $d(z',u') + d(u',u) < d(z',z)$. From the inequalities $d(u',z') + d(z',z) \leq d(u',u)$ and $d(z',u') + d(u',u) < d(z',z)$, we obtain that $d(u',z') < 0$, a contradiction. ■

Since x is the apex of x with respect to u and v, the shortest paths $I(x,u')$ and $I(x,v')$ intersect only in x, and their union is the (u',v')-path passing via x. By Claim 20, z' belongs to the complementary (u',v')-path P of C and $I(x,z') \cap \{u',v'\} \neq \varnothing$. Hence, the set $Z(x,u,v)$ is non-empty. Indeed, let w be a vertex of $I(x,z)$ at distance r_1+1 from x. Then, either z' is the gate of w in C or w is a vertex of the path P, showing that $w \in Z(x,u,v)$. Consequently, $Z(x,u,v) \neq \varnothing$, and thus, at least one of the vertices s,t exists. If s (t, respectively) exists, then its gate s' (t', respectively) in C belongs to the path P. Since T does not satisfy the non-clashing condition for B_1 and B_2, if s (t, respectively) exists, then $s \notin B_2$ ($t \notin B_2$, respectively), and thus, $z \neq s$ ($z \neq t$, respectively).

Claim 21 $z' \in I(u',s') \cup I(v',t')$

Proof Suppose first that $C \nsubseteq B_1$. If the claim does not hold, then z' belongs to the (s',t')-path of C that does not contain x. This implies that removing $s = s'$ and $t = t'$ disconnects z and u. Since $u,z \in B_2$ and $s,t \notin B_2$, we obtain a contradiction.

Suppose now that $C \subseteq B_1$. Since $z' \notin I(x,u') \cup I(x,v')$, u' and v' disconnect x from z' and z. Consequently, $I(x,z) \cap \{u',v'\} \neq \varnothing$. Without loss of generality, assume that $u' \in I(x,z')$ and consider a vertex $w \in I(x,z)$ such that $d(x,w) = r_1 + 1$. Since $C \subseteq B_1$, $w \notin C$ and z' disconnects x and w, *i.e.*, $w' = z'$. Therefore, $w \in Z^u(x,u,v)$, and thus, by the definition of s, we have $d(u',s') \geq d(u',w')$. Consequently, $w' = z' \in I(u',s')$, and we are done. ■

Without loss of generality, assume that $z' \in I(u', s')$. By Claim 20, $z' \neq u'$. Since $u' \in I(z', x) \subseteq I(s', x)$, $z' \in I(x, z)$, and $s' \in I(x, s)$, we obtain that $d(x, z) = d(x, u') + d(u', z') + d(z', z)$ and $d(x, s) = d(x, u') + d(u', z') + d(z', s') + d(s', s)$. Since $d(x, z) \geq r_1 + 1 = d(x, s)$, from the two previous equalities we conclude that $d(z', z) \geq d(z', s') + d(s', s)$. Recall that the vertex y belongs to $C(u, v)$, and thus, either $y = y' \in C$ or $y' \in \{u', v'\}$. Consequently, $d(y, z) = d(y, y') + d(y', z') + d(z', z)$. Since $s \notin B_2$, $r_2 < d(y, s) = d(y, y') + d(y', s') + d(s', s) \leq d(y, y') + d(y', z') + d(z', s') + d(s', s) \leq d(y, y') + d(y', z') + d(z', z) = d(y, z) \leq r_2$, a contradiction (by the triangle inequality and the inequality $d(z', s') + d(s', s) \leq d(z', z)$ established above). This contradiction establishes that under the conditions of the theorem, we must have $B_1 = B_2$.

The second assertion is a consequence of the first one since two balls with distinct diameters are distinguished by the diametral pair of the ball with the larger diameter. Finally, $\mathrm{VCD}(\mathcal{B}(\mathfrak{K})) \leq 3$ follows from the fact that trees of cycles $\mathfrak{K}$ cannot be contracted to K_4 and the result of Bousquet and Thomassé (2015); Chepoi et al. (2007) that if a graph G does not contain K_{d+1} as a minor, then $\mathrm{VCD}(\mathcal{B}(G)) \leq d$. ■

We continue with the proof of the theorem for δ-hyperbolic graphs.

Theorem 16 *For a δ-hyperbolic graph $\mathfrak{H}$, T is an NCTM$^+_{2\delta}$ of size 2 for $\mathcal{B}(\mathfrak{H})$.*

Proof The proof is similar to the proof for trees. Clearly, $T(x, r) \subseteq B_r(x)$ for any vertex x and any radius r, and thus, the map T satisfies the inclusion condition. Hence, since $|T(x, r)| = 2$ for any ball $B_r(x)$ with $r \geq 1$, T is non-clashing for any pair of balls that includes a ball of radius 0. Now, assume that $B_{r_1}(x)$ and $B_{r_2}(y)$ are not δ-identical, and suppose, without loss of generality, that there exists $z \in B_{r_2}(y) \setminus B_{r_1+2\delta}(x)$. Let $T(y, r_2) = \{u, v\}$ and note that, since $\mathfrak{H}$ is δ-hyperbolic, we have $d(x, z) + d(u, v) \leq \max\{d(x, u) + d(v, z), d(x, v) + d(u, z)\} + 2\delta$. Without loss of generality, assume that $d(x, z) + d(u, v) \leq d(x, v) + d(u, z) + 2\delta$. Since $u, z \in B_{r_2}(y)$ and since $\{u, v\}$ is a diametral pair of $B_{r_2}(y)$, $d(u, z) \leq d(u, v)$. Consequently, $r_1 + 2\delta < d(x, z) \leq d(x, v) + 2\delta$, and thus, $v \notin B_{r_1}(x)$. Therefore, $v \in (B_{r_2}(y) \setminus B_{r_1}(y)) \cap (T(x, r_1) \cup T(y, r_2))$, establishing that T satisfies the non-clashing condition for $B_{r_1}(x)$ and $B_{r_2}(y)$. ■