Oracle-Efficient Hybrid Online Learning with Unknown Distribution

Changlong Wu  
CSoI, Purdue University  
WUCHANGL@HAWAII.EDU

Jin Sima  
ECE Department, UIUC  
JSIMA@ILLINOIS.EDU

Wojciech Szpankowski  
CSoI, Purdue University  
SZPAN@PURDUE.EDU

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Abstract

We study the problem of oracle-efficient hybrid online learning when the features are generated by an unknown i.i.d. process and the labels are generated adversarially. Assuming access to an (offline) ERM oracle, we show that there exists a computationally efficient online predictor that achieves a regret upper bounded by $\tilde{O}(T^{3/4})$ for a finite-VC class, and upper bounded by $\tilde{O}(T^{p+1/p})$ for a class with $\alpha$ fat-shattering dimension $\alpha^{-p}$. This provides the first known oracle-efficient sublinear regret bounds for hybrid online learning with an unknown feature generation process. In particular, it confirms a conjecture of Lazaric and Munos (2012). We then extend our result to the scenario of shifting distributions with $K$ changes, yielding a regret of order $\tilde{O}(T^{4/5}K^{1/5})$. Finally, we establish a regret of $\tilde{O}((K^{2/3}(\log |\mathcal{H}|)^{1/3})^2 + K) \cdot T^{2/5}$ for the contextual $K$-armed bandits with a finite policy set $\mathcal{H}$, i.i.d. generated contexts from an unknown distribution, and adversarially generated costs.

Keywords: Hybrid online learning, ERM oracle, oracle-efficiency, relaxation, random playout

1. Introduction

We study the problem of hybrid stochastic-adversary online learning, where the features are assumed to be sampled from an unknown stochastic source while the labels are selected adversarially. Recent advancements Lazaric and Munos (2009); Rakhlin et al. (2012); Haghtalab et al. (2020, 2022a,b); Block et al. (2022); Wu et al. (2023a,b) have demonstrated that such hybrid settings provide a fundamental paradigm shift beyond the classical worst-case adversarial online setting to accommodate broader stochastic scenarios, while still preserving the capacity to handle various adversarial situations and maintain minimal assumptions on the expert class.

We are interested in oracle-efficient regret minimization methods as in Kakade and Kalai (2005). Here, we assume that the learner has access to an Empirical Risk Minimization (ERM) optimization oracle which, given any sequence of feature-label pairs, identifies the expert within the class that achieves the minimal cumulative loss. This effectively reduces the online learning problem to a batch learning problem, for which algorithms such as gradient descent have been highly successful in computing ERM optimization, even in complex classes like neural networks. Previous studies have applied this methodology in various online learning scenarios, including transductive online learning as in Kakade and Kalai (2005), online learning with smooth adversary samples as in Rakhlin et al. (2012); Haghtalab et al. (2022a); Block et al. (2022), and contextual bandits with a known i.i.d. feature generation distribution as in Rakhlin and Sridharan (2016); Syrgkanis et al. (2016); Banihashem et al. (2023). However, all of these works have assumed some form of access...
to a sampling oracle for the feature generation process. This may not be realistic when feature generation is costly, such as in medical data, or when the underlying probability law is unknown a priori. Other studies, like Lazaric and Munos (2009); Wu et al. (2023a,b), do address scenarios with unknown distributions, but provide only computationally inefficient prediction rules.

This paper initiates the study of oracle-efficient hybrid online learning without assuming any access to the underlying probability law of the feature generation process. For the clarity of presentation, we will mainly focus on scenarios of online learning where features are generated by an unknown i.i.d. process. However, we will also consider extensions to other scenarios, such as shifting distributions and contextual multi-armed bandits. Our approach also provides a general methodology that concentrates on the feature efficiency in online learning.

**Problem formulation.** Let \( \mathcal{X} \) be an instance (feature) space and \( \mathcal{H} \subset [0,1]^\mathcal{X} \) be a function class mapping \( \mathcal{X} \rightarrow [0,1] \). We consider the following hybrid online learning scenario. Nature selects an (unknown) distribution \( \mu \) over \( \mathcal{X} \) at the start of the game. At each time step \( t \), Nature independently samples \( x_t \sim \mu \) and selects adversarially a \( y_t \in [0,1] \), but reveals only \( x_t \). A predictor then (randomly) generates \( \hat{y}_t \in [0,1] \) based on \( x^t, y^{t-1} \), where \( x^t = \{x_1, \ldots, x_t\} \) and \( y^{t-1} = \{y_1, \ldots, y_{t-1}\} \). Nature then reveals \( y_t \), and the predictor incurs a loss \( \ell(\hat{y}_t, y_t) \), for a predefined loss function \( \ell : [0,1]^2 \rightarrow \mathbb{R}^+ \). Here, we assume that the loss \( \ell \) is convex in its first argument and \( L \)-Lipschitz in both arguments, e.g., the absolute loss \( \ell(\hat{y}, y) = |\hat{y} - y| \). A prediction rule is a function \( \Phi \) that takes inputs from \( (\mathcal{X} \times [0,1])^* \times \mathcal{X} \) and generates a distribution over \( [0,1] \). For any prediction rule \( \Phi \) and function class \( \mathcal{H} \), we define the hybrid minimax regret as:

\[
\tilde{r}_T(\mathcal{H}, \Phi) = \sup_{\mu} \sup_{y_t \in [0,1]} \mathbb{E}_{\hat{y}_t} \sum_{t=1}^{T} \ell(\hat{y}_t, y_t) - \inf_{h \in \mathcal{H}} \sum_{t=1}^{T} \ell(h(x_t), y_t),
\]

where \( x_t \sim \mu \) and \( \hat{y}_t \sim \Phi(x^t, y^{t-1}) \) for \( t \in [T] \). Our goal is to find an oracle-efficient prediction rule \( \Phi \) that minimizes \( \tilde{r}_T(\mathcal{H}, \Phi) \).

1.1. Results and Techniques

In this work, we provide the first known oracle-efficient sub-linear regret bounds for the hybrid online learning with unknown feature generation distributions.

**Theorem 1 (Informal)** Let \( \mathcal{H} \subset [0,1]^\mathcal{X} \) be a class with Rademacher complexity \( O(T^{q/2}) \). Then, there exists an oracle-efficient predictor with at most \( O(\sqrt{T \log T}) \) calls to the ERM oracle that achieves the hybrid minimax regret of order \( \tilde{O}(T^{\frac{3}{2} - \frac{q}{2q+2}}) \). In particular, for a VC-class, this implies a regret of \( \tilde{O}(T^{\frac{3}{4}}) \), and for classes with \( \alpha \)-fat shattering dimension \( \alpha^{-P} \), it results in a regret of \( \tilde{O}(T^{\max\{\frac{3}{4}, \frac{\log \log T}{\log T}\}}) \), where \( \tilde{O} \) hides poly-log factors.

To the best of our knowledge, the regret bounds presented in Theorem 1 are the first known oracle-efficient sub-linear regret bounds for hybrid online learning with unknown feature generation distributions and generic (non-parametric) hypothesis classes. In particular, our \( \tilde{O}(T^{\frac{3}{4}}) \) bound confirms a conjecture of Lazaric and Munos (2012) regarding the oracle-efficient regret bounds for finite-VC classes under absolute loss \(^1\). Note that, it was demonstrated by (Block et al., 2022, Theorem 7) that for known feature generation distributions, an \( \tilde{O}(T^{\max\{\frac{1}{2}, \frac{\log T}{T}\}}) \) regret bound is achievable for a class

\(^1\) They also obtained an \( O(\sqrt{T \log T}) \) bound using a computationally inefficient covering-based approach.
with an $\alpha$-fat shattering dimension of order $\alpha^{-p}$. However, to the best of our knowledge, such a chaining-based bound was not known for the unknown distribution case, even in the information-theoretical sense. The closest comparison is the regret bound obtained in Wu et al. (2023b) that matches our $O(T^{\frac{n+1}{p+2}})$ bound, but established via an (inefficient) one-step covering approach.

At a high level, our oracle-efficient prediction rule is based on the relaxation and random play-out techniques, as introduced in Rakhlin et al. (2012). However, a distinguishing feature of our setup is that we are not able to access the sampling oracle of the underlying feature generating process. Our main idea is to employ an epoch-based approach. We partition the time horizon into a set of carefully designed epochs. At each epoch, we estimate the underlying distribution $\mu$ by $\hat{\mu}$ using samples observed in prior epochs. We then use the estimated distribution $\hat{\mu}$ to generate the hallucinated samples as needed in the relaxation framework for the current epoch. Observe, however, that the estimation $\hat{\mu}$ can arbitrarily deviate from $\mu$ under total variation, as we do not make any structural assumption on $\mu$, and the adversary knows $\hat{\mu}$ when generating adversary samples. Therefore, the randomness matching argument, as in Rakhlin et al. (2012); Block et al. (2022), will not work. To overcome this issue, we introduce a surrogate relaxation based on $\hat{\mu}$ and relate it to the regret via a novel concept of approx-admissibility, which is further controlled by a novel symmetrization argument. The regret will then follow by carefully designing the epochs to balance the error introduced by the approx-admissibility and the Rademacher complexity of the class restricted to each epoch.

Tighter bounds for special cases. Going beyond the general result in Theorem 1, we show in Theorem 23 (Appendix F) that an oracle-efficient $O(T^{\max\{\frac{1}{2}, \frac{p+1}{p+2}\}})$ regret is achievable for finite fat-shattering classes of order $p$ if we assume a weaker oblivious adversary. This is tight up to poly-logarithmic factors. Furthermore, for the class $\mathcal{H}$ of all Lipschitz functions $[0, 1]^d \to [0, 1]$ we establish in Theorem 15 the (optimal) regret $O(T^{\max\{\frac{1}{2}, \frac{d-1}{d}\}})$ against the adaptive adversary.

Shifting distributions. Our next result drops the i.i.d. assumption and allows distributions to change over time. We assume that the number of changes is upper bounded by $K$ and that the possible distributions and change points are completely unconstrained and unknown a priori. We show that the oracle-efficient regret is upper bounded by $O(T^{\frac{d}{2} K^{\frac{1}{2}}})$ for a finite VC class. Note that an $O(\sqrt{KT \log T})$ bound was demonstrated by Wu et al. (2023a). However, their algorithm relies on constructing an exponentially large covering, thus being computationally inefficient.

Contextual $K$-arm Bandits. Finally, we establish an $O((K^{\frac{d}{2}} (\log |\mathcal{H}|)^{\frac{1}{2}} + K \sqrt{\log K}) \cdot T^{\frac{1}{2}})$ oracle-efficient regret bound for the contextual $K$-armed bandits with a finite policy set $\mathcal{H}$, where the contexts are generated by an unknown i.i.d. process and the costs are selected adversarially. The closest comparison to this result is the $O(T^{\frac{d}{2}} (K \log |\mathcal{H}|)^{\frac{1}{2}})$ bound established in Banihashem et al. (2023), only for the known i.i.d. distribution case. Notably, our result answers positively a question of Banihashem et al. (2023) regarding relaxing the sampling access to the context distribution.

2. Notation and Preliminaries

Oracle-Efficient Predictors. We adopt the following mixed-ERM oracle from Block et al. (2022). Let $(x_1, y_1), \cdots, (x_m, y_m) \in \mathcal{X} \times [0, 1], C \in \mathbb{R}^+, \epsilon_n \in \{-1, +1\}^n$, and $\tilde{x}_n \in \mathcal{X}^n$. The mixed ERM oracle is the task of finding $\inf_{h \in \mathcal{H}} \left\{ \sum_{i=1}^{m} \ell(h(x_i), y_i) + C \sum_{j=1}^{n} \epsilon_j h(\tilde{x}_j) \right\}$. Note that the
unintuitive parts \( eh(x) \) can be interpreted as an absolute loss since \( eh(x) = |h(x) - \frac{(1-\epsilon)}{2}| - \frac{1-\epsilon}{2} \) for \( \epsilon \in \{-1, +1\} \). Therefore, the mixed ERM oracle is reduced to a regular (weighted) ERM oracle if \( \ell \) is the absolute loss. Moreover, the weight \( C \) can be understood as repeating the same sample \( C \) times (rounding to an integer if necessary). We say a predictor \( \Phi \) is oracle-efficient if the running time of computing \( \hat{y}_t \sim \Phi(x^t, y^{t-1}) \) is polynomial with respect to \( t \) by accessing a mixed ERM oracle (with each oracle call treated as unit time) for any \( x^t, y^{t-1} \).

**Adaptive v.s. Oblivious.** Note that our formulation in (1) assumes that the generation of \( y_t \)s is adaptive, since the selection of \( y_t \) at each time step depends on all prior information \( x^t, y^{t-1} \), and \( y^t \). For comparison, we also introduce a weaker notion of adversary, namely, the oblivious adversary, which selects the \( y_t \)s based only on the current instance \( x_t \) (see Theorem 23 in Appendix F). It turns out that the adaptive nature of the adversary constitutes the main obstacle in our analysis.

**Hybrid Contextual Bandits.** We now formulate the contextual \( K \)-arm bandits within our framework. Let \( \mathcal{D}_K \) be the set of all probability distributions over \([K]\). A policy set \( \mathcal{H} \) is a class of functions \( \mathcal{X} \rightarrow [K] \). We consider the following bandit setup: Nature selects some \( \mu \) at the start of the game. At each time step \( t \), Nature samples \( x_t \sim \mu \) and selects adversarially a cost vector \( c_t \in [0, 1]^K \), but reveals only \( x_t \). A predictor then selects a distribution \( q_t \in \mathcal{D}_K \) based on the history observed thus far and samples \( \hat{y}_t \sim q_t \). Nature reveals only \( c_t[\hat{y}_t] \) which is also the predictor incurred loss. The goal is to find an oracle-efficient prediction rule \( \Phi : (\mathcal{X} \times [0, 1])^* \times \mathcal{X} \rightarrow \mathcal{D}_K \) that minimizes: \( \tilde{r}^\text{bandit}_T(\mathcal{H}, \Phi) = \sup_\mu \mathbb{E}_{x_1} \sup_\ldots \sup_{x_T} \sup_{c_1} \sup_{c_T} \mathbb{E}_{q^T} \left[ \sum_{t=1}^{T} \langle q_t, c_t \rangle - \inf_{h \in \mathcal{H}} \sum_{t=1}^{T} c_t[h(x_t)] \right] \), where \( q_t = \Phi(x^t, c_1[\hat{y}_1], \ldots, c_{t-1}[\hat{y}_{t-1}]). \) Here, the ERM oracle is to find \( \inf_{h \in \mathcal{H}} \sum_{i=1}^{m} c_i[h(x_i)] \) for any \( (x_1, c_1), \ldots, (x_m, c_m) \in \mathcal{X} \times \mathbb{R}^K \), as Rakhlin and Sridharan (2016).

### 3. Oracle-Efficient Regret Bounds for Online Learning

In this section, we focus on bounding the hybrid minimax regret for online learning as in (1) with a generic hypothesis class \( \mathcal{H} \), using oracle-efficient predictors by accessing to an mixed ERM oracle. We first recall the following standard notion of Rademacher complexity:

**Definition 2** Let \( \mathcal{H} \subset [0, 1]^\mathcal{X} \) be a function class and \( T \in \mathbb{N}^+ \). The Rademacher complexity of \( \mathcal{H} \) at horizon \( T \) is defined to be \( \text{Rad}_T(\mathcal{H}) = \sup_{x^T \in \mathcal{X}^T} \mathbb{E}_{\epsilon^T} \left[ \sup_{h \in \mathcal{H}} \sum_{t=1}^{T} \epsilon_t h(x_t) \right] \), where \( \epsilon_t \) is i.i.d. sampled from the uniform distribution over \( \{\pm 1\} \).

We are now ready to state our first main result:

**Theorem 3** Let \( \mathcal{H} \subset [0, 1]^\mathcal{X} \) be a class with \( \text{Rad}_T(\mathcal{H}) \leq O(T^q) \) for some \( q \in [\frac{1}{2}, 1] \), and let \( \ell \) be a \( L \)-Lipschitz loss that is convex in its first argument. Then there exists an oracle-efficient prediction rule \( \Phi \) with at most \( O(L \sqrt{T \log T}) \) calls to the ERM oracle per round, such that

\[
\tilde{r}_T(\mathcal{H}, \Phi) \leq O(L \sqrt{\log(LT)} \cdot T^{\frac{3-q}{2q}}).
\]

In particular, for a binary-valued class with finite VC-dimension, we have \( \tilde{r}_T(\mathcal{H}, \Phi) \leq O(L \sqrt{\mathbb{VC}(\mathcal{H}) \log(LT)} \cdot T^{\frac{3}{4}}) \), and for a real-valued class \( \mathcal{H} \) with an \( \alpha \)-fat shattering dimension of order \( \alpha^{-p} \) for \( p > 0 \) (Alon et al., 1997), we have \( \tilde{r}_T(\mathcal{H}, \Phi) \leq O(LT^{\max\left(\frac{p+1}{4}, \frac{1}{p+2}\right)}) \).
The rest of this section is devoted to establishing Theorem 3. At a high level, we will partition the time horizon into a set of epochs and make predictions at each epoch using features in prior epochs as side-information. Our main proof technique is an epoch-based predictor introduced in Section 3.2, together with a novel approx-admissibility framework for handling prediction with side-information, as developed in Section 3.1.

3.1. Regret Analysis with Side-Information

We first consider a hypothetical scenario where we assume the predictor has access to some side-information \( x_{0:M} \) sampled i.i.d. from the same distribution \( \mu \). It is crucial to note that this information is known to the adversarial as well, i.e., the adversary’s strategy could also depend on \( x_{0:N} \), which turns out to be the main obstacle in our analysis.

Formally, we consider the following learning game proceeds over a horizon of length \( M \): At the start of the game, Nature selects an unknown distribution \( \mu \) over \( \mathcal{X} \), samples an i.i.d. sample \( x_{0:N+1} \) of size \( N \) from \( \mu \) and reveals \( x_{0:N+1} \) to a predictor; At each time step \( j \in [M] \), Nature samples \( x_j \) and selects adversarially \( y_j \in [0,1] \) (depends on \( x_{N+1} \) and \( y^{j-1} \)) but reveals only \( x_j \); The predictor then (randomly) generates \( \hat{y}_j \in [0,1] \) based on \( x_{N+1} \) and \( y^{j-1} \); Nature reveals \( y_j \) and the predictor incurs loss \( \ell(\hat{y}_j, y_j) \), for some predefined convex and \( L \)-Lipschitz loss.

**Predictor via surrogate relaxation.** Let \( \hat{\mu}_N \) be the empirical distribution \( \hat{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \), where \( \delta_x \) is the Dirac measure on \( x \). For any time step \( j \in [M] \) and horizon \( M \) satisfying \( M \leq N/2 \), we construct the following randomized prediction rule:

1. Sample (internally) the hallucinated samples \( \hat{x}_{j+1}, \ldots, \hat{x}_M \) from \( \hat{\mu}_N \) without replacement \(^2\) and \( \epsilon_{j+1}, \ldots, \epsilon_M \) i.i.d. from the uniform distribution over \( \{-1,+1\} \);

2. Make prediction

\[
\hat{y}_j = \arg \min_{\hat{y} \in [0,1]} \sup_{y \in [0,1]} \left\{ \ell(\hat{y}, y) + \sup_{h \in \mathcal{H}} \left[ 2L \sum_{i=j+1}^{M} \epsilon_i h(\hat{x}_i) - \ell(h(x_j), y) - \sum_{i=1}^{j-1} \ell(h(x_i), y_i) \right] \right\}.
\]

(2)

Note that the main difference from the classical random play-out techniques such as in Rakhlin et al. (2012); Block et al. (2022) is that the hallucinated samples are generated from \( \hat{\mu}_N \) instead of \( \mu \). Crucially, our sampling is performed without replacement (not i.i.d.), which is essential for our following analysis (Lemma 9). More generally, one may also replace the estimation \( \hat{\mu}_N \) with other estimation rules instead of the empirical distribution we used here. This could provide tighter bounds if the distribution \( \mu \) is well structured, see Section 3.3. The following lemma shows that the predictor \( \hat{y}_j \) can be computed efficiently by accessing to a mixed-ERM oracle.

**Lemma 4** The predictor \( \hat{y}_j \) can be computed up to error \( \pm \frac{1}{L \sqrt{M}} \) by making at most \( O(L \sqrt{M} \log M) \) mixed-ERM oracle calls. Moreover, for binary valued class \( \mathcal{H} \) with \( y \in \{0,1\} \) and absolute loss, we need only 2 (regular) ERM oracle calls to compute \( \hat{y}_j \) exactly.

\(^2\) For technical reasons, we assume here that \( \hat{x}_{j+1} \) is sampled from \( \hat{\mu}_N \) without replacement. Equivalently, \( \hat{x}_{j+1} \) is sampled uniformly from all (permuted) subsequeunces of \( x_{0:N+1} \) of length \( M - j \).
**Lemma 5 (Approx-Admissibility)** Let \( \hat{\mu} \) be obtained by a "backward tracing" argument. The detailed proof is deferred to Appendix B.

**Analysis of the regret.** Denote by \( \Phi \) the prediction rule derived from (2). We consider the following analogous hybrid minimax regret as in (1) with the additional side-information:

\[
\tilde{r}_{M,N}^{\text{side}}(H, \Phi) = \sup_{\mu} \mathbb{E}_{x_{N+1}} \sup_{y_1} \mathbb{E}_{x_1} \cdots \mathbb{E}_{x_M} \sup_{y_M} \mathbb{E}_{\hat{y}_M} \left[ \sum_{j=1}^{M} \ell(\hat{y}_j, y_j) - \inf_{h \in H} \sum_{j=1}^{M} \ell(h(x_j), y_j) \right],
\]

where the randomness of \( \hat{y}_j \)'s is over the \( \hat{x}_i \)'s as in (2), while \( x_j \)'s are sampled i.i.d. from \( \mu \).

To proceed, we first introduce the following key concept. Let \( (x_1, y_1), \ldots, (x_M, y_M) \in X \times [0, 1] \) be any realization of the feature-label pairs. We write \( L_j^h = \sum_{i=1}^{j} \ell(h(x_i), y_i) \) to simplify our discussion. The surrogate relaxation is defined as

\[
R_j = \mathbb{E}_{\hat{x}, \epsilon} \left[ \sup_{h \in H} 2L \sum_{i=j+1}^{M} \epsilon_i h(\hat{x}_i) - L_j^h \right],
\]

where \( \hat{x}_i \)'s and \( \epsilon_i \)'s are generated the same way as in (2). We also define the following variation that replaces the single \( \hat{x}_{j+1} \) with a sample \( x \sim \mu \):

\[
\tilde{R}_j = \mathbb{E}_{x \sim \mu} \mathbb{E}_{\hat{x}, \epsilon} \left[ \sup_{h \in H} 2L \epsilon_{j+1} h(x) + 2L \sum_{i=j+2}^{M} \epsilon_i h(\hat{x}_i) - L_j^h \right].
\]

Note that the main technique for proving the relaxation based regret bounds, such as Rakhlin et al. (2012), is through the concept of admissibility, which essentially asserts that \( \mathbb{E}_{x} \sup_{y_j} \mathbb{E}_{\epsilon, \hat{x}} [\ell(\hat{y}_j, y_j) + R_j] \leq R_{j-1} \). However, a major technical step for establishing such an result is based on the so-called randomness matching argument by leveraging the fact that the hallucinated samples used to define the relaxation are the same as the actual feature generating process. This, unfortunately, is not true in our case since the empirical distribution \( \hat{\mu}_N \) can deviate arbitrarily from \( \mu \) under total variation, regardless of how large the sample size \( N \) is. We instead establish the following approx-admissibility of our surrogate relaxation, with the proof deferred to Appendix A.

**Lemma 5 (Approx-Admissibility)** Let \( \hat{y}_j \) be as in (2), then for all \( j \in [M] \) we have:

\[
\mathbb{E}_{x_j} \sup_{y_j} \mathbb{E}_{\epsilon, \hat{x}} [\ell(\hat{y}_j, y_j) + R_j] \leq \tilde{R}_{j-1}.
\]

We are now ready to state our first main technical lemma of this section, which follows from Lemma 5 by a "backward tracing" argument. The detailed proof is deferred to Appendix B.

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3. Throughout the paper, we use the convention \( \mathbb{E}_{x, \epsilon} \equiv \mathbb{E}_{x_{j+1}, \epsilon_{j+1}} \) to simplify notation.
Lemma 6 (Regret Bound via Approx-Admissibility) Let $\Phi$ be the predictor as in (2). Then for any class $\mathcal{H} \subset [0,1]^X$ with a convex and $L$-Lipschitz loss $\ell$, we have

$$
\tilde{r}_{\mathcal{H}}^N(\mathcal{H}, \Phi) \leq \mathbb{E}_{x^0_{-N+1}} \left[ \tilde{R}_0 + \sum_{j=1}^{M-1} \mathbb{E}_{x^j} \sup_{y^j} (\tilde{R}_j - R_j) \right],
$$

where $x^M_{-N+1}$ are sampled i.i.d. from $\mu$ and $R_j$, $\tilde{R}_j$ are defined as in (4) and (5).

Remark 7 Note that the decomposition presented in Lemma 6 holds whenever the approx-admissibility condition of Lemma 5 is satisfied. We believe this could be applicable to a broader set of problems and is of independent interest.

Bounding the relaxations. By Lemma 6, we know that the regret $\tilde{r}_{\mathcal{H}}^N(\mathcal{H}, \Phi)$ can be upper bounded by $\tilde{R}_0$ and the discrepancies between $R_j$ and $\tilde{R}_j$. Clearly, by the definition of $\tilde{R}_j$, we have $\tilde{R}_0 \leq 2L \text{Rad}_M(\mathcal{H})$, where $\text{Rad}_M(\mathcal{H})$ is the Rademacher complexity of $\mathcal{H}$ as in Definition 2. To bound the discrepancies, for any $j \in [M-1]$, $x^j, \tilde{x}^+_{j+2} \in \mathcal{X}^*$, $\epsilon^M_{j+1} \in \{\pm 1\}^*$ and $y^j \in [0,1]^2$, we define the following function:

$$
\tilde{f}_{x^j, \tilde{x}^M_{j+2}, \epsilon^M_{j+1}, y^j}(x) = \sup_{h \in \mathcal{H}} \left\{ 2L \epsilon_{j+1} h(x) + 2L \sum_{i=j+2}^{M} \epsilon_i h(\tilde{x}_i) - L_i^h \right\}.
$$

The following fact is a consequence of our definitions.

Fact 1 We have $R_j = \mathbb{E}_{\tilde{x}} \tilde{f}_{x^j, \tilde{x}^M_{j+2}, \epsilon^M_{j+1}, y^j}(\tilde{x}_j+1)$ and $\tilde{R}_j = \mathbb{E}_{x \sim \mu} \mathbb{E}_{\tilde{x}, \epsilon} \tilde{f}_{x^j, \tilde{x}^M_{j+2}, \epsilon^M_{j+1}, y^j}(x)$.

Let $z_j = (x^j, \tilde{x}^M_{j+2}, \epsilon^M_{j+1})$. We now observe the following key properties of the functions $f_{x^j, y^j}(x)$, which demonstrates that $f_{x^j, y^j}(x)$ has sensitivity upper bounded by $4L$ and is Lipschitz on $y^j$. We refer to Appendix D for the detailed proof.

Proposition 8 For any $z_j$ and $y^j$, we have $\sup_{x, x'} |f_{x^j, y^j}(x) - f_{x^j, y^j}(x')| \leq 4L$. Moreover, for all $z_j$, $x$ and $y^j$, $y^j \in [0,1]^2$, we have $|f_{x^j, y^j}(x) - f_{x^j, y^j}(x)| \leq j L ||y^j - y^j||_{\infty}$.

Note that Proposition 8 and Fact 1 immediately imply that $\tilde{R}_j - R_j \leq 4L ||\mu - \hat{\mu}_N||_{TV}$.

Unfortunately, we are unable to bound the total variation distance $||\mu - \hat{\mu}_N||_{TV}$ due to the lack of any structure we impose on $\mu$. We instead establish the following key technical result, which bounds the discrepancies via a Rademacher sum of the functions $f_{x^j, y^j}$. This result constitutes the main technical ingredient in our following analysis.

Lemma 9 For all $j \in [M-1]$, $M \leq N/2$ and $B = N - M + j + 1$, we find

$$
\mathbb{E}_{x^0_{-N+1}} \mathbb{E}_{x^j} \sup_{y^j}(\tilde{R}_j - R_j) \leq \sup_{x^M_{-N+1} \sim \mu^B, z_j} \mathbb{E}_{\epsilon^B} \left[ \sup_{y^j} \frac{1}{B} \sum_{i=1}^{B} \epsilon_i (f_{x^j, y^j}(x^j_i) - f_{x^j, y^j}(x^j_{-N+i})) \right],
$$

where $x^M_{-N+1}, x^B, z_j$ run over all possible values and $\epsilon^B$ is distributed uniformly over $\{\pm 1\}^B$.

4. Using the fact that $E_{x \sim \mu}[f(x)] - E_{x \sim \nu}[f(x)] \leq \sup_{x, x'} |f(x) - f(x')| \cdot ||\mu - \nu||_{TV}$.
Proof [Sketch] We highlight only the main idea here and refer to Appendix C for the complete proof. By Fact 1, we can upper bound the discrepancies by $\mathbb{E}_{x_{N+1}} \mathbb{E}_{x_j} \sup_{y_i} [\mathbb{E}_{x} f_{x,j,y} (x)] - \mathbb{E}_{x_{N+1}} [f_{x_j,y} (x_{j+1})]$, where $z_j = (x^j, \tilde{x}^M_{j+2}, \epsilon^M_j)$. Note that $\tilde{x}^M_{j+1}$ is sampled uniformly from $x^0_{-N+1}$ without replacement as in (2). Therefore, the randomness of $\tilde{x}^M_{j+1}$ can be described as follows: we first sample $\tilde{x}^M_{j+2}$ from $x^0_{-N+1}$ and then sample $\tilde{x}_{j+1}$ uniformly from the remaining samples in $x^0_{-N+1}$. Now, the key observation is that, by symmetries of $x^0_{-N+1}$ (which are i.i.d.), we can fix $\tilde{x}^M_{j+2}$ being the last $M-j-1$ samples in $x^0_{-N+1}$. Therefore, we have $\mathbb{E}_{x_{N+1}} [f_{x_j,y} (x_{j+1})] = \frac{1}{B} \sum_{i=1}^B f_{z_j,y} (x_{-N+i})$, where $B = N-M+j+1$. Since $z_j$ is decoupled from $x_{-N+1}^0$ by our construction, we obtain the upper bound $\mathbb{E}_{x_j} \mathbb{E}_{x_{-N+1}^0} \sup_{y_i} [\mathbb{E}_{x} f_{x_j,y} (x)] - \frac{1}{B} \sum_{i=1}^B f_{z_j,y} (x_{-N+i})].$

The lemma then follows by symmetrization with $\mathbb{E}_{x} f_{x_j,y} (x)$ (see Appendix C).

For any $j \in [M-1]$ and $z_j$ as above, we define the following function class $G_{z_j}$:

$$G_{z_j} = \{g_{z_j,y} (x, x') \defeq f_{x_j,y} (x') - f_{x_j,y} (x) : y \in [0,1]^j, (x, x') \in X^2\}.$$  

(10)

Lemma 9 essentially states that the discrepancy between $R_1$ and $\tilde{R}_j$ is upper bounded by the Rademacher complexity of the class $G_{z_j}$, as $\mathbb{E}_{x_{-N+1}^0} \sup_{y_i} (R_j - \tilde{R}_j) \leq \sup_{z_j} \frac{1}{B} \text{Rad}_B (G_{z_j}).$

The following lemma provides a useful bound on such Rademacher complexities.

Lemma 10 Let $G_{z_j}$ be as in (10), $M \leq N/2$ and $B = N - M + j + 1$. Then

$$\sup_{z_j} \frac{1}{B} \text{Rad}_B (G_{z_j}) \leq O \left( \sqrt{\frac{j L^2 \log(jLB)}{B}} \right) \leq O \left( \sqrt{\frac{2j L^2 \log(jLN/2)}{N}} \right).$$  

(11)

Proof Let $C \subset [0,1]^j$ be a covering of $[0,1]^j$ with norm $L_\infty$ radius $\frac{1}{2} jLB$. We have $|C| \leq (jLB)^j$.

By the second part of Proposition 8, we know that the class $G'_{z_j} \defeq \{g_{z_j,y} : y \in C\}$ forms a uniform $L_\infty$-covering of $G_{z_j}$ with radius $\frac{1}{2} B$. Therefore, $\frac{1}{B} \text{Rad}_B (G_{z_j}) \leq \frac{1}{B} \text{Rad}_B (G'_{z_j}) + \frac{1}{B}$. The first inequality then follows by a simple application of Massart’s lemma (Shalev-Shwartz and Ben-David, 2014, Lemma 26.8) over $G'_{z_j}$, since $|G'_{z_j}| \leq |C| \leq (jLB)^j$ and $\sup_{(x,x') \in X^2} \{g_{z_j,y} (x, x')\} \leq 4L$ for all $g_{z_j,y} \in G_{z_j}$ due to the first part of Proposition 8. The second inequality is implied by that $B \geq N/2$ and the fact that the function $\frac{\log B}{B}$ is monotone decreasing.

Putting everything together, we arrive at:

Theorem 11 Let $\Phi$ be the predictor as in (2) and $M \leq N/2$. Then for any class $H \subset [0,1]^X$ with a convex and $L$-Lipschitz loss $\ell$, the predictor $\Phi$ can be computed efficiently with access to at most $O(L \sqrt{M \log M})$ mixed-ERM oracle calls per round such that

$$\tau_{M,N}^\text{side} (H, \Phi) \leq 2L \text{Rad}_M (H) + \sqrt{M} + O \left( \sqrt{\frac{M^3 L^2 \log(MLN)}{N}} \right).$$  

(12)

Proof The regret bound follows directly from Lemma 4, Lemma 6 and Lemma 9. We then invoke Lemma 10 to bound the discrepancies by noticing that $j \leq M$. 

5. Note that the "complexity" of $G_{z_j}$ arises from the $y \in [0,1]^j$. 

Remark 12 Note that Theorem 11 shows that if \( N \gg M^2 \log M \) then the regret with side-information is reduced to the Rademacher complexities of \( \mathcal{H} \), and thus matches the case when the distribution is known in advance. However, in reality such side-information is not available for the unknown distribution case, which can only be obtained from prior samples.

3.2. Proof of Theorem 3: the Epoch Approach

We are now equipped with all the technical tools to prove Theorem 3, with the only missing ingredient of constructing the side-information. For this purpose, we employ an epoch-based approach, resembling those used in Lazaric and Munos (2009); Wu et al. (2023a), but in a completely different context. We partition the time horizon into epochs, with epoch \( n \) of length \( M(n) \). Let \( S(n) = \sum_{i=1}^{n-1} M(i) \) be the total time steps after \( n - 1 \) epochs. We will use the features observed up to time \( S(n) \) as the side-information introduces in Section 3.1 and apply the predictor constructed in (2) to make the prediction during the \( n \)th epoch.

\[
N := S(n) \quad M(n) \quad \text{Epoch 1} \quad \cdots \quad \text{Epoch } n \quad \cdots
\]

To this end, our main technical part is to optimize the epoch length \( M(n) \) that balances the trade-off in (12) and achieving the minimal total regret. Let \( \Phi \) be the predictor derived from (2), which we write as \( \Phi(x^0_{-N+1}, x^j, y^{j-1}) \) for the side-information \( x^0_{-N+1} \), features \( x^j \) and labels \( y^{j-1} \) observed thus far. We define the following epoch predictor \( \Psi \): for any epoch \( n \) and time step \( j \) during such epoch, we set

\[
\Psi(x^{S(n)+j}, y^{S(n)+j-1}) = \Phi \left( x^{S(n)}, x^{S(n)+j}, y^{S(n)+j-1} \right).
\]

Let \( S^{-1}(T) \) be the largest number \( n \) such that \( S(n) < T \). The following lemma upper bounds the hybrid minimax regret (1) of \( \Psi \) using the regrets with side information (3) incurred by \( \Phi \). Note that this is not immediately obvious since we have reused the side-information among different epochs.

Lemma 13 For any \( \mathcal{H} \) and convex \( L \)-Lipschitz loss \( \ell \), we have

\[
\tilde{r}_T(\mathcal{H}, \Psi) \leq \sum_{n=1}^{S^{-1}(T)} \tilde{r}_{M(n), S(n)}^{\text{side}}(\mathcal{H}, \Phi).
\]

Proof Define the operator \( Q_i^T \equiv \mathbb{E}_{x_i} \sup_{y_i} \mathbb{E}_{\hat{y}_i} \cdots \mathbb{E}_{x_j} \sup_{y_j} \mathbb{E}_{\hat{y}_j} \), where \( \hat{y}_t \sim \Psi(x^t, y^{t-1}) \) for all \( t \in [T] \). We have (truncate the last \( S(n+1) \) above \( T \) if necessary):

\[
\tilde{r}_T(\mathcal{H}, \Psi) = Q_i^T \sup_{h \in \mathcal{H}} \left[ \sum_{n=1}^{S^{-1}(T)} \sum_{j=S(n)+1}^{S(n+1)} \ell(\hat{y}_j, y_j) - \ell(h(x_j), y_j) \right] \\
\leq \sum_{n=1}^{(a)} \sum_{j=S(n)+1}^{S(n+1)} \ell(\hat{y}_j, y_j) - \ell(h(x_j), y_j)
\]
any given $z$ as binary valued classes.

$M$, $S(n) \leq T$ we have

$$\tilde{r}_{\text{side}}^{M(n), S(n)}(H, \Phi) \leq O \left( L M(n)^q + \sqrt{\frac{M(n)^3 L^2 \log(LT^2)}{S(n)}} \right).$$

Let $M(n) = n^\alpha$ for some $\alpha > 0$ to be determined later. We have $S(n) = \sum_{i=1}^{n-1} i^\alpha = \Theta(n^{\alpha+1})$ by integration approximation, and $S^{-1}(T) \leq O(T^{1/(\alpha+1)})$. This implies that $\tilde{r}_{\text{side}}^{M(n), S(n)}(H, \Phi) \leq O(Ln^{\alpha q} + L\sqrt{\log(LT^2)}n^{\alpha - \frac{1}{2}}).$ By Lemma 13 and integration approximation again, we conclude

$$\tilde{r}_T(H, \Phi) \leq O \left( L T^{\frac{\alpha+1}{\alpha+1}} + L\sqrt{\log(LT^2)}T^{\frac{\alpha+1}{\alpha+1}} \right).$$

Optimizing $\arg \min_{\alpha > 0} \max \{ \frac{\alpha q + 1}{\alpha+1}, \frac{\alpha+1}{\alpha+1} \}$, we find (14) is minimized when taking $\alpha = \frac{1}{2(1-q)}$.

Plugging back to (14), we find $\tilde{r}_T(H, \Phi) \leq O \left( L\sqrt{\log(LT^2)}T^{\frac{2-q}{2q - q+1}} \right).$ This completes the proof of the first part. The second and third parts follow by the facts that $\text{Rad}_T(H) \leq O(\sqrt{\text{VC}(H)T})$ for finite-VC class (Wainwright, 2019, Example 5.24), and $\text{Rad}_T(H) \leq \tilde{O}(T^{\max \{ \frac{1}{2p+1}, \frac{1}{p-1} \}})$ for classes with $\alpha$-fat shattering dimension of order $\alpha^{-p}$ (Block et al., 2022). This completes the proof and the big-O notations and $M(n) \leq S(n)/2$ are justified by noting that $\alpha \geq 1$ since $q \geq \frac{1}{2}$.

### 3.3. Tighter Bounds for Special Classes

As demonstrated in Section 3.1, the main technical obstacle for analyzing the regret is to upper bound the discrepancies between $\tilde{R}_j$ and $R_j$ as in Lemma 6. It was shown in Lemma 9 that such discrepancies can be upper bounded by the Rademacher complexity of the class $G_{z_j}$ in (10). We demonstrate in this section how to leverage the structural information of $G_{z_j}$, leading to tighter regret bounds for certain special classes, when compared to the general bounds from Theorem 3.

**Binary valued classes.** Let $H \subseteq \{0, 1\}^X$ be a binary valued class and $\ell(\hat{y}, y) = |\hat{y} - y|$. For any given $z_j$ (assume, w.l.o.g., $z_{j+1} = 1$) and $y^j \in \{0, 1\}^j$, the function $f_{z_j, y^j}$ can be expressed as $f_{z_j, y^j}(x) = \sup_{h \in H} \{2h(x) + F(h)\}$ (see definition in (8)), where $F(h)$ is a discrete valued function taking values in $[-2M, 2M]$. Define $H_0 = \{h \in H : F(h) = \sup_{h' \in H} F(h')\}$ and $H_1 = \{h \in H : F(h) = \sup_{h' \in H} F(h') - 1\}$. Let $h_0(x) = \sup_{h \in H_0} \{h(x)\}$, $h_1(x) = \sup_{h \in H_1} \{h(x)\}$ and $\hat{h} = \arg \max_{h \in H} F(h)$. The following structural characterization of $f_{z_j, y^j}$ holds:
Fact 2 For any $x$, if $h^0(x) = 1$ then $f_{z_j, y^j}(x) = F(\hat{h}) + 2$; if $h^0(x) = 0$ and $h^1(x) = 1$ then $f_{z_j, y^j}(x) = F(\hat{h}) + 1$; else $f_{z_j, y^j}(x) = F(\hat{h})$.

Theorem 14 Let $\mathcal{H} \subset \{0,1\}^X$, $\mathcal{F}^i = \{ f_{\mathcal{H}^i}(x) = \sup_{h \in \mathcal{H}^i} \{ h(x) \} : \mathcal{H}^i \subset \mathcal{H} \}$, $\mathcal{F} = \{ f_{\mathcal{H}^0}(x) = \inf_{h \in \mathcal{H}^0} \{ h(x) \} : \mathcal{H}^0 \subset \mathcal{H} \}$ be two classes of functions and $\ell$ be the absolute loss. Then there exists an oracle-efficient predictor $\Phi$ satisfying $\tilde{r}_T(\mathcal{H}, \Phi) \leq O(\sqrt{\max\{VC(\mathcal{F}^u), VC(\mathcal{F})\}} \sqrt{T})$.

Proof Sketch By Fact 2, we know $f_{z_j, y^j}$ is completely determined by the functions $h^0, h^1$. Therefore, the "complexity" of $G_{z_j}$ is reduced to that of $\{h^0, h^1\}$s. The proof then follows by the key observation that $h^0, h^1 \in \mathcal{F}^u$ and techniques in Section 3.1 & 3.2. See Appendix D for details.

Note that for the threshold functions $\mathcal{H} = \{1\{x \geq a\} : a, x \in [0,1]\}$ we have $\mathcal{F}^u = \mathcal{F}^i = \mathcal{H}$. Theorem 14 implies an oracle efficient $O(\sqrt{T})$ regret, which matches the information-theoretical lower bound and is tighter than the covering-based $O(T \log T)$ bound implied by Lazaric and Munos (2009). Another example is the class of indicators of intervals with bounded length $\{1\{x \in [a, b]\} : b - a \geq \gamma, [a, b] \subset [0,1]\}$, for which we have $VC(\mathcal{F}^i) = 2$ and $VC(\mathcal{F}^u) \leq O(1)$.

Lipschitz functions. Let $\mathcal{X} = [0,1]^d$ and $\mathcal{H} \subset [0,1]^X$ be the class of all 1-Lipschitz functions under $L_\infty$ norm. Assume $\ell(\hat{y}, y) = |\hat{y} - y|$ is the absolute loss. Let $\mu$ and $\hat{\mu}_N$ be the true and empirical distributions, respectively, as in Section 3.1. By Fact 1 and assuming that $x_{i+1}^M$ is sampled i.i.d. from $\hat{\mu}_N$, we have $\mathbb{E}_{x_{i+1}^M} \sup_{y, z} (\hat{R}_j - R_j) \leq 2W_1(\mu, \hat{\mu}_N)$, where $W_1(\mu, \hat{\mu}_N) = \sup_{y, z} (\mathbb{E}_{x \sim \mu} [f_{z_j, y^j}(x)] - \mathbb{E}_{x \sim \hat{\mu}_N} [f_{z_j, y^j}(x)])$. By the same argument as Proposition 8 (second part) and Lipschitz property of $h$, we have:

Fact 3 For all $z_j, y^j$ and $x, x'$, $|f_{z_j, y^j}(x) - f_{z_j, y^j}(x')| \leq 2||x - x'||_\infty$.

Theorem 15 Let $\mathcal{H}$ and $\ell$ be as above. Then, there exists an oracle-efficient predictor $\Phi$ such that $\tilde{r}_T(\mathcal{H}, \Phi) \leq \tilde{O}(T^{\max\{\frac{1}{2}, \frac{d-1}{d}\}})$, and this bound is tight up to poly-logarithmic factors.

Proof By Fact 3, we know that for all $z_j, y^j$ the function $f_{z_j, y^j}(x)$ is 2-Lipschitz. Therefore, by Kantorovich-Rubinstein duality (Villani, 2003) we have $\sup_{y, z} (\mathbb{E}_{x \sim \mu} [f_{z_j, y^j}(x)] - \mathbb{E}_{x \sim \hat{\mu}_N} [f_{z_j, y^j}(x)]) \leq 2W_1(\mu, \hat{\mu}_N)$, where $W_1(\mu, \hat{\mu}_N) = \sup_{y, z} (\mathbb{E}_{x \sim \mu} [f_{z_j, y^j}(x)] - \mathbb{E}_{x \sim \hat{\mu}_N} [f_{z_j, y^j}(x)])$ is the Wasserstein 1-distance with $\Gamma(\mu, \hat{\mu}_N)$ being the class of all coupling between $\mu, \hat{\mu}_N$. Therefore, we have $\mathbb{E}_{x_{i+1}^M} \sup_{y, z} (\hat{R}_j - R_j) \leq 2\mathbb{E}_{x_{i+1}^M} W_1(\mu, \hat{\mu}_N)$, i.e., the discrepancy is upper bounded by the convergence rate of empirical distribution under Wasserstein 1-distance. Invoking (Fournier and Guillin, 2019, Thm 1) and boundedness of $\mathcal{X}$, we have $\mathbb{E}_{x_{i+1}^M} W_1(\mu, \hat{\mu}_N) \leq \tilde{O}(N^{-1/d})$. Let $\Phi$ be the predictor in (2). By Lemma 6 and $\text{Rad}_M(\mathcal{H}) \leq \tilde{O}(M^{\max\{\frac{1}{2}, \frac{d-1}{d}\}})$ (Wainwright, 2019), we have $\hat{r}_{M,N}(\mathcal{H}) \leq \tilde{O}(M^{\max\{\frac{1}{2}, \frac{d-1}{d}\}} + MN^{-1/d})$. The result then follows by Lemma 13 with $M(n) = 2^n$ (which ensures $N = S(n) = M(n) - 1$). The last part follows by that the $\epsilon$-metric entropy of $\mathcal{H}$ is $\Theta(\frac{1}{\epsilon^d})$ (Wainwright, 2019).

Remark 16 Note that, if we assume certain structure on $\mu$ that admits a computationally efficient estimator $\hat{\mu}_N$ that satisfies $||\mu - \hat{\mu}_N||_TV \leq O(\frac{1}{\sqrt{N}})$ (such as for Gaussian distributions (Ashtiani et al., 2018)), then the (optimal) $O(\text{Rad}(\mathcal{H}) + \sqrt{T})$ bound is achievable for any class $\mathcal{H} \subset [0,1]^X$. 


4. Shifting Distributions

In this section, we consider a scenario where the underlying feature-generating distribution is allowed to change over time. We assume that the total number of changes is upper bounded by $K$, while the selection of distributions and change points are completely arbitrary. It was demonstrated by Wu et al. (2023a) that for finite-VC classes and absolute loss, the regret grows as $O(\sqrt{KT}\log T)$ under such feature generation processes. However, their algorithm depends on the construction of an exponentially sized cover, making it computationally inefficient. We will demonstrate in this section an oracle-efficient algorithm that achieves a slightly worse regret.

We first observe that if we know the positions of the change points, then we can simply run our oracle-efficient predictor from Theorem 3 on each of the segments independently, leading to $\tilde{O}(T^{3/4}K^{1/4})$ regret. Since there are only $T^K$ possible configurations of the change points, we can therefore run an expert algorithm to aggregate each of such configurations, leading to an $\tilde{O}(\sqrt{KT}\log T + T^{3/4}K^{1/4})$ regret. Unfortunately, this approach has computational cost dominated by $\Omega(T^K)$ and therefore not efficient for large $K$. To address this issue, we instead partition the time horizon into epochs with fixed length $B$ and run our oracle-efficient algorithm on each of the epochs independently. The rationale behind this approach is that, if we select $B$ small enough, there will be at least $T/B - K$ epochs with i.i.d. sampling. By tuning the epoch length $B$, we arrive at:

**Proposition 17**

Let $\mathcal{H} \subset \{0, 1\}^X$ be a binary valued class with finite VC-dimension under a convex and L-Lipschitz loss. Then there exists an oracle-efficient predictor $\Phi$ such that if the features are generated by the process with change cost $K$ and the labels are generated adversarially, then the hybrid minimax regret as (1) is upper bounded by $O\left(\sqrt{\text{VC}(\mathcal{H})\log(\frac{LT}{B})}K^{1/4}T^{3/4}\right)$.

**Proof** We partition the time horizon into $\frac{T}{B}$ epochs each of length $B$. Let $\Phi$ be the predictor from Theorem 3 that we run on each epochs independently. By independence, we know that the total regret is upper bounded by the sum of regrets incurred on each of the epochs. Note that, if an epoch does not contain a change point, then the regret is upper bounded by $O(L\sqrt{\text{VC}(\mathcal{H})}\log(\frac{LT}{B})B^{1/2})$ by Theorem 3, else we naively upper bounded by $B$. Since there can be at most $K$ epochs containing a change point, the total regret is upper bounded by $O\left((\frac{T}{B} - K) L\sqrt{\text{VC}(\mathcal{H})}\log(\frac{LT}{B})B^{1/2} + KB\right)$ by optimizing over $B$: We find that $B = T^{3/4}K^{-3/4}$ attains the minimum and the result follows.

5. Contextual $K$-arm Bandits

We now briefly discuss the extension to the contextual $K$-arm bandits as introduced in Section 2, leaving details to Appendix E. The basic idea follows the same path as in the online learning case, where we partition the time steps into epochs, at each epoch we use the sample observed thus far to estimate the underlying distribution and use the estimated distribution to generate the hallucinated samples as needed in the relaxation based algorithms.

**Bandit predictor with side-information.** Let $(x_1, c_1), \ldots, (x_M, c_M)$ be any realization of the context-cost pairs and $x^0_{-N+1}$ be the side-information with $x^M_{-N+1}$ sampled i.i.d. from an (unknown) distribution $\mu$. At each time step $j \in [M]$, we construct the following prediction rule adapted from Syrgkanis et al. (2016) by generating the hallucinated samples from $\tilde{\mu}_N$ instead of $\mu$: 
1. Sample $\tilde{x}^M_{j+1}$ from $\hat{\mu}_N$ without replacement; $\epsilon^M_{j+1}$ i.i.d. from uniform distribution over $\{\pm 1\}^K$; $Z^M_{j+1}$ i.i.d. from distribution over $\{0, \frac{1}{M}\}$ such that $\Pr[Z_i = \frac{1}{M}] = \gamma K$, where $\gamma$ is a parameter to be tuned; Let $e_k$ be the standard base of $\mathbb{R}^K$ with coordinate $k$ being 1;

2. Let $D_K$ be the class of distributions over $[K]$. Find (using ERM oracle):

$$\hat{q}_j = \arg\min_{q \in D_K} \sup_{p_j \in D'} \mathbb{E} \hat{c}_j \sim p_j \left\{ \langle q, \hat{c}_j \rangle - \inf_{h \in H} \left( \sum_{i=1}^j \hat{c}_i[h(x_i)] + \sum_{i=j+1}^M 2\epsilon_i[h(x_i)]Z_i \right) + \gamma(M-j)K \right\},$$

where $D'$ is the class of distributions over $\{\frac{1}{\gamma}e_k : k \in [K]\} \cup \{0\}$ such that the probability equals $\frac{1}{\gamma}e_k$ is upper bounded by $\gamma$ for all $k \in [K]$; and $0 \in \mathbb{R}^K$ is the all 0s vector;

3. Define $q_j = (1 - \gamma K)\hat{q}_j + \gamma 1$ and make prediction $\hat{y}_j \sim q_j$, where 1 is the all 1s vector.

Here, the $\hat{c}_i$ at step 2 is an unbiased estimation of $c_i$ as $\hat{c}_i = \frac{1}{\gamma}I_i e_{\hat{y}_i}$, where $I_i$ is the indicator that takes value 1 w.p. $\frac{c_i[q]}{q}[q]$ with $\hat{y}_i$ and $q_i$ being the predictions at step $i \leq j - 1$.

**Analysis of regret.** Our key idea is to define a surrogate relaxation $R_j$ and $\tilde{R}_j$ for the predictor $\hat{q}_j$ and establish a bandit version of decomposition for the regret with side-information as in Lemma 6. This is achieved via the concept of approx-admissibility as in Lemma 5 and a careful adaption of the admissibility proof from Syrgkanis et al. (2016). The technical challenge then boils down to bounding the discrepancies between $R_j$ and $\tilde{R}_j$, as in Lemma 6. To this end, we employ a technique similar to Lemma 9 that relates the discrepancies to a Rademacher sum and show that the sensitivity of the functions in the sum is upper bounded by $O(K)$. Crucially, the sensitivity is independent of $\gamma$, and therefore the regret with side information is optimized at $\gamma = (\log |H|/KM)^{\frac{1}{3}}$. By leveraging a similar epoch approach as in the proof of Theorem 3, and setting the epoch length to $n^{\frac{3}{2}}$, we arrive at the main result of this section (see Appendix E for a detailed proof):

**Theorem 18** Let $H \subset [K]^X$ be a finite policy set. Then there exists an oracle-efficient predictor $\Phi$ such that the hybrid bandit minimax regret defined in Section 2 is upper bounded by

$$r^\text{bandit}(H, \Phi) \leq O \left( \left( K^{\frac{2}{3}} (\log |H|)^{\frac{1}{3}} + K \sqrt{\log K} \right) \cdot T^{\frac{4}{5}} \right).$$

**Remark 19** Note that, our primary focus here is on the dependency on $T$. We believe a more careful selection of the epoch length and employing techniques from Banihashem et al. (2023) could result in a better dependency on $K$. We leave it as an open problem to improve the $\frac{4}{5}$ exponent of $T$.

### 6. Additional Related Work

The relaxation-based approach was first introduced by Rakhlin et al. (2012), providing a generic method for constructing sequential prediction algorithms (albeit potentially inefficient) for a wide range of online learning scenarios. Rakhlin et al. (2012) demonstrated that an oracle-efficient online learning algorithm is feasible via the so-called random play-out approach, provided one can access a sampling oracle for future features. This includes applications such as transduction online learning (Kakade and Kalai, 2005; Cesa-Bianchi and Shamir, 2013) and known i.i.d. feature generation distributions (Rakhlin and Sridharan, 2016; Syrgkanis et al., 2016; Banihashem et al., 2023). A more
sophisticated scenario, the smooth adversarial setting, was investigated by Rakhlin et al. (2011); Haghtalab et al. (2022b); Block et al. (2022); Haghtalab et al. (2022a). In this setting, the future sampling distribution is not directly accessible but can be stochastically controlled via a coupling argument introduced by Haghtalab et al. (2022b). However, this approach still requires access to a sampling oracle of the underlying reference measure.

An alternative technical approach, the follow the perturbed leader (FTPL) algorithm, has been widely used in the literature for obtaining oracle-efficient algorithms (Kakade and Kalai, 2005; Haghtalab et al., 2022a; Block et al., 2022; Bhatt et al., 2023). A key technical benefit of this approach, compared to the relaxation-based approach, is that the prediction rule is proper (i.e., the prediction can be generated by a function within the hypothesis class) and involves fewer ERM oracle calls. However, it also suffers from higher regret guarantees for general fat-shattering classes Block et al. (2022). Interestingly, the FTPL-based approaches share some technical similarities with our work as well. For instance, our Lemma 6 is similar to the decomposition of regret as presented in (Cesa-Bianchi and Lugosi, 2006, Chapter 4.3). Additionally, the way we control the discrepancies of our surrogate relaxation in Lemma 9 is similar in spirit to methods employed by Haghtalab et al. (2022a); Block et al. (2022) for bounding their ”generalization errors”. However, it is unclear whether this FTPL-based approach can be adapted to our unknown distribution setting.

Concurrent work. We note also a recent concurrent work by Block et al. (2024), which specifically studied the ERM rule for smoothed adversaries with an unknown reference measure under the assumption of realizable labels and with $L_2$ loss. However, this work does not provide sublinear regret bounds for adversarially generated labels as investigated in our paper.

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References


Appendix A. Proof of Lemma 5

In this section, we establish the *approx-admissibility* of our predictor in (2). The reasoning follows closely to the arguments as in (Rakhlin et al., 2012, Lemma 11&12) but needs careful adaption for handling the hallucinated samples \( \tilde{x} \)s generated from \( \tilde{\mu}_N \). We have

\[
\mathbb{E}_{x_j} \sup \mathbb{E}_{y_j, \tilde{x}} \left[ \ell(\hat{y}_j, y_j) + R_j \right] \stackrel{(a)}{=} \mathbb{E}_{x_j} \sup \mathbb{E}_{y_j, \tilde{x}} \left[ \ell(\hat{y}_j, y_j) + \sup_{h \in \mathcal{H}} \sum_{i=1}^{M} \epsilon_i h(\tilde{x}_i) - L_j^h \right]
\]

\[
\leq \mathbb{E}_{x_j} \mathbb{E}_{y_j} \sup \mathbb{E}_{x_j} \left[ \ell(\hat{y}_j, y_j) + \sum_{i=1}^{M} \epsilon_i h(\tilde{x}_i) - \ell(h(x_j), y_j) - L_j^h \right]
\]

\[
\leq \mathbb{E}_{x_j} \mathbb{E}_{y_j} \inf \sup \mathbb{E}_{y_j} \left[ \ell(\hat{y}, y_j) + \sum_{i=1}^{M} \epsilon_i h(\tilde{x}_i) - \ell(h(x_j), y_j) - L_j^h \right]
\]

\[
= \mathbb{E}_{x_j} \mathbb{E}_{y_j} \inf \sup \mathbb{E}_{y_j} \left[ \ell(\hat{y}_j, y_j) + \sum_{i=1}^{M} \epsilon_i h(\tilde{x}_i) - L_j^h + \ell(\hat{y}, y_j) - \ell(h(x_j), y_j) \right]
\]

\[
\leq \mathbb{E}_{x_j} \mathbb{E}_{y_j} \inf \sup \mathbb{E}_{y_j} \left[ \ell(\hat{y}_j, y_j) + \sum_{i=1}^{M} \epsilon_i h(\tilde{x}_i) - L_j^h + \partial \ell(\hat{y}, y_j)(\hat{y} - h(x_j)) \right]
\]

\[
\leq \mathbb{E}_{x_j} \mathbb{E}_{y_j} \inf \sup \mathbb{E}_{y_j} \left[ \ell(\hat{y}_j, y_j) + \sum_{i=1}^{M} \epsilon_i h(\tilde{x}_i) - L_j^h + g_j(\hat{y} - h(x_j)) \right]
\]

where (a) follows by the definition of \( R_j \) and that \( \hat{y}_j \) has the same randomness as \( R_j \) (i.e., the \( \tilde{x} \)s and \( \epsilon \)s); (b) is due to definition of \( \hat{y}_j \); (c) is due to convexity of \( \ell \); (d) is due to \( L \)-Lipschitz property of \( \ell \); (e) follows by that the inner function is convex w.r.t. \( g_j \) and thus the \( \sup_{g_j \in [-L, L]} \) is attained on the boundary \( \{-L, L\} \). We have

\[
\mathbb{E}_{x_j} \mathbb{E}_{x_j} \left[ \inf \sup \sum_{i=1}^{M} \epsilon_i h(\tilde{x}_i) - L_j^h + g_j(\hat{y} - h(x_j)) \right]
\]

\[
\stackrel{(a)}{=} \mathbb{E}_{x_j} \mathbb{E}_{x_j} \left[ \inf \sup \sum_{i=1}^{M} \epsilon_i h(\tilde{x}_i) - L_j^h + g_j(\hat{y} - h(x_j)) \right]
\]

\[
\stackrel{(b)}{=} \mathbb{E}_{x_j} \mathbb{E}_{x_j} \left[ \sup \inf \sum_{i=1}^{M} \epsilon_i h(\tilde{x}_i) - L_j^h + g_j(\hat{y} - h(x_j)) \right]
\]

\[
\stackrel{(c)}{=} \mathbb{E}_{x_j} \mathbb{E}_{x_j} \left[ \sup \inf \sum_{i=1}^{M} \epsilon_i h(\tilde{x}_i) - L_j^h - g_j h(x_j) \right]
\]

17
where $(a)$ follows by $\sup_{g_j \in \{-L, L\}} \equiv \sup_{d_j \in \Delta(\{-L, L\})} \mathbb{E}_{g_j \sim d_j}$ where $\Delta(\{-L, L\})$ is the set of all probability distributions over $\{-L, L\}$; $(b)$ follows by the minimax theorem and noticing that the inner expectation is bi-linear w.r.t. $\hat{y}$ and $d_j$; $(c)$ follows by the fact that $g_j \hat{y}$ is independent of $\sup_h$; $(d)$ follows by that the $\sup_h$ term is independent of $\hat{y}$ and introducing an i.i.d. copy $g_j'$ of $g_j$; $(e)$ follows by the fact that replacing $\hat{y}$ with $h(x_j)$ does not decrease the inf term; $(f)$ is due to $\sup \mathbb{E} \leq \sup \mathbb{E} \sup$; $(g)$ is due to symmetries of $g_j, g_j'$ and $\epsilon_j$ is uniform over $\{-1, 1\}$; $(h)$ follows by $\sup(A + B + C) \leq \sup(A/2 + B) + \sup(A/2 + C) = \left(\sup(A/2B) + \sup(A + 2C)\right)/2$, the linearity of expectation and symmetries of $B, C$; $(i)$ follows by that the inner expectation takes the
same value for all $g_j \in \{-L, L\}$ and therefore the $\sup_{d_j} E_{g_j \sim d_j}$ can be eliminated. This completes the proof.

\textbf{Appendix B. Proof of Lemma 6}

Denote $Q_j \equiv E_{x_{-N+1}}^0 \sup_{y_1} E_{\hat{y}_1} \cdots E_{x_j} \sup_{y_j} E_{\hat{y}_j}$ for notation convenience. We have

\[
\tilde{r}_{\text{side}}^{\text{de}}(\mathcal{H}, \Phi) \overset{(a)}{=} Q_M \left[ \sum_{j=1}^M \ell(\hat{y}_j, y_j) + R_M \right] \]

\[
\overset{(b)}{=} Q_{M-1} \left[ \sum_{j=1}^{M-1} \ell(\hat{y}_j, y_j) + E_{x_M} \sup_{y_M} E_{\hat{y}_M} [\ell(\hat{y}_M, y_M) + R_M] \right] \]

\[
\overset{(c)}{\leq} Q_{M-1} \left[ \sum_{j=1}^{M-1} \ell(\hat{y}_j, y_j) + \tilde{R}_{M-1} \right] \]

\[
= Q_{M-1} \left[ \sum_{j=1}^{M-1} \ell(\hat{y}_j, y_j) + R_{M-1} + (\tilde{R}_{M-1} - R_{M-1}) \right] \]

\[
\overset{(d)}{\leq} Q_{M-1} \left[ \sum_{j=1}^{M-1} \ell(\hat{y}_j, y_j) + R_{M-1} \right] + E_{x_{-N+1}}^0 E_{x_{M-1}} \sup_{y_{M-1}} (\tilde{R}_{M-1} - R_{M-1}) \]

\[
\overset{(e)}{\leq} E_{x_{-N+1}}^0 \tilde{R}_0 + \sum_{j=1}^{M-1} E_{x_{-N+1}}^0 E_{x_{j}} \sup_{y_j} (\tilde{R}_j - R_j),
\]

where (a) follows by definition of $R_M$; (b) follows by extracting the last layer of $Q_M$; (c) follows by Lemma 5 and noticing that $\hat{y}_j$ has the same randomness as $R_j$; (d) follows by the facts that $\sup(A + B) \leq \sup A + \sup B$, $\sup E \leq E \sup$, the linearity of expectation and $\tilde{R}_{M-1} - R_{M-1}$ is independent of $\hat{y}_j$ for all $j \leq M - 1$; (e) follows by repeating the same arguments for another $M - 1$ steps. This completes the proof.

\textbf{Appendix C. Proof of Lemma 9}

We have

\[
E_{x_{-N+1}}^0 E_{x_{j}} \sup_{y_j} (\tilde{R}_j - R_j) \overset{(a)}{=} E_{x_{-N+1}}^0 E_{x_{j}} \sup_{y_j} E_{x_{M-1}, \epsilon_{j+1}} E_{x_{j+1} \sim \mu} [f_{x_{j}, y_j}(x) - f_{x_{j}, y_j}(\tilde{x}_{j+1})] \]

\[
\leq E_{x_{-N+1}}^0 E_{x_{j}} \sup_{y_j} E_{x_{j+1}} \sup_{y_j} E_{x_{j+1} \sim \mu} [f_{x_{j}, y_j}(x) - f_{x_{j}, y_j}(\tilde{x}_{j+1})] \]

\[
\overset{(b)}{=} E_{x_{-N+1}}^0 E_{x_{j}} \sup_{y_j} E_{x_{j+1}} \sup_{y_j} E_{x_{j+1} \sim \mu} [f_{x_{j}, y_j}(x) - f_{x_{j}, y_j}(\tilde{x}_{j+1})] \]

\[
= E_{x_{-N+1}}^0 E_{x_{j}} \sup_{y_j} E_{x_{j+1}} \sup_{y_j} E_{x_{j+1} \sim \mu} [f_{x_{j}, y_j}(x) - f_{x_{j}, y_j}(\tilde{x}_{j+1})] \]

\[
\overset{(c)}{=} E_{x_{j}} E_{x_{-N+1}}^0 \sup_{y_j} \left[ E_{x \sim \mu} [f_{x_{j}, y_j}(x)] - \frac{1}{B} \sum_{i=1}^{B} f_{x_{j}, y_j}(x_{-N+i}) \right] \]
\[ (d) \quad \mathbb{E}_{\mathbf{z}_j} \mathbb{E}_{x_{-N+1}^{-B}} \sup_{y^j} \left[ \frac{1}{B} \sum_{i=1}^{B} \mathbb{E}_{X'_i \sim \mu} [f_{\mathbf{z}_j, y^j}(x'_i)] - f_{\mathbf{z}_j, y^j}(x_{-N+i}) \right] \]

\[ = \mathbb{E}_{\mathbf{z}_j} \mathbb{E}_{x_{-N+1}^{-B}} \sup_{y^j} \left[ \frac{1}{B} \sum_{i=1}^{B} f_{\mathbf{z}_j, y^j}(x'_i) - f_{\mathbf{z}_j, y^j}(x_{-N+i}) \right] \]

\[ \leq \mathbb{E}_{\mathbf{z}_j} \mathbb{E}_{x_{-N+1}^{-B}} \mathbb{E}_{x'^B} \sup_{y^j} \left[ \frac{1}{B} \sum_{i=1}^{B} \epsilon'_j(f_{\mathbf{z}_j, y^j}(x'_i) - f_{\mathbf{z}_j, y^j}(x_{-N+i})) \right] \]

\[ = \mathbb{E}_{\mathbf{z}_j} \mathbb{E}_{x_{-N+1}^{-B}} \mathbb{E}_{x'^B} \sup_{y^j} \left[ \frac{1}{B} \sum_{i=1}^{B} \epsilon'_j(f_{\mathbf{z}_j, y^j}(x'_i) - f_{\mathbf{z}_j, y^j}(x_{-N+i})) \right] \]

where (a) follows by Fact 1 (in Section 3.1); (b) follows by definition of \( \mathbf{z}_j \); (c) follows by Fact 4 below and taking \( B = N - M + j + 1 \); (d) follows by introducing \( B \) fresh i.i.d. samples \( x'^B \sim \mu^B \); (e) follows by symmetries of \( x'^B \) and \( x_{-N+1}^{-B} \) (which are independent of \( \mathbf{z}_j \)) and introducing the \( i.i.d. \) random variables \( \epsilon^B \) uniform over \( \{-1, 1\}^B \).

**Fact 4** Let \( B = N - M + j + 1 \), then

\[ \mathbb{E}_{x_{-N+1}^{0}} \mathbb{E}_{\mathbf{z}_j} \sup_{y^j} \mathbb{E}_{X \sim \mu} [f_{\mathbf{z}_j, y^j}(x)] - \mathbb{E}_{\tilde{x}_{j+1}} [f_{\mathbf{z}_j, y^j}(\tilde{x}_{j+1})] \]

\[ = \mathbb{E}_{\mathbf{z}_j} \mathbb{E}_{x_{-N+1}^{-B}} \sup_{y^j} \left[ \mathbb{E}_{X \sim \mu} [f_{\mathbf{z}_j, y^j}(x)] - \frac{1}{B} \sum_{i=1}^{B} f_{\mathbf{z}_j, y^j}(x_{-N+i}) \right]. \]

**Proof** Note that \( \mathbf{z}_j = (x^j, \tilde{x}_{j+2}^M, \epsilon^M_{j+1}) \), where \( \tilde{x}_{j+1}^M \) are sampled uniformly from \( x_{-N+1}^{0} \) without replacement, and \( x^j, \epsilon_{j+1}^M \) are independent of \( x_{-N+1}^{0} \). Therefore, we have

\[ \mathbb{E}_{x_{-N+1}^{0}} \mathbb{E}_{\mathbf{z}_j} \sup_{y^j} \mathbb{E}_{X \sim \mu} [f_{\mathbf{z}_j, y^j}(x)] - \mathbb{E}_{\tilde{x}_{j+1}} [f_{\mathbf{z}_j, y^j}(\tilde{x}_{j+1})] \]

\[ = \mathbb{E}_{x'^{M} \mathbf{z}_{j+2}^{j}} \mathbb{E}_{x_{-N+1}^{0}} \mathbb{E}_{x_{j+2}^{M}} \sup_{y^j} \mathbb{E}_{X \sim \mu} [f_{\mathbf{z}_j, y^j}(x)] - \mathbb{E}_{\tilde{x}_{j+1}} [f_{\mathbf{z}_j, y^j}(\tilde{x}_{j+1})] \]

\[ \overset{(\star)}{=} \mathbb{E}_{x'^{M} \mathbf{z}_{j+1}^{j}} \mathbb{E}_{x_{-N+1}^{0}} \mathbb{E}_{I} \sup_{y^j} \left[ \mathbb{E}_{X \sim \mu} [f_{\mathbf{z}_j, y^j}(x)] - \frac{1}{B} \sum_{i \in I} f_{\mathbf{z}_j, y^j}(x_{-N+i}) \right] \]

where the key step (\( \star \)) follows by noticing that the randomness of \( \tilde{x}_{j+2}^M \) is equivalent to selecting a random index set \( I \subset [N] \) uniformly with size \( |I| = M - j - 1 \) and the index of \( \tilde{x}_{j+1} \) (in \( x_{-N+1}^{0} \)) is then uniform over \( [N] \setminus I \). 6, where the size of \( [N] \setminus I \) is \( B = N - M + j + 1 \); Therefore,

\[ \mathbb{E}_{\tilde{x}_{j+1}} [f_{\mathbf{z}_j, y^j}(\tilde{x}_{j+1})] = \frac{1}{B} \sum_{i \in [N] \setminus I} f_{\mathbf{z}_j, y^j}(x_{-N+i}). \]

6. By the definition of sampling without replacement.
Note that $x^0_{-N+1}$ is an i.i.d. sample, by symmetries, we can fix $I = \{B + 1, \cdots, N\}$ (i.e., we take $\hat{x}^M_{j+2}$ being $x^0_{-N+B+1}$) and therefore $\hat{x}^M_{j+2}$ can be decoupled from $x^0_{-N+1}$, leading to

\[
\begin{align*}
\mathbb{E}_{x^M_{j+2}} \mathbb{E}_{x^0_{-N+1}} \mathbb{E}_I \sup_{y^j} \left[ \mathbb{E}_{x \sim \mu} [f_{z_j, y^j}(x)] - \frac{1}{B} \sum_{i \in [N] \setminus I} f_{z_j, y^j}(x_{-N+i}) \right] \\
= \mathbb{E}_{x^M_{j+2}} \mathbb{E}_{x^0_{-N+1}} \mathbb{E}_I \sup_{y^j} \left[ \mathbb{E}_{x \sim \mu} [f_{z_j, y^j}(x)] - \frac{1}{B} \sum_{i=1}^B f_{z_j, y^j}(x_{-N+i}) \right] \\
= \mathbb{E}_{x^M_{j+2}} \mathbb{E}_{x^0_{-N+1}} \sup_{y^j} \left[ \mathbb{E}_{x \sim \mu} [f_{z_j, y^j}(x)] - \frac{1}{B} \sum_{i=1}^B f_{z_j, y^j}(x_{-N+i}) \right].
\end{align*}
\]

This completes the proof of the Fact.

**Appendix D. Omitted Proofs**

In this section, we collect all other proofs that are omitted from the main text.

**Proof [Proof of Proposition 8]** Denote

\[
F(h) = 2L \sum_{i=j+2}^M \epsilon_i h(\tilde{x}_i) - L^h_j.
\]

Let $\hat{h} = \arg \max_{h \in \mathcal{H}} F(h)$ (find an approximation if necessary). We claim that for any $x \in \mathcal{X}$,

\[
F(\hat{h}) - 2L \leq \sup_{h \in \mathcal{H}} \{2\epsilon_{j+1} L h(x) + F(h)\} \leq F(\hat{h}) + 2L.
\]

This will complete the proof of the first part. To see the upper bound, we have

\[
\sup_{h \in \mathcal{H}} \{2\epsilon_{j+1} L h(x) + F(h)\} \leq \sup_h \{2\epsilon_{j+1} L h(x)\} + \sup_h F(h) \leq 2L + F(\hat{h}),
\]

since $h(x) \in [0, 1]$. For the lower bound, we have

\[
\sup_{h \in \mathcal{H}} \{2\epsilon_{j+1} L h(x) + F(h)\} \geq 2\epsilon_{j+1} L \hat{h}(x) + F(\hat{h}) \geq F(\hat{h}) - 2L,
\]

since sup do not increase by replacing $h$ with any specific $\hat{h}$ and $\hat{h}(x) \in [0, 1]$.

To prove the second part, for any given $h \in \mathcal{H}$, we denote

\[
g_h(y^j) = 2L \epsilon_{j+1} h(x) + 2L \sum_{i=j+2}^M \epsilon_i h(\tilde{x}_i) - L^h_j.
\]

Note that, $y^j$ only appears in the $L^h_j$ term. By definition of $L^h_j$ and $L$-Lipschitz property of the loss $\ell$, we have

\[
\forall h \in \mathcal{H}, \quad |g_h(y^j) - g_h(y'^j)| \leq jL||y^j - y'^j||_{\infty}.
\]
Let \( \hat{h} = \arg \max_h g_h(y^j) \), we have
\[
\sup_h g_h(y^j) - \sup_h g_h(y^{j'}) \leq g_h(y^j) - g_h(y^{j'}) \leq jL||y^j - y^{j'}||_\infty.
\]
Let \( \hat{h}' = \arg \max_h g_h(y^{j'}) \), we have
\[
\sup_h g_h(y^j) - \sup_h g_h(y^{j'}) \geq g_{h'}(y^j) - g_{h'}(y^{j'}) \geq -jL||y^j - y^{j'}||_\infty.
\]
The proposition follows by noticing that \( f_{\hat{x},y'}(x) = \sup_h g_h(y^j) \).

**Proof** [Proof of Fact 2] Note that \( f_{\hat{x},y'}(x) = \sup_h \{2h(x) + F(h)\} \). If \( h^0(x) = 1 \), then \( \exists h \in \mathcal{H}^0 \) such that \( h(x) = 1 \) and \( F(h) = F(\hat{h}) \), thus \( f_{\hat{x},y'}(x) \geq 2 + F(\hat{h}) \). Clearly, we also have \( f_{\hat{x},y'}(x) \leq 2 \sup_h h(x) + \sup_h F(h) \leq 2 + F(\hat{h}) \), the first case follows. If \( h^0(x) = 0 \) and \( h^1(x) = 1 \), then there exists \( h \in \mathcal{H}^1 \) such that \( h(x) = 1 \) and \( F(h) = F(\hat{h}) - 1 \), thus \( f_{\hat{x},y'}(x) \geq F(\hat{h}) - 1 + 2 = F(\hat{h}) + 1 \). On the other-hand, since \( h^0(x) = 0 \), we have for all \( h \in \mathcal{H}^0 \), \( 2h(x) + F(h) = F(\hat{h}) \). For any other \( h \in \mathcal{H} \), we have \( 2h(x) + F(h) = F(\hat{h}) \). Therefore, \( f_{\hat{x},y'}(x) \leq F(\hat{h}) + 1 \), this completes the second case. Finally, if both \( h^0(x) = h^1(x) = 0 \), we have for any \( h \in \mathcal{H}^0 \), \( 2h(x) + F(h) = F(\hat{h}) \), i.e., \( f_{\hat{x},y'}(x) = F(\hat{h}) \). Moreover, for any \( h \in \mathcal{H}^0 \), it is easy to verify that \( 2h(x) + F(h) \leq F(\hat{h}) \). This completes the proof.

**Proof** [Proof of Theorem 14] Assume, w.o.l.g., \( \epsilon_{j+1} = 1 \). The functions \( h^0, h^1 \) as in Fact 2 are within \( \mathcal{F}^u \). For any \( x \in \mathcal{X} \) and \( \hat{\mu} \) uniform over \( x \), there exists a \( \gamma \)-cover \( C_\gamma \) of \( \mathcal{F}^u \) under distance \( d_{\hat{\mu}}(f_1, f_2) = \Pr_{x \sim \hat{\mu}}[f_1(x) \neq f_2(x)] \) such that \( |C_\gamma| \leq O\left(\frac{1}{\sqrt{VC(\mathcal{F}^u)}}\right) \) (Haussler, 1995). By Fact 2, there exists a function \( \mathcal{T} : (\mathcal{F}^u)^2 \rightarrow [0, 1, 2] \) such that for any \( f_{\hat{x},y'} \), there exist \( h^0, h^1 \in \mathcal{F}^u \) such that \( f_{\hat{x},y'}(x) = T(h^0(x), h^1(x)) + c_{\hat{x},y'} \), where \( c_{\hat{x},y'} = F(\hat{h}) \) as in Fact 2. Therefore, the function class \( C' \) def \( \{T(h^0, h^1) : h^0, h^1 \in C_\gamma\} \) forms a \( 2\gamma \)-cover of \( \{(f_{\hat{x},y'}(x) - c_{\hat{x},y'}) : y^j \in [0, 1]\} \) under distance \( d_{\hat{\mu}}(f_1, f_2) \) and \( |C'| \leq O\left(\frac{1}{\sqrt{VC(\mathcal{F}^u)}}\right) \). This implies that the function class \( C'' \) def \( \{g(x', x) = f(x') - f(x) : f \in C', (x', x) \in \mathcal{X}^2\} \) forms a \( 4\gamma \)-cover of
\[
G_{\hat{x}} = \{g_{\hat{x},y'}(x', x) = f_{\hat{x},y'}(x') - f_{\hat{x},y'}(x) : y^j \in [0, 1], (x', x) \in \mathcal{X}^2\}
\]under distance \( d_{\hat{\nu}}(g_1, g_2) = \Pr_{(x', x) \sim \hat{\nu}}[g_1(x', x) \neq g_2(x', x)] \) for any distribution \( \hat{\nu} \) uniform over a fixed pairing of \( \mathcal{X}^2 \) and \( |C''| \leq O\left(\frac{1}{\sqrt{VC(\mathcal{F}^u)}}\right) \). We have by the chaining argument (Wainwright, 2019, Example 5.24) that \( \text{Rad}_N(G_{\hat{x}}) \leq O\left(\sqrt{VC(\mathcal{F}^u)}N\right) \). This implies by Lemma 6 & 9 that
\[
\hat{r}_{M, N}(\mathcal{H}, \Phi) \leq O\left(\sqrt{VC(\mathcal{H})M + M\sqrt{VC(\mathcal{F}^u)}\sqrt{N}}\right).
\]
Taking \( M(n) = 1.5^n \) in (13), we have \( N = S(n) = 2 \cdot 1.5^n - 3 \), which ensures \( M(n) \leq S(n)/2 + O(1) \) (as required for (15) to hold). Invoking Lemma 13, we conclude
\[
\hat{r}_T(\mathcal{H}, \Psi) \leq O\left(\sqrt{VC(\mathcal{H})} + \sqrt{VC(\mathcal{F}^u)}\sum_{n=1}^{\lceil \log_{1.5}(T) \rceil} 1.5^{n/2} \right) \leq O\left(\sqrt{VC(\mathcal{F}^u)}T\right),
\]
where the last inequality follows by \( \mathcal{H} \subset \mathcal{F}^u \). This completes the proof and the case for \( \epsilon_{j+1} = -1 \) is symmetric with \( \mathcal{F}^l \). ■
Appendix E. Analysis of Contextual $K$-arm Bandits

In this appendix, we provide a detailed analysis of the contextual $K$-arm bandit problem with the contexts generated by an unknown i.i.d. process and costs generated adversarially, as defined in Section 2. Let $\mathcal{H} \subset [K]^X$ be a finite policy set. Following the same steps as in the online learning case, we first consider the scenario with side-information as in Section 3.1. Let $(x_1, c_1), \ldots, (x_M, c_M)$ be any realization of the feature-loss pairs and $x_0_{-N+1}$ be the side-information with $x_{-N+1}^M$ sampled i.i.d. from an (unknown) distribution $\mu$. We consider the following surrogate relaxation:

$$R_j = E_{x, \epsilon, Z} \left[ - \inf_{h \in \mathcal{H}} \left( 2\epsilon_{j+1}[h(\tilde{x}_{j+1})]Z_{j+1} + \sum_{i=1}^{j} \hat{c}_i[h(x_i)] + \sum_{i=j+2}^{M} 2\epsilon_i|h(\tilde{x}_i)|Z_i \right) + \gamma(M - j)K \right],$$

where $\epsilon$s are (vectors) i.i.d. uniform over $\{-1, 1\}^K$, $\tilde{x}_i$s are sampled from $\tilde{\mu}_N$ (i.e., the empirical distribution over $x_0_{-N+1}$) without replacement and $Z_i$s are i.i.d. with $Z_i \in \{0, \frac{1}{\gamma}\}$ such that $\Pr[Z_i = \frac{1}{\gamma}] = \gamma K$ and $\gamma$ is a parameter that needs to be tuned. Moreover, $\hat{c}_i$ is a random vector constructed from the prediction $\hat{y}_i \in [K]$ and distribution $q_i$ (see construction below) as $\hat{c}_i = \frac{1}{\gamma} I_i e_k$, where $e_k$ is the standard base of $\mathbb{R}^K$ with coordinate $k$ being 1 and $I_i$ is the indicator that takes value 1 w.p. $\frac{\gamma c_k[y_i]}{q_i[y_i]}$. Similarly, we define the variational relaxation as

$$\tilde{R}_j = E_{\tilde{x}, \epsilon, Z} \left[ - \inf_{h \in \mathcal{H}} \left( 2\epsilon_{j+1}[h(\tilde{x})]Z_{j+1} + \sum_{i=1}^{j} \hat{c}_i[h(x_i)] + \sum_{i=j+2}^{M} 2\epsilon_i|h(\tilde{x}_i)|Z_i \right) + \gamma(M - j)K \right].$$

Prediction rule. The prediction rule at step $j$ is given as follows:

1. Sample the data $\tilde{x}_t$s, $\epsilon_t$s and $Z_t$s as in the definition of $R_j$;
2. Let $\mathcal{D}_K$ be the class of distribution over $[K]$. Find

$$\hat{q}_j = \arg\min_{q \in \mathcal{D}_K} \sup_{p_j \in \mathcal{D}'} E_{\hat{c}_j \sim p_j} \left\{ \langle q, \hat{c}_j \rangle - \inf_{h \in \mathcal{H}} \left( \sum_{i=1}^{j} \hat{c}_i[h(x_i)] + \sum_{i=j+1}^{M} 2\epsilon_i|h(\tilde{x}_i)|Z_i \right) + \gamma(M - j)K \right\},$$

where $\mathcal{D}'$ is the class of distributions over $\{1/\gamma e_k : k \in [K]\} \cup \{0\}$ s.t. $\forall k \in [K], p[k] \leq \gamma$ and $\langle q, \hat{c}_j \rangle$ is the scalar product;
3. Define $q_j = (1 - \gamma K)\hat{q}_j + \gamma 1$ and make prediction $\hat{y}_j \sim q_j$.

Note that, the prediction rule is iterative in the sense that the predictions $q_j$s and $\hat{y}_j$s are used to construct $\hat{c}_j$s for the predictors in the following steps.

Let $\Phi$ be the prediction rule as define above. The bandit minimax regret with side-information is defined as follows:

$$\tilde{r}_{M,N}^\text{bandit}(\mathcal{H}, \Phi) = \sup_{\mu} E_{x_0_{-N+1}} E_{x_1} \cdots E_{x_{M}} \sup_{c_1} \cdots \sup_{c_{M}} \left[ \sum_{j=1}^{M} \langle q_j, c_j \rangle - \inf_{h \in \mathcal{H}} \sum_{j=1}^{M} c_j[h(x_j)] \right],$$

where $E_{q_j}$ is over all the internal randomness used to construct the predictor, including $\tilde{x}$, $\epsilon$, $Z$, $I_j$ and the randomness of $\hat{y}_j \sim q_j$. It is important to note that the selection of costs $c_j$s is oblivious to
the predictions \( \hat{y}_j \)'s, although it still (adaptively) depends on the contexts \( x_j \)'s, which differs from the online learning case as in (1).

We have the following bandit version of approx-admissibility.

**Lemma 20** For the predictors \( q_j \)'s and \( \hat{y}_j \)'s, we have

\[
\mathbb{E}_{x_j} \sup_{c_j} \mathbb{E}_{\hat{y}_j} [c_j \hat{y}_j] + R_j \leq \tilde{R}_{j-1}.
\]

**Proof** [Sketch] We sketch only the high-level ideas. The proof essentially follows the same arguments as in Lemma 5 and the admissibility proof of Syrgkanis et al. (2016, Theorem 3), by noticing that the primary distinction between our \( R_j \) and the relaxation described in Syrgkanis et al. (2016) is the randomness of \( \tilde{x}_i \)s. Their entire proof of admissibility is conducted by conditioning on such randomness, which only enters in the final step. Therefore, it can be applied in parallel to our case, as in the proof of Lemma 5.

Similarly, we have the following decomposition as in Lemma 6.

**Lemma 21** For the predictor \( \Phi \) and any given \( \mu \), the bandit minimax regret with side-information is upper bounded by

\[
\tilde{r}_{M,N}^{\text{bandit}}(\mathcal{H}, \Phi) \leq \mathbb{E}_{x_0} \sup_{\tilde{c}_j} \left[ \tilde{R}_0 + \sum_{j=1}^{M-1} \mathbb{E}_{x_j, \tilde{y}_j} \sup_{c_j} (\tilde{R}_j - R_j) \right].
\]

**Proof** We first observe that \( \hat{c}_j \) is an unbiased estimation of \( c_j \) for all \( j \in [M] \), i.e., \( \mathbb{E}_{\tilde{y}_j} [\hat{c}_j[k]] = c_j[k] \) for all \( k \in [K] \). Therefore,

\[
\mathbb{E}_{\tilde{y}_M} \left[ \sum_{j=1}^{M} \langle q_j, c_j \rangle - \inf_{h \in \mathcal{H}} \sum_{j=1}^{M} c_j[h(x_j)] \right] = \mathbb{E}_{\tilde{y}_M} \left[ \sum_{j=1}^{M} \langle q_j, c_j \rangle - \inf_{h \in \mathcal{H}} \sum_{j=1}^{M} c_j[h(x_j)] \right]
\]

\[
= \sup_{h \in \mathcal{H}} \mathbb{E}_{\tilde{y}_M} \left[ \sum_{j=1}^{M} \langle q_j, c_j \rangle - \hat{c}_j[h(x_j)] \right]
\]

\[
\leq \mathbb{E}_{\tilde{y}_M} \sup_{h \in \mathcal{H}} \left[ \sum_{j=1}^{M} \langle q_j, c_j \rangle - \hat{c}_j[h(x_j)] \right]
\]

\[
= \mathbb{E}_{\tilde{y}_M} \left[ \sum_{j=1}^{M} \langle q_j, c_j \rangle - \inf_{h \in \mathcal{H}} \sum_{j=1}^{M} \hat{c}_j[h(x_j)] \right].
\]

7. This assumption was also made implicitly in prior literature for contextual bandits, such as Rakhlin and Sridharan (2016); Syrgkanis et al. (2016); Banihashem et al. (2023).
where \((a)\) follows by that \(\hat{c}_j\)'s are unbiased estimations of \(c_j\)'s. This implies that

\[
\tau^{\text{bandit}}_{M,N}(\mathcal{H}, \Phi) \leq \mathbb{E}_{x_{N+1}} \mathbb{E}_{x_1} \cdots \mathbb{E}_{x_M} \sup_{c_1} \mathbb{E}_{\tilde{g}_M} \left[ \sum_{j=1}^M \langle q_j, c_j \rangle + R_M \right] \\
\leq \mathbb{E}_{x_{N+1}} \mathbb{E}_{x_1} \mathbb{E}_{\tilde{g}_1} \cdots \mathbb{E}_{x_M} \mathbb{E}_{\tilde{g}_M} \left[ \sum_{j=1}^M c_j \langle \tilde{y}_j, c_j \rangle + R_M \right],
\]

where the second inequality follows by \(\sup \mathbb{E} \leq \mathbb{E} \sup \) and \(\mathbb{E}_{\tilde{g}_j}[c_j \langle \tilde{y}_j \rangle] = \langle q_j, c_j \rangle\). The lemma then follows from Lemma 20 and the same argument as in the proof of Lemma 6.

In order to analyze the discrepancy between \(R_j\) and \(\hat{R}_j\), we define the following function:

\[
f_{\mathbf{z}_j, \tilde{c}^j}(x) = \mathbb{E}_{Z_{j+1}} \left[ \inf_{h \in \mathcal{H}} \left( 2\epsilon_{j+1} [h(x)] Z_{j+1} + \sum_{i=1}^j \tilde{c}_i [h(x_i)] + \sum_{i=j+2}^M 2\epsilon_i [h(x_i)] Z_i \right) \right],
\]

where \(Z_j\) is composed of all other variables that define \(R_j\) except \(Z_{j+1}\) and \(\tilde{c}^j\).

The following key property bounds the sensitivity of \(f_{\mathbf{z}_j, \tilde{c}^j}\):

**Lemma 22** We have for any \(\mathbf{z}_j, \tilde{c}^j\) and \(x, x' \in X\)

\[|f_{\mathbf{z}_j, \tilde{c}^j}(x) - f_{\mathbf{z}_j, \tilde{c}^j}(x')| \leq 4K.\]

**Proof** Clearly, if \(Z_{j+1} = 0\) then the sensitivity is 0, else, \(Z_{j+1} = \frac{1}{q}\) and the sensitivity is upper bounded by \(\frac{4}{q}\) using the same argument in the proof of Proposition 8. Since \(Z_{j+1} \neq 0\) happens w.p. \(\leq \gamma K\), the expected function has sensitivity upper bounded by \(\frac{4}{q} \times \gamma K = 4K\).

We now observe that

\[
\mathbb{E}_{x_{j+1}} \sup_{\tilde{c}^j} \sup_{\tilde{c}_j} (\hat{R}_j - R_j) \leq \mathbb{E}_{x_{j+1}} \sup_{\tilde{c}_j} (\hat{R}_j - R_j),
\]

since \(R_j\) and \(\hat{R}_j\) depend only on \(\tilde{c}_j\). Notably, the support set size of \(\tilde{c}_j\)s is \textit{finite} and upper bounded by \(K + 1\). Invoking Lemma 9, Lemma 22 and a simple application of Massart’s lemma gives

\[
\mathbb{E}_{x_{j+1}} \sup_{\tilde{c}^j} (\hat{R}_j - R_j) \leq O \left( \sqrt{\frac{K^2 j \log K}{N}} \right). \tag{17}
\]

Note that this upper bound is \textit{independent} of the parameter \(\gamma\). Invoking (Syrgkanis et al., 2016, Theorem 3) and taking \(\gamma = (\log |\mathcal{H}| / KM)^{\frac{1}{2}}\), we find \(\hat{R}_0 \leq O((KM)^{\frac{1}{2}} (\log |\mathcal{H}|)^{\frac{1}{2}})\). This, together with Lemma 21 and (17), implies that

\[
\tau^{\text{bandit}}_{M,N}(\mathcal{H}, \Phi) \leq O \left( (KM)^{\frac{3}{2}} (\log |\mathcal{H}|)^{\frac{1}{2}} + K \sqrt{\frac{M \log K}{N}} \right).
\]

Setting \(M(n) = n^2\) and using a similar argument as Lemma 13, we arrive at our main result of this section

\[
\tau^\Phi_{M,N}(\mathcal{H}, \Phi) \leq O \left( (K^2 (\log |\mathcal{H}|)^{\frac{1}{2}} + K \sqrt{\log K} \cdot T^{\frac{1}{2}} \right),
\]

where \(\Phi\) can be computed \textit{efficiently} by accessing to an ERM oracle (with computational cost same as Syrgkanis et al. (2016)).
Appendix F. Oblivious Adversaries

In this section, we provide the regret analysis for online learning against an oblivious adversary as introduced in Section 2. We follow the same online learning game as in (1) with the exception that the adversary fixes functions \( f_1, \ldots, f_T : \mathcal{X} \to [0, 1] \) before the game and sets the adversary labels \( y_t = f_t(x_t) \) for each time step \( t \in [T] \). Formally, for any expert class \( \mathcal{H} \) and prediction rule \( \Phi \), we are interested in the following oblivious minimax regret:

\[
\tilde{r}_T^{ob}(\mathcal{H}, \Phi) = \sup_{f_1, \ldots, f_T \in [0,1]^X} \sup_\mu \mathbb{E}_{x^T} \mathbb{E}_{y^T} \left[ \sum_{t=1}^T \ell(\hat{y}_t, f_t(x_t)) - \inf_{h \in \mathcal{H}} \sum_{t=1}^T \ell(h(x_t), f_t(x_t)) \right],
\]

where \( x^T \) are sampled i.i.d. from \( \mu \) and \( \hat{y}_t \sim \Phi(x', y^{t-1}) \) for \( t \in [T] \). For the clarity of presentation, we assume that \( \ell(\hat{y}, y) = |\hat{y} - y| \) is the absolute loss. We now ready to state the main result of this appendix:

**Theorem 23** Let \( \mathcal{H} \subset [0, 1]^X \) be a class of Rademacher complexity \( \text{Rad}_T(\mathcal{H}) = O(T^q) \) for some \( q \in [\frac{1}{2}, 1] \) and \( \ell \) be the absolute loss. Then there exists an oracle-efficient prediction rule \( \Phi \) with at most \( O(\sqrt{T} \log T) \) calls to the ERM oracle per round, such that \( \tilde{r}_T^{ob}(\mathcal{H}, \Phi) \leq O(T^q) \). In particular, for finite-VC class \( \mathcal{H} \), we have \( \tilde{r}_T^{ob}(\mathcal{H}, \Phi) \leq O(\sqrt{VC(\mathcal{H})T}) \). For a class \( \mathcal{H} \) with \( \alpha \)-fat shattering dimension \( O(\alpha^{-p}) \) for some \( p > 0 \), we have \( \tilde{r}_T^{ob}(\mathcal{H}, \Phi) \leq O(T^{\max(\frac{1}{2}, \frac{p}{p-1})}) \).

**Proof** We will follow the same path as the regret analysis for the non-oblivious adversaries as established in Section 3. We first consider the scenario with side-information \( x^{0: N+1} \), and define for any predictor \( \Phi \) the following oblivious minmax regret with side-information:

\[
r_{M,N}^{ob,side}(\mathcal{H}, \Phi) = \sup_{f_1, \ldots, f_M \in [0,1]^X} \sup_\mu \mathbb{E}_{x^{M,N+1}} \mathbb{E}_{\hat{y}^{M,N+1}} \left[ \sum_{j=1}^M \ell(\hat{y}_j, f_j(x_j)) - \inf_{h \in \mathcal{H}} \sum_{j=1}^M \ell(h(x_j), f_j(x_j)) \right],
\]

where \( x^{M,N+1} \) are sampled i.i.d. from \( \mu \). Let \( \Phi \) be the predictor as in (2) and \( R_j \) and \( \tilde{R}_j \) be the same surrogate relaxations as in (4) and (5). We claim that:

\[
r_{M,N}^{ob,side}(\mathcal{H}, \Phi) \leq \sup_{f^M} \sup_\mu \mathbb{E}_{x^{0:N}} \left[ \tilde{R}_0 + \sum_{j=1}^{M-1} \mathbb{E}_x [\tilde{R}_j - R_j] \right]. \tag{18}
\]

To see this, we find

\[
r_{M,N}^{ob,side}(\mathcal{H}, \Phi) = \sup_{f^M} \sup_\mu \mathbb{E}_{x^{M,N+1}} \mathbb{E}_{\hat{y}^M} \left[ \sum_{j=1}^M \ell(\hat{y}_j, f_j(x_j)) - \inf_{h \in \mathcal{H}} \sum_{j=1}^M \ell(h(x_j), f_j(x_j)) \right]
= \sup_{f^M} \sup_\mu \mathbb{E}_{x^{M,N+1}} \mathbb{E}_{\hat{y}^M} \left[ \sum_{j=1}^M \ell(\hat{y}_j, f_j(x_j)) + R_M \right]
= \sup_{f^M} \sup_\mu \mathbb{E}_{x^{M-1,N+1}} \mathbb{E}_{\hat{y}^{M-1}} \left[ \sum_{j=1}^{M-1} \ell(\hat{y}_j, f_j(x_j)) + \mathbb{E}_{x^M, \hat{y}^M} [\ell(\hat{y}_M, f_M(x_M)) + R_M] \right]
\]
Lemma 5 \[(a) \sup_{f_M} \sup_{\mu} \mathbb{E}_{X_{M-1}} \mathbb{E}_{y_{M-1}} \left[ \sum_{j=1}^{M-1} \ell(y_j, f_j(x_j)) + \mathbb{E}_{x_M} \sup_{y_M} \mathbb{E}_{y_M} [\ell(y_M, y_M) + R_M] \right] \]
\[(b) \sup_{f_M} \sup_{\mu} \mathbb{E}_{X_{M-1}} \mathbb{E}_{y_{M-1}} \left[ \sum_{j=1}^{M-1} \ell(y_j, f_j(x_j)) + \bar{R}_{M-1} \right] \]
\[= \sup_{f_M} \sup_{\mu} \mathbb{E}_{X_{M-1}} \mathbb{E}_{y_{M-1}} \left[ \sum_{j=1}^{M-1} \ell(y_j, f_j(x_j)) + R_{M-1} + \bar{R}_{M-1} - R_{M-1} \right] \]
\[= \sup_{f_M} \sup_{\mu} \left( \mathbb{E}_{X_{M-1}} \mathbb{E}_{y_{M-1}} \left[ \sum_{j=1}^{M-1} \ell(y_j, f_j(x_j)) + R_{M-1} \right] + \mathbb{E}_{X_{N+1}} (\bar{R}_{M-1} - R_{M-1}) \right) \]
\[(c) \sup_{f_M} \sup_{\mu} \mathbb{E}_{X_{N+1}} \left[ \bar{R}_0 + \sum_{j=1}^{M-1} \mathbb{E}_{x_j} [\bar{R}_j - R_j] \right] \]

where \((a)\) follows by that replacing \(f_M(x_M)\) with \(\sup_{y_M}\) do not decrease the value; \((b)\) follows by Lemma 5; \((c)\) follows by repeating the same argument for another \(M - 1\) steps.

Now, the key observation is that \(\mathbb{E}_{X_{N+1}} [\bar{R}_j - R_j] = 0\) for all \(j \in [M - 1]\) whenever \(N \geq M - 1\). This follows by the same argument as in the proof of Lemma 9 by noticing that the \(\sup_{y_M}\) is outside the expectation \(\mathbb{E}_{x_M}\) for oblivious adversaries. Moreover, this argument holds for all \(B = N - M + 1 \geq 1\), i.e., \(N \geq M - j\) (since by our assumption \(N \geq M - 1\) and \(j \geq 1\)). Therefore, we have

\[
\tilde{r}^{ob, side}_{M,N}(\mathcal{H}, \Phi) \leq \mathbb{E}_{X_{N+1}} \left[ \bar{R}_0 \right] \leq \text{Rad}_M(\mathcal{H}) \leq O(M^q),
\]

whenever \(N \geq M - 1\). By the epoch approach as in Section 3.2 and taking the epoch length \(M(n) = 2^n\) (which ensures \(S(n) \geq M(n) - 1\)) we conclude

\[
\tilde{r}^{ob}_{T}(\mathcal{H}, \Psi) \leq \sum_{n=1}^{\lfloor \log T \rfloor} 2^{nq} \leq O(T^q),
\]

where \(\Psi\) is the epoch predictor derived from \(\Phi\) as \((13)\). The theorem now follows by Lemma 4 and noticing that the computational error only contributes \(O(\sqrt{T})\) to the regret. \(\blacksquare\)

**Remark 24** Theorem 23 demonstrates that the oblivious minimax regret with unknown i.i.d. feature generation process is equivalent to the regret achievable with known feature generation distribution and non-oblivious adversaries (Block et al., 2022, Thm 7), which also matches the information-theoretical lower bound (upto poly-logarithmic factors).

**Remark 25** It is interesting to note that the proof of Theorem 23 can also be applied to the semi-adaptive adversaries that selects the adversary label \(y_j\) depending on \(x_j[B]\) for some \(B \geq 0\). This provides a way to interpolate all the ranges of regret from \(\tilde{O}(T^{\max\left(\frac{1}{2}, \frac{p+1}{2p}\right)})\) to \(\tilde{O}(T^{\max\left(\frac{1}{2}, \frac{p+1}{2p+1}\right)})\).